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Sir *ISAAC NEWTON*'s
Two TREATISES
OF THE
Quadrature of CURVES,
AND

ANALYSIS by Equations of an infinite Number of Terms,
explained :

CONTAINING

The Treatises themselves, translated into *English*, with a large Commentary; in which the Demonstrations are supplied where wanting, the Doctrine illustrated, and the whole accommodated to the Capacities of Beginners, for whom it is chiefly designed.

By JOHN STEWART, A. M. Professor of Mathematicks
in the Marishal College and University of *Aberdeen*.



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ST. JAMES NEWTON'S
TWO TREATISES

OF THE
QUESTIONS OF CURVATURE

AND

A NEW METHOD OF DETERMINING THE QUANTITY OF THE
CURVATURE OF A SURFACE

CONTAINING
THE THEOREMS, PROPOSITIONS, AND COROLLARIES
WHICH ARE NECESSARY TO THE UNDERSTANDING
OF THE THEORY OF CURVATURE

BY JOHN NEWTON, ESQ.
OF THE MIDDLE TEMPLE, ESQUIRE



To his *GRACE*

THOMAS *Duke* of LEEDS.

MY LORD,

WHEN a stranger implores your GRACE's protection to a mathematical work, he is encouraged to it from the patronage, with which you are pleased to favour all the liberal sciences; and not from any peculiar regard vouchsafed to this more than others. For among the many virtues so universally acknowledged in your GRACE's character, your zeal for the general advancement of learning is eminently conspicuous, as being the *President* and life of a *Society* founded solely for its encouragement. A *Society* so generous and disinterested in its institution, that neither Greece nor Rome could ever boast of a nobler design; and so applauded by foreign nations, that they would gladly imitate it, were they blessed with such a *Maecenas* to inspire them with a like ardour for the promoting of science. For whatever entertainment and delight knowledge affords the human mind, and how far soever it elevates those, who are possessed of it, above the rest of their species;

DEDICATION.

yet they only, who communicate it to others, have the merit of rendering it beneficial to mankind.

THE great regard, which your honoured *Society* retain for the works of the illustrious *Newton*, has engaged them to condescend so far, as to countenance his *Commentator*. The same motive animated my indeavours to illustrate and explain those principles, which are the foundation of the greatest and most surprising discoveries, either of the present, or any preceding age. And this being the first book of the kind, which the *Society* have taken under their care, as the dignity of the author justifies their choice; so, I would hope, it may render the public more favourable to my performance. I shall however have this satisfaction, that while I offer a small tribute to the memory of Sir *Isaac Newton*; I have the honour to pay my sincerest acknowledgements of gratitude to your GRACE, and with the highest esteem and veneration to subscribe myself,

May it please your GRACE,

Your GRACE's most obliged,

Most devoted, and

Most obedient humble Servant,

JOHN STEWART.

The P R E F A C E.

SUCH as have duly read and considered Sir *Isaac Newton's* Treatise of the *Quadrature of Curves*, and his other *Works* allied to it, must be sensible, that he has greatly extended the Limits of Geometry, by discovering a Method of investigating Truth, different from all those Methods, which Geometricians had made use of before his Time; and which leads to the Discovery of the most hidden and remote Truths in Mathematicks, and Philosophy.

But his invaluable Discoveries, especially such of them as are contained in this Treatise, are delivered by himself in such a concise Way (the Effect of his vast Genius) that they still lye beyond the Reach of not a few, who have made some considerable Progress in the Mathematical Sciences: or at least they find such Stops and Difficulties, before they are well entered into this Work, that they give it over; either apprehending that those Difficulties are so great, that they cannot surmount them at all; or, if they are able, that it will require more Time and Pains, than they can afford to bestow upon such abstruse and deep Speculations. Yet the same Persons, moved by the Character of the Author and the Work, would willingly
I bestow

bestow any reasonable Time and Pains to understand these Discoveries, were they furnished with proper and sufficient Helps for that Purpose. This Consideration may justly make one wonder, why among the vast Number of Books and Performances, which have been sent abroad into the World, since the *Quadrature of Curves* was first published, in the year 1704, no one has ever obliged the publick (so far as I know) with a Commentary upon this Work, or any Explication of it; by which Means it happens, that it's great Worth and Excellency still lye in a great Measure concealed from the Bulk even of those, who have made some tolerable Proficiency in the Study of Mathematicks.

I do not pretend to assign all the Causes, that may have concurred to produce this Effect. It certainly cannot be owing to this, that learned Men are not sensible of the Necessity and Usefulness of such an Undertaking. One Reason perhaps may be, that they, who are best qualified for it, and could execute it to best Purpose, having once engaged so deep in the Study of Geometry, as the full understanding of our Author's Treatise requires, forthwith affect to be Authors themselves, rather than labour in the lower Office of explaining and illustrating the original Discoveries of others. Notwithstanding it must be owned, that one may be as usefully employed, to say no more, in this latter Office, as in the former, which is esteemed higher. Since it has been justly enough observed, that this Science is now brought to such a Degree of Perfection, that the Application of these abstruse and sublime Speculations to some useful Purposes, either in common Life or Philosophy, is much more wanted, and would tend more to the Good of Mankind; than the carrying them to yet a greater

greater Height, without making any Application of them for the Purposes mentioned, by which the Good of *Society* might be promoted, and the *wise* and *excellent* Constitution of Things in this material World better understood.

This naturally enough leads me to answer an Objection, that some perhaps may be ready to start, both against our Author's *Treatise*, and this *Attempt* to make it more universally known. *Cui usui* is the Word: Why spend so much Time and Labour upon such abstract geometrical Speculations? We acknowledge, say such, the Pursuit and Acquisition of Knowledge is an agreeable Exercise to the Mind of Man; but this alone is not sufficient to account for wasting away so much Time, which our Nature and Circumstances, as *reasonable* Beings and Members of *Society*, require to be otherwise employed. As such, others have an Interest in Us, and may justly demand a very considerable Share of our Time and Labour. *Non nobis solum nati sumus, ortusque nostri partem patria vindicat, partem amici.* Cic.

In answer to which, it is readily acknowledged, that other *rational* Beings have a very considerable *Interest* in our Time and Labours. But if the Matter be rightly considered, it will appear, that this *Interest* and *Concern* of theirs is by no Means overlooked or neglected; on the contrary, it is consulted and regarded. The *natural* Desire of Knowledge found in all Mankind is a great *Excitement* to the Pursuit and Acquisition of it, and at the same Time a sufficient *Indication*, that *Nature* designed it for a Part of our Employment; which is still confirmed by the great Satisfaction and Delight, the Mind of Man enjoys in the Discovery and Possession of Truth. *Ex quatuor autem locis, in quos honesti naturam vimque divisimus,*

divisimus, primus ille, qui in veri cognitione consistit, maxime naturam attingit humanam. Omnes enim trahimur & ducimur ad cognitionis et scientiæ cupiditatem, in qua excellere pulchrum putamus. Cic. Whoever therefore contributes to the Advancement of Knowledge in others, by bestowing some Time and Pains to make way for the Entrance of it into their Minds; so far *administers* to their *Happiness*, and consults their *Advantage*. And this holds especially in the present Case, if the natural Tendency and genuine Effect of mathematical Knowledge pursued and acquired be right'y considered, and attended to; as it strengthens our intellectual Powers, by forming in the Mind an Habit of thinking *closely*, and reasoning *accurately*, even upon other Subjects¹.

But further, every one, who is the least conversant in these Matters, knows, that natural Philosophy never was, nor can be successfully prosecuted or advanced, but by the

¹ Would you have a Man reason well, you must use him to it betimes, exercise his Mind in observing the Connection of Ideas, and following them in train. Nothing does this better than Mathematicks; which therefore, I think, should be taught all those, who have the Time and Opportunity, not so much to make them Mathematicians, as to make them reasonable Creatures. *Locke, On the Conduct of the Understanding.* And a little after: I have mentioned Mathematicks as a way to settle in the Mind an Habit of reasoning closely, and in Train; not that I think it necessary, that all Men should be deep Mathematicians; but that having got the way of reasoning, which that Study necessarily brings the Mind to, they might be able to transfer it to other Parts of Knowledge, as they shall have occasion.—I think the Study of Mathematicks of infinite Use even to grown Men; first by experimentally convincing them, that to make any one reason well, it is not enough to have Parts, wherewith he is satisfied, and that serve him well enough in his ordinary Course. A Man in those Studies will see, that however good he may think his Understanding; yet in many Things, and those very visible, it may fail him. This would take off that Presumption, which most Men have in that Part.—The Study of Mathematicks would shew them the Necessity there is in reasoning to separate all the distinct Ideas, and see the Habitues that all those concerned in the present Enquiry have to one another, and to lay by those which relate not to the Proposition in Hand, and wholly to leave them out of the reckoning. This is that, which in other Subjects besides Quantity, is absolutely necessary to just reasoning.

In geometria partem fatentur esse utilem teneris ætatibus: agitari namque animos, atque acui ingenia, & celeritatem percipiendi inde venire concedunt. Quintil.

Help of Geometry and Arithmetick; for it is by Means of these Sciences, that we discover the *Constitution*, and investigate the *Laws*, according to which this material World, and all Things in it, are so *wisely* framed, maintained and preserved, in that admirable and beautiful *Order* and *Harmony*, wherein we behold them: which is the true Notion and Design of natural Philosophy. Accordingly you will find, that the greatest Masters in these Sciences have most successfully laboured in the promoting and perfecting of natural Knowledge. Witness our noble Author, who in that admirable Performance, his *Mathematical Principles of natural Philosophy*, hath happily shewn us, of what great Use abstract Mathematical Knowledge may be, for investigating the *Forces* of natural Bodies; and thence explaining and demonstrating the *Phænomena* and *Laws*, which Nature observes in her Operations, both in the Solar System, and upon this our Earth. And as a great Part of the Discoveries contained in that Book, is owing to, and founded upon, the *Doctrine of Fluxions* (the Invention of the same happy Genius) as was observed some Time ago by the noble and learned Marquis *D'Hospital*²; so we find in particular, that he often proceeds upon the Quadrature of Curves as a *Postulatum*, or Principle already known and granted. See *Propos.* 46, 53, 54, 56, 81. *Book I.* and many other Places. By which he hath shewn, that the most sublime Parts of Geometry, and particularly the *Doctrine of Fluxions*, and the Quadrature of Curves, are of infinite Use in true Philosophy.

² C'est encore une Justice dûë au scavant Monsieur *Newton*, & que Monsieur *Leibnitz* lui a renduë lui-même: qu'il avoit aussi trouvé quelque chose de semblable au Calcul différentiel, comme il paroît par l'excellent livre intitulé, *Philosophiæ naturalis Principia Mathematica*, qu'il nous donna en 1687: lequel est presque tout de ce calcul. *Anal. des infiniment Petits. Pref.*

P R E F A C E

Wherefore by cultivating Geometry, and studying even it's highest Parts, several noble and useful Purposes are served and promoted. Our rational and intellectual Powers are greatly enlarged and strengthened. Many Things useful and convenient in common Life are discovered. Natural Philosophy in all it's Branches is happily advanced. And what is most considerable of all, the Mind is thereby led to perceive and observe the infinite Knowledge and Wisdom, Power and Goodness of the *Almighty* Creator, and *bountiful* Preserver and Governor of this Universe. The Knowledge and Contemplation of whose Works and Perfections is the most noble and delightful Exercise of our *reasonable* Powers, and naturally leads the Mind to thank and praise, reverence and admire, love and imitate this supreme Being; which is the greatest Happiness and Perfection of any reasonable Creature, and the true End of all our Knowledge, and of our very Existence itself.

It is chiefly with a Design of promoting these noble Ends, that I publish to the World this *Commentary* upon Sir Isaac Newton's *Quadrature of Curves*, and his *Analysis by Equations of an infinite Number of Terms*: And though my Attempt should fail of the designed Effect, I shall at least have the Pleasure of designing well, which is something to a virtuous Mind.

Sir

TABLE OF CONTENTS.

Contents of Sir Isaac Newton's Treatise of the Quadrature of Curves.

Introduction to the Quadrature of Curves, Page 1.

GENESIS of Mathematical Quantities, *Art. 1.*

The original, and true Notion of Fluxions, *Art. 2.*

Fluxions very nearly as the Augments of the Flowing Quantities, generated in equal, but very small Particles of Time: and speaking accurately, in the first Ratio of the nascent Augments, *Art. 3.*

This explained and illustrated by the Description of plane Figures: whose Fluxions are as their Ordinates, *Art. 4.*

The Proportion of the Fluxions of the Absciss, Ordinate, and Curve Line, in Curves, *Art. 5.*

Fluxions in the last Ratio of the evanescent Parts, *Art. 6.*

The Proportion of the Fluxions of Solids; and their Surfaces, *Art. 7.*

Examples of the Method of Fluxions, *Art. 8.*

The Quantity x flowing uniformly, to find the Fluxion of x^n , *Art. 11.*

The Fluxions of all Sorts of Quantities may be collected by the Method of prime and ultimate Ratios: and no Necessity of introducing Figures infinitely small into Geometry, *Art. 12.*

The Quadrature of Curves, Page 5.

The Notation of Fluents and Fluxions: and an Account of their Gradation and different Orders, *Art. 13.*

CONTENTS.

Prop. 1. Prob. *An Equation being given involving any Number of flowing Quantities, to find the Fluxions, Art. 15.*

A fuller Explication of this Prop. with the Way of finding second, third, and subsequent Orders of Fluxions, *Art. 18.*

Prop. 2. Prob. *To find such Curves as can be squared, Art. 22.*

Prop. 3. Th. *When the Area of a Curve is $z^{\theta} R^{\lambda}$ (z being the Absciss, and $R = e + fz^n + gz^{2n} + \&c.$) what it's Ordinate is equal to: demonstrated, Art. 23.*

Prop. 4. Th. *When the Area of a Curve is $z^{\theta} R^{\lambda} S^{\mu}$ (z and R , being the same as before; and $S = k + lz^n + mz^{2n} \&c.$) what the Ordinate is equal to, Art. 25.*

Prop. 5. Th. *When the Ordinate of a Curve is $z^{\theta-1} R^{\lambda-1} \times a + bz^n + cz^{2n} \&c.$ what the Area will be equal to: demonstrated, Art. 26.*

Every Ordinate of a Curve is resolved into a Series two different Ways: with the Uses of this for finding the Areas of Curves; and when they may be squared, or not, *Art. 28.*

When the Ordinate is a Product contained under a rational Factor and an irreducible surd Factor, what to be done, *Art. 29.*

When the Ordinate is an irreducible rational Fraction, whose Denominator is made up of two or more Terms, what to be done, *Art. 30.*

When the Ordinate is an irreducible Fraction, whose Denominator is a Product contained under a rational Factor, and an irreducible surd Factor, what to be done, *Art. 31.*

Prop. 6. Th. *When the Ordinate of a Curve is $z^{\theta-1} R^{\lambda-1} S^{\mu-1} \times a + bz^n + cz^{2n} \&c.$ what the Area is equal to, Art. 32.*

Prop. 7. Th. *If z and R denote the same Things as before, and in the Ordinate of any Curve $z^{\theta \pm n\sigma} R^{\lambda \pm \tau}$, the given Quantities $\theta, \eta, \lambda, e, f, g, \&c.$ remain the same; and for σ and τ be wrote successively any integral Numbers; and if there be given the Areas of one, two, three, $\&c.$ of the innumerable Curves whose Ordinates are so designed, when the Ordinates are Binomials, Trinomials, Quadrinomials, $\&c.$ respectively, under the Vinculum of the Root; the Areas of all these Curves shall be given, Art. 33.*

Prop. 8. Th. *If in the Ordinate of any Curve $z^{\theta \pm n\sigma} R^{\lambda \pm \tau} S^{\mu \pm \nu}$, the Quantities $\theta, \eta, \lambda, \mu, e, f, g, k, l, m, \&c.$ remaining the same for σ, τ, ν , be wrote any whole Numbers successively: and if the Areas of*

of two of the Curves be given, which are designed by the Ordinates thus arising, when R and S are Binomials: or if three, four, &c. of these Areas be given, when R and S taken together consist of five Nomes, six Nomes, &c. respectively; the Areas of all the Curves shall be given, Art. 39.

Prop. 9. Th. The Areas of all those Curves are equal, whose Ordinates are reciprocally as the Fluxions of their Abscisses, Art. 40.

Cor. 1. Innumerable Curves found whose Areas are equal among themselves, Art. 41.

Cor. 2. How a Curve whose Ordinate is $z^{\theta-1} \times e + fz^n + gz^{2n} \&c.$ is transformed into another equal Curve. Art. 42.

Cor. 3. How a Curve whose Ordinate is $z^{\theta-1} \times a + bz^n + cz^{2n} \&c.$ $\times e + fz^n + gz^{2n} \&c.$, is transformed into another equal Curve, Art. 43.

Cor. 4. How a Curve whose Ordinate is $z^{\theta-1} \times a + bz^n + cz^{2n} \&c.$ $\times e + fz^n + gz^{2n} \&c.$ $\times k + lz^n + mx^{2n} \&c.$, is transformed into another equal Curve, Art. 44.

Cor. 5. The Curve whose Ordinate is $z^{\theta-1} \times e + fz^n + gz^{2n} \&c.$, transformed by putting $\frac{1}{z} = x$, Art. 45.

Cor. 6. The Curve whose Ordinate is $z^{\theta-1} \times e + fz^n + gz^{2n} \&c.$ $\times k + lz^n + mx^{2n} \&c.$, transformed the same Way, Art. 46.

Cor. 7. A Curve the Relation of whose Absciss and Ordinate is defined by a certain affected Equation, made to pass into another equal to it, where that Relation is exprest by an Equation not affected, Art. 48.

Cor. 8. A Curve having the Relation of the Absciss and Ordinate defined by an affected Equation of a certain Kind, transformed into another, where that Relation is exprest by an Equation less affected, Art. 49.

Cor. 9. How a Curve whose Ordinate is

$\pi z^{\theta-1} \times ve + v + \eta \times fz^n + v + 2\eta \times gz^{2n} \&c.$ $\times$
 $a + b \times ez^n + fz^{2n} + gz^{3n} \&c.$, is transformed into another
 whose Ordinate is $x^9 \times a + bx^n$, Art. 50.

Cor.

Cor. 10. How a Curve having $\pi R^{\lambda} + \phi R^{\lambda-1} S^{\mu-1} \times a S^{\nu} + b R^{\gamma}$, for it's Ordinate, is transformed into another whose Ordinate is $x^{\delta} \times a + b x^{\gamma}$, Art. 51.

Prop. 10. Prob. To find the most simple Curves, with which any Curve may be geometrically compared, whose Ordinate, from the Absciss given, is determined by an Equation not affected, Art. 52.

Case 1. When the Ordinate is $ax^{\delta-1}$, Art. 53.

Case 2. When the Ordinate is $ax^{\delta-1} \times e + fz^{\gamma} + gz^{2\gamma} \&c.$, Art. 54.

Case 3. When the Ordinate is $x^{\delta-1} \times a + bx^{\gamma} + cx^{2\gamma} \&c. \times e + fz^{\gamma} + gz^{2\gamma} \&c.$, Art. 55.

Case 4. When the Ordinate is $x^{\delta-1} \times a + bx^{\gamma} + cx^{2\gamma} \&c. \times e + fz^{\gamma} + gz^{2\gamma} \&c.$, Art. 56.

Case 5. When the Ordinate consists of different Parts: what to be done, Art. 57.

Cor. 1. Every Curve whose Ordinate is the affected square Root of it's own Equation, may be compared with the most simple Figures, rectilinear, or curvilinear, Art. 58.

Cor. 2. Every Curve, whose Ordinate, being defined by an affected Equation, passes into an Equation not affected, is either squared, or compared with the most simple Figures: and by this Means every Curve whose Equation consists of three Terms, is compared with the most simple Figures, Art. 59.

Cor. 3. Every Curve, whose Ordinate is defined by an affected Equation, which can be made to pass into an affected quadratic Equation, is either squared, or compared with the most simple Figures, Art. 60.

Schol. When Curves are to be squared, it is most convenient for Practice to have Tables of Quadratures, Art. 61.

A Table of the more simple Kind of Curves that may be squared, Page 24.

A Table of the more simple Kind of Curves that may be compared with the Ellipsis and Hyperbola, Page 25.

How the Series's of Curves of the first Tab. may be continued in inf. Art. 62.

How the same Thing may be done in Tab. 2. Art. 63.

How these Series's of Curves use to vary, Art. 64.

Prop.

CONTENTS

xv

Prop. 11. Th. *What Relation the Areas of Curves have, which constitute a Series, in which the Ordinate of any Curve is equal to the Area of the preceding applied to Unity, Art. 65.*

Schol. First, second, third, &c. Fluxions compared with the Terms of infinite converging Series's, *Art. 69.*

These Fluxions expounded by the Ordinates of Curves, *Art. 71.*

In what Cases of fluxional Equations, the Fluents may be found by the Quadrature of Curves, *Art. 75.*

Every Fluent collected from a first Fluxion, may be augmented or diminished by any constant Quantity, &c. *Art. 82.*

How the Fluents collected from the Fluxions, may be ascertained, *Art. 83.*

Conclusion.

Contents of the Explication of the Quadrature of Curves.

SECTION I.

Notes on the Introduction to the Quadrature of Curves, Page 33.

Introduction to the Explication of the Quadrature of Curves, Art. 1.

The Method of Fluxions reduced to two general Problems relating to a moving Body, *Art. 2.*

The Design, Use, and Extent of the Doctrine of the Quadrature of Curves, *Art. 3.*

The Doctrine of prime and ultimate Ratios explained and vindicated, *Art. 6.*

Some Lemmas relating to Motion, *Art. 22.*

The Author's fundamental Prop. with Respect to the Proportion of Fluxions, demonstrated, *Art. 28.*

That the Fluxions of plain Figures are as their Ordinates, when the Ordinates move with the same uniform Motion along the Absciss: demonstrated by the Method of prime and ultimate Ratios, *Art. 29.*

The same Thing demonstrated from other indisputed Principles, *Art. 31.*

That the Fluxions of plain Figures are, universally, as the Rectangles contained under the Ordinates and Fluxions of the Absciss: demonstrated, *Art. 35.*

The same Thing demonstrated otherwise, *Art. 36.*

The Areas of plain Figures, which begin to be described at the same Time, are always equal, if the Rectangles contained under their Ordinates and the Fluxions of their Absciss be always equal: and conversely, *Art. 38.*

If

- If any flowing Quantities whatsoever be always equal, their Fluxions shall be equal to one another at every the same Instant of Time: and if the Fluxions be at all Times equal, and the flowing Quantities arise at the same Time; or vanish at the same Time; the flowing Quantities at every the same Instant of Time are equal, *Art. 40.*
- The Proportion of the Fluxions of the Absciss, Ordinate, and Curve-Line, of any plain curvilinear Figure, demonstrated to be the same with the Proportion of the Subtangent, Ordinate, and Tangent, *Art. 41.*
- The Method of drawing Tangents; and the Rectification of Curve Lines thence derived, *Art. 42.*
- The Proportion of the Fluxions of solid Figures investigated and demonstrated, *Art. 43.*
- The Proportion of the Fluxions of their Curve Surfaces investigated and demonstrated, *Art. 45.*
- The Cubature of Solids; and plaining of Curve Surfaces, founded thereupon, *Art. 46.*
- Demonstration of the Example *Art. 10* of the Quadrature of Curves, *Art. 47.*
- Sir *Isaac Newton's* Rule for finding the Fluxion of any Power of a flowing Quantity demonstrated algebraically, and geometrically, *Art. 49.*
- The Rule for the Quadrature of simple Curves thence deduced and demonstrated, *Art. 52.*
- Sir *Isaac Newton* vindicated from a false Accusation of bad Reasoning, *Art. 53.*

Notes on the Quadrature of Curves.

S E C T. II.

Notes on Articles 13, 14 of the Quadrature of Curves, Page 61.

- The different Orders of Fluxions explained; and the Reality of them asserted and demonstrated, *Art. 55.*

S E C T. III.

Notes on Articles 15—25 of the Quadrature of Curves, Page 68.

Explication of Prop. 1. of the Quadrature of Curves, Art. 72.

- The Investigation and Demonstration of the Rule for finding the Fluxion of any flowing Quantity, which is the Product of any Number of Quantities, *Art. 75.*

The

C O N T E N T S.

xvii

The Rules for finding the Fluxions of fractional and surd Quantities, demonstrated, *Art.* 84.

The Method of deriving Equations for expressing the Relations of second, third, &c. Fluxions, *Art.* 88.

The Investigation of the Relations of Fluxions, by Means of fluential and fluxional Equations, further prosecuted: together with their Limitations, *Art.* 89.

An Enumeration of general Problems whose Solution is derived from the Method of Fluxions, *Art.* 100.

Which of these Problems are explained and resolved in this Explication, *Art.* 102.

Explication of Prop. 2. of the Quadrature of Curves, Page 83.

That Prop. illustrated by several Examples, *Art.* 107.

Demonstration of Prop. 4. of the Quadrature of Curves, Page 85.

The same further explained and illustrated, *Art.* 110.

S E C T. IV.

Notes on Articles 26—31 of the Quadrature of Curves, Page 89.

Some Lemmas serving to demonstrate several Things contained in the Author's Remarks upon Prop. 5. Quad. of Curves, *Art.* 119.

Explication of that Prop. and the Author's Remarks upon it, Page 91.

How the Ordinate of a Curve is expressed in two different Forms, *Art.* 127.

Hence a two fold Area arises: illustrated by Example, *Art.* 128.

Exceptions of Sir Isaac Newton's to his general Rule for determining when the Series for the Area of a Curve, doth converge or not, illustrated: and that those Exceptions are not always true, shewn by Examples, *Art.* 132.

Rules for determining when the Series for the Area of Curves, converges or not, in the Case of a binomial Curve, thus generally expressed

$$ax^{b-1} \times e + fz^n)^{a-1} = y, \text{ Art. 133.}$$

The Case of a Curve whose Ordinate is a rational irreducible Fraction whose Denominator is made up of two or more Terms (mentioned *Art.* 30. Quad. of Curves) explained and illustrated, *Art.* 136.

That no Curve whose Ordinate is such a Fraction, and involves in its Denominator any prime Divisor that has no fellow, can be squared: demonstrated, *Art.* 139.

b

A further

A further Limitation in this Case, besides that mentioned by the Author, *Art.* 140.

The Case of a Curve whose Ordinate is an irreducible Fraction, having it's Denominator made up of a rational Factor and an irreducible furd Factor, explained and illustrated, *Art.* 141.

A Catalogue of certain Classes of Curves belonging to Prop. 5. Quad. of Curves, proposed; in order to investigate general Rules for discovering when they may be squared, or not, *Art.* 142.

Curves of this Class $ax^{\theta-1} = y$, always quadrable, unless when it is $\theta = 0$, *Art.* 143.

Curves of this Class $ax^p + bx^q + cx^r \&c. = y$, always quadrable; unless p , q , or r , $\&c.$ be nothing, *Art.* 145.

Hence the Areas of all Curves, in which the Relation of the Absciss and Ordinate is defined by any algebraical Equation, however affected, may be found either exactly, or approximately, *Art.* 146.

Rule for finding when Curves of this Class $ax^{\theta-1} \times e + fz^n)^{\lambda-1} = y$, may be squared, *Art.* 147.

Curves doubly quadrable defined, *Art.* 148.

Rule for finding when Curves of this Class $ax^{\theta-1} \times e + fz^n)^{\lambda-1} = y$, are quadrable or not; and when they are doubly quadrable, *Art.* 149.

Rules for finding when a Trinomial Curve of this Class $ax^{\theta-1} \times e + fz^n + gz^{2n})^{\lambda-1} = y$, is quadrable or not; and when it is doubly quadrable, *Art.* 150.

Rule for finding when Curves of this Class $x^{\theta-1} \times e + fz^n)^{\lambda-1} \times a + bx^n = y$, are quadrable or not; and when they are doubly quadrable, *Art.* 162.

Rule for finding the same Things in Curves of this Class $x^{\theta-1} \times e + fz^n)^{\lambda-1} \times a + bx^n + cz^{2n} = y$, *Art.* 163.

Rule for finding the same Things in Curves of this Class $x^{\theta-1} \times e + fz^n)^{\lambda-1} \times a + bx^n + cz^{2n} \&c. = y$, *Art.* 164.

Rule for finding the same Things in Curves of this Class $x^{\theta-1} \times e + fz^n + gz^{2n})^{\lambda-1} \times a + bx^n + cz^{2n} \&c. = y$, *Art.* 165.

A Rule for finding when any Curve of this Form $x^{\theta-1} \times e + fz^n)^{\lambda-1} \times a + bx^n + gz^{2n} \&c. = y$, may be squared, *Art.* 166.

CONTENTS.

xix

SECT. V.

Of the Position and Limits of the Areas of Curves, Page 133.

A general Rule for determining the initial Limits of curvilinear Areas: with their Position and Extent, *Art. 168.*

A general Rule for determining the Points of the Absciss in which the Curve meets with it, *Art. 170.*

A general Rule for finding the Points where the Curve decussates the Base or Absciss, *Art. 173.*

How the Area of a Curve must be estimated, when it lies partly above and partly below the Base, *Art. 174.*

That there are sometimes different initial Limits of the Area: and what to be done in that Case, *Art. 177.*

That the Area found by the Method of Quadratures, sometimes lies partly upon one Side, and partly upon the other Side of the Ordinate: when this happens: and what to be done in that Case, *Art. 181.*

Sometimes no initial Limit of the Area can be found: in what Case this happens, *Art. 184.*

What to be done when that happens, *Art. 185.*

How to find the Area adjacent to any given Part of the Base, *Art. 187.*

The Case of the hyperbolical Space adjacent to the Asymptote in the Apollonian Hyperbola, considered and explained, *Art. 190.*

Of changing the Beginning of the Absciss; with its Use; and Effect, *Art. 192.*

SECT. VI.

Notes on Article 32 of the Quadrature of Curves.

Demonstration of Prop. 6. Quad. of Curves, Page 194.

SECT. VII.

Notes on Articles 33—39 of the Quadrature of Curves.

Explication of Prop. 7. Quad. of Curves, Page 156.

Demonstration of Case 1. of that Prop. when $R = e + fx^n$ a Binomial, *Art. 196.*

Two general Theorems for deriving the Areas of all such Curves, when the Exponent of $x^{\theta \pm n}$ is to be increased or diminished by n , *Art. 202.*

C O N T E N T S.

Explication of Case 1. Prop. 7. when $R = e + fz^n + gz^{2n}$ a Trinomial, *Art.* 204.

Three general Theorems for deriving the Areas of all such Curves, when the Index of $z^{\pm n}$ is to be increased or diminished by n , *Art.* 206.

A Case supposed by Sir *Isaac*, not sufficiently understood: with some general Observations upon it, *Art.* 209.

Demonstration of Case 2. Prop. 7. accommodated to Curves in which it is $R = e + fz^n$ a Binomial, *Art.* 210.

Two general Theorems for deriving the Areas of all such Curves, when the Index of $R^{\pm 1}$ is increased or diminished by Units, *Art.* 211.

Cases first and second of Prop. 7. illustrated by an Example, *Art.* 213.

Two general Theorems for deriving the Areas of all those Curves in which $R = e + fz^n + gz^{2n}$ a Trinomial; and the Index of $R^{\pm 1}$ is increased or diminished by Units, *Art.* 215.

Cases third and fourth of Prop. 7. illustrated, 218.

Demonstration of Prop. 8. *Quad. of Curves: divided into four different Cases*, *Art.* 219.

S E C T. VIII.

Notes on Articles 40—51 Quadrature of Curves, Page 174.

The Demonstration of Prop. 9. *Quad. of Curves*, referred to *Art.* 38. of the *Explication*, *Art.* 224.

Cor. 1. Prop. 9. illustrated by Examples, in the Transformation of Curves, *Art.* 225.

To find any Number of Curves whose Areas bear any assignable Relation to the Area of any known Curve, *Art.* 226.

Cor. 2. Prop. 9. demonstrated, *Art.* 227.

Cor. 3. demonstrated, *Art.* 228.

Cor. 4. demonstrated, *Art.* 229.

Cor. 5. demonstrated, *Art.* 230.

Cor. 6. demonstrated, *Art.* 231.

Cor. 7. demonstrated, *Art.* 233.

Cor. 8. demonstrated, *Art.* 234.

Cor. 9. demonstrated, *Art.* 235.

Cor. 10. corrected, and demonstrated, *Art.* 237.

The Necessity of the Correction evinced, *Art.* 240.

S E C T.

CONTENTS.

xxi

S E C T. IX.

Notes on Articles 52—64 Quadrature of Curves, Page 185.

Prop. 10. *Quad. of Curves, and its Corollaries explained and illustrated, Art. 242.*

The five different Cases of that Prop. explained, *Art. 243.*

Cor. 1. explained, *Art. 252.*

What to be done when the Ordinate of the Curve has different Values, *Art. 253.*

Cor. 2. demonstrated, *Art. 254.*

That every Curve whose Equation consists of three Terms however affected, is either quadrable, or may be compared with the most simple Curves, illustrated by an Example, *Art. 255.*

What to be done when the Equation defining the Relation of the Absciss and Ordinate is such that none of the preceding Methods are sufficient for finding its Area, *Art. 258.*

A general Account of the Design and Use of the Author's two Tables of Quadratures, *Art. 259.*

Explication of Tab. 1. Art. 260.

Demonstration of the Quadrature of Curves belonging to Form 1. Tab. 1. *Art. 261.*

Demonstration of the Quadrature of Curves belonging to Form 2. Tab. 1. and that they are doubly quadrable, *Art. 262.*

Demonstration of the Quadrature of Curves belonging to Form 3. Tab. 1. *Art. 264.*

A general Theorem for finding the Areas of all Curves of Form 3. with the Demonstration of Sir Isaac's Rule for carrying on the Series of Curves of this Form *in inf.* *Art. 265.*

Demonstration of the Quadrature of Curves belonging to Form 4. Tab. 1. *Art. 266.*

How the Quadrature of Curves of Forms 3 and 4. Tab. 1. may be continued forward *in inf.* by Means of the Theor. *Art. 202*, but that they cannot be continued from the first Species backwards, *Art. 268.*

A general Theorem for expressing the Areas of all Curves of Form 4, by which Sir Isaac Newton's Observation (*Art. 62. Quad. of Curves*) for continuing the Progression of Curves of this Form is demonstrated, *Art. 269.*

When the Index of the Radical in Form 3 is $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{4}$, &c. and in Form 4 is $-\frac{3}{2}$, $-\frac{5}{2}$, $-\frac{7}{2}$, &c. what to be done, *Art. 271.*

Explication

Explication of Tab. 2. Art. 272.

Demonstration of the Quadrature of Curves belonging to Species 1.
Form 1. Tab. 2. *Art. 274.*

How the Series of Curves belonging to Form 1. may be carried on *in inf.* by the Canon, *Art. 211.* Explication, or by Division: or by a general Expression exhibiting all their Areas, *Art. 275.*

Demonstration of the Quadrature Species 1. Form 3. Tab. 2. with the Construction of the conic Sections, *Art. 276.*

The Matter farther illustrated and explained by an Example, *Art. 279.*

How the Areas of the Curves belonging to Species 1. Form 3. are found, by supposing a different Relation of Abscisses from that of the former: likewise how the Areas of all the Curves of that Form may be found, *Art. 280.*

Demonstration of the Quadrature of Curves belonging to Species 2.
Form 5. Tab. 2. *Art. 281.*

How the Series of Curves belonging to that Form may be continued *in inf.* *Art. 282.*

Demonstration of the Quadrature of Curves belonging to Form 7. and Species 1. Form 11. Tab. 2. *Art. 283.*

General Remarks with Respect to the Use and Application of the Author's Tables of Quadratures, Page 212.

- 1° How the Series of Curves belonging to all the Forms are to be continued *in inf.* both forwards and backwards, under some Limitations, *Art. 285.*
- 2° What to be done when the Signs of the Quantities $e, f, g,$ &c. are changed, *Art. 286.*
- 3° With Respect to the Proportions and Limitations of these Quantities, *Art. 287.*
- 4° With Respect to the Construction of the conic Sections requisite for the Quadrature of the Curves in Tab. 2. *Art. 288.*
- 5° What to be done when the numeral Indexes $\eta, \theta,$ become negative, *Art. 314.*
- 6° What to be done when the defining Equation of any Curve is compounded of several Equations, *Art. 315.*
- 7° With Respect to the initial Limits of the Areas, *Art. 316.*
- 8° How any proposed Curve is to be managed, that its Area may be found by the Tables, *Art. 317.*
- 9° With Respect to the Continuation of the Progressions of the Curves of the several Forms of both Tables; without the Limitations in Remark 1. *Art. 318.*

The

The Application of the Author's Tables of Quadratures in a Variety of Examples, *Art.* 319.

The Fig. of Subtangents in any Curve, when the Subtangents are ordi-
nately applied to the right Series of the Arches, demonstrated to be
equal to the Area of the Curve, *Art.* 326.

What the Fig. of Secants in a Circle is equal to, when the Secant is
ordinately applied to the right Sine of the Arch, *Art.* 327.

What the Figure of Secants in a Circle, Ellipse, or Hyperbola, is equal
to, when the Secant of the Arch is ordinally applied to the versed
Sine, *Art.* 328.

That the Meridian Line in Mercator's Projection, is equal to the Dif-
ference betwixt the logarithmical Radius and logarithmical Tangent
of the half Complement of the Latitude, multiplied by 2.302585
&c. demonstrated, *Art.* 329.

The same Thing demonstrated in another Way, *Art.* 330.

To find the Areas of Mechanical Curves, *Art.* 331.

How the Areas of Spirals may be found, *Art.* 335.

To find the Areas of Curves generated by the Revolution of a Radius
about a fixt Point, *Art.* 336.

S E C T. X.

*Of the Reduction of the Hyperbola and Ellipse; together with all curvi-
linear Areas, and Fluents, which depend upon these conic Sections, to
the Measures of Ratios, and Angles; or Logarithms, and circular
Arches, Page 246.*

The Definition and fundamental Property of the logistical or logarith-
mical Curve, *Art.* 337.

The Analogy betwixt the Abscisses of the logarithmical Curve, and
the hyperbolic Spaces adjacent to the Assymptote; as both are the
Measures of Ratios, *Art.* 338.

The Principles of logarithmical Arithmetic demonstrated, *Art.* 340.

To find the Logarithm of any given Ratio: and construct a Table of
Logarithms, *Art.* 341.

Different Expressions of the Ratios of which hyperbolic Sectors, or
hyperbolic Spaces adjacent to the Assymptote, are the Measures or
Logarithms, *Art.* 344.

What Proportion the Logarithms or Measures of the same Ratio in
different Systems bear to one another, *Art.* 346.

Definition of the *Modulus* of a System of Logarithms, *Art.* 348.

The

The Modulus of the common System of Logarithms, called *Briggs's* Logarithms, *Art.* 349.

The hyperbolical Logarithm of any Ratio multiplied by the Modulus of any other System, makes the Logarithm of that Ratio in that System, *Art.* 350.

Definition of the *Ratio modularis*: that it is the same in every System: and how to be found, *Art.* 351.

A Logarithm being given, to find the Number or Ratio to which it belongs, *Art.* 352.

How any hyperbolical Spaces may be found by Means of a logarithmical Canon, *Art.* 355.

The Areas of all Curves capable of being compared with the Hyperbola; and all Fluents whose Fluxions are analogous to the Fluxions of those Areas, found and expressed by Logarithms or Measures of Ratios, *Art.* 357.

How any elliptical Area may be reduced to a circular Area: and so be found by Means of the trigonometrical Canon, and the known Proportion of the Radius of a Circle to its Circumference, *Art.* 358.

Elliptical Sectors one System of Measures of Angles, *Art.* 361.

The Modulus of that System, what, *Art.* 362.

The Areas of elliptical Spaces found by the trigonometrical Canon: illustrated by an Example, *Art.* 364.

The Areas of all Curves capable of being compared with the Ellipse; and all Fluents whose Fluxions are analogous to the Fluxions of those Areas, found and expressed by the Measures of Angles, *Art.* 365.

The Analogy betwixt the Measures of Angles, and Measures of Ratios illustrated, *Art.* 366.

The Connection betwixt Sir *Isaac Newton's* Method of expressing Fluents in his *Quadrature of Curves*, and Mr. *Cotes's* Method in his *Harmonia Mensurarum*, explained by an Example, *Art.* 372.

S E C T. XI.

Demonstration of Prop. 11. of the Quadrature of Curves, Page 268.

S E C T. XII.

Explication of the last Schol. of the Quadrature of Curves: with the Application of the preceding Doctrine to the Solution of some Problems, Page 272.

The Relation and Analogy betwixt the various Orders of Differences and Fluxions, enquired into: upon which Subject there seems to be some

- some Error or Mistake in the Author's Schol. with an Account of Dr. Keil's Correction, *Art.* 382.
- The Foundation of the Agreement betwixt the differential Method of Mr. Leibnitz and Sir Isaac Newton's Method of Fluxions, as to their Conclusions, *Art.* 383.
- Some Remarks with Respect to the two general Problems including the direct and inverse Method of Fluxions, *Art.* 385.
- Fluents pure or mixt: with the Difference betwixt them: and how the true Fluent, although mixt, may, in certain Circumstances, be found, *Art.* 387.
- The same fluxionary Expression may have two different pure Fluents: how these Fluents may be found by the Rules formerly delivered for finding the Areas of Curves, *Art.* 388.
- The Relation of the flowing Quantities the same, whether one, or other; or none of them be supposed to flow with a constant or uniform Fluxion, *Art.* 289.
- The Principle upon which the finding the Fluents from the Fluxions, by the Quadrature of Curves, is founded, *Art.* 390.
- Prob. To determine the Maxima and Minima of Quantities, *Art.* 391.
- To find whether the Ordinate of a Curve arrives at a Maximum, or Minimum; or at a Point of contrary Flexure, or Retrogression, when the Fluxion of the Ordinate vanishes, or is infinite, *Art.* 394.
- This Matter illustrated by several Examples, *Art.* 395.
- To find such a Part of a spherical Superficies as can be illuminated in its farther Part, by Light coming from a great Distance, and refracted by the nearer Hemisphere, *Art.* 398.
- Prob. To draw Tangents to all sorts of Curves, *Art.* 400.
- Examples hereof in geometrical Curves, *Art.* 401.
- The Points of contrary Flexure how determined, *Art.* 402.
- How Tangents are drawn to mechanical Curves, and Spirals: with Examples, *Art.* 403.
- Prob. To find the Length of Curve Lines: with Examples in geometrical and mechanical Curves, *Art.* 406.
- To find as many Curves as you please, whose Lengths are equal to known Right Lines, *Art.* 415.
- Prob. To find the Areas of Curve Surfaces: with Examples, *Art.* 416.
- Prob. To find the Contents of Solids generated by the Revolution of Plane Figures: with Examples, *Art.* 432.
- What Fluents may be found by the Quadrature of Curves, *Art.* 435.

The various Cases of fluxionary Equations, from which the Relations of the Fluents can be found by Means of the Quadrature of Curves, explained and illustrated, *Art.* 436.

In finding the Relation of the Fluents, one of them is supposed to flow with an uniform Fluxion, which Fluxion is put equal to Unity: and that flowing Quantity is supposed to be assumed in order to determine the other, *Art.* 444.

How the Fluents found by the preceding Operations may be ascertained, *Art.* 445.

The Author's Conclusion expressive of the Greatness of his Modesty, and Vastness of his Understanding, *Art.* 446.

Contents of the Analysis by Equations of an infinite Number of Terms.

The Author's own Account of the Design and Contents of this Treatise, *Art.* 1.

The Quadrature of simple Curves, Rule 1. Page 321.

The Quadrature of compound Curves, Rule 2. Page 322.

First Examples of this Rule: in which the Exponents of the Powers of x are all positive, *Art.* 5.

Second Examples: in which these Exponents are all negative, *Art.* 6.

Third Examples: in which these Exponents are partly positive, partly negative, *Art.* 8.

The Quadrature of all other Curves; Rule 3. Page 224.

Examples of this Rule, where you divide, *Art.* 12.

Examples where the Square Root must be extracted, *Art.* 15.

Examples by the Resolution of Equations, *Page 328.*

The numeral Resolution of affected Equations, Art. 21.

Equations wanting any Terms, or not, managed with the same Ease, *Art.* 28.

How one kind of Quantities may be most commodiously substituted for another, in an Equation, *Art.* 29.

The literal Resolution of affected Equations, Page 330.

Another Way of resolving them; when the Area is so much the nearer the Truth the greater that x is, Page 332.

Some Remarks with Respect to what is to be done, when the preceding Methods fail, *Art.* 35.

All Problems concerning the Length of Curves; the Quantity, and Surface of Solids; and the Centers of Gravity, reducible to the Quadrature of Curves, *Art.* 36.

The

CONTENTS.

xxvii

The Application of what hath been said to other Problems of that Kind,

Page 335.

To find the Lengths of Curves, Art. 38.

To find the Converse of these Things, Art. 42.

To find the Base from the Area given, Art. 43.

To find the Base from the Length of the Curve given, Art. 45.

Concerning the Continuation of the Series of the Progressions, Art. 47.

The Application of what has been said to mechanical Curves, Art. 48.

The Demonstration of the Quadrature of simple Curves belonging to Rule 1.

Page 340.

To find those Curves which can be squared, Art. 56.

The Demonstration of the Resolution of affected Equations, Art. 57.

Contents of Analysis by Equations &c. explained.

S E C T. I.

Introduction: containing an Account of the Rise and Progress of the Contemplation of Infinites; and the Quadrature of Curves: together with some general Account of the Contents of Sir Isaac Newton's two Treatises of the Quadrature of Curves; and Analysis by infinite Equations, Page 344.

The Foundation of the Method of Exhaustions laid down by Euclid, Art. 2.

The Method of Exhaustions made Use of by Euclid and Archimedes, Art. 3.

This Method illustrated by an Example taken from Archimedes, Art. 4.

The Gregorian and Newtonian Methods of infinite converging Series, lying in Embryo in the Ancients Method of Exhaustions, Art. 6.

The Analogy, Gradation, and Difference of the Method of Exhaustions; the Gregorian, and Newtonian Forms of converging Series, Art. 7.

Some Account of the Method of Indivisibles, introduced by Bonaventura Cavallerius, Art. 8.

An Account of Dr. Wallis's Arithmetic of Infinites, Art. 9.

Where Dr. Wallis's Method stopt, Art. 11.

A large Account of the Original of Sir Isaac Newton's Method of infinite Series taken from one of his Letters to Mr. Oldenburg: and delivered in his own Words. Where also some Account is given of Mr. James Gregory's Discoveries, Art. 14.

An Account of the Contents of the Analysis by Equations of an infinite Number of Terms, Art. 16.

The Treatise of the Quadrature of Curves a further Improvement :
with an Account of what it contains, *Art. 17.*

When the Methods laid down for finding the Areas of Curves in the
Quadrature of Curves fails, Recourse must be had to the Methods
laid down in this *Analysis*, *Art. 20.*

S E C T. II.

*Explication of the first Part of the Author's third Rule, which shews the
Method of finding the Areas of those Curves, which require the Reduction
of complicate Fractions and Radicals: contained in Art. 12—20.*
Page 364.

The Author's first Example upon Reduction of complicate Fractions by
Division, explained, *Art. 25.*

The Author's second Example illustrated, *Art. 27.*

How the Division in his last Example upon this Head is performed by
two different Ways: with the Effects of both as to the Area of the
Curve, *Art. 28.*

That the Reduction of specious Quantities into infinite Series, resembles
certain Operations in Decimal Arithmetic: with an Account of the
Agreement and Difference betwixt numeral and specious converging
Series's, *Art. 29.*

The Reduction of Radicals into infinite Series, by extraction of Roots,
resembles the like Operation in Decimal Arithmetic, *Art. 32.*

The Operation in the Author's Example, *Art. 15.* represented and ex-
plained more fully, *Art. 33.*

How the Law of Progression of Series's is found, *Art. 37.*

Circular and Hyperbolical Segments compared, *Art. 38.*

The Computation of the Area of a Circle, *Art. 40.*

The Rectification of an elliptical Arch; being an Explication of the
Author's Example at *Art. 18.* *Art. 42.*

What Uses the Author's Binomial Theorem serves for, *Art. 44.*

This famous Theorem exhibited, *Art. 45.*

How it was first found out, *Art. 46.*

The Application of it in a Variety of Examples, *Art. 47.*

Mr. James Gregory's Series for the Length of an elliptical Arch investi-
gated, and compared with one of Sir Isaac Newton's for the same
Purpose, *Art. 52.*

The Binomial Theorem serves for reducing complex Fractions into
infinite Series; and for raising to any perfect Powers: as well as
extracting Roots: with Examples, *Art. 54.*

The

The Series for the curvilinear Area; or Fluent, found by Help of the Binomial Theorem, may run out into an infinite Series, although the Curve be geometrically quadrable; *i. e.* the Area of the Curve; or Fluent, assignable in finite Terms, by Prop. 5 and 6. Quad. Curv. Art. 56.

How the same Thing may be discovered by the Transmutation of Curves, Art. 57.

The same Thing found by Cor. 9 and 10. Prop. 9. Quad. Curv. Art. 59.

S E C T. III.

Explication of the numeral Resolution of affected Equations contained in Art. 21—29, Page 388.

An Explanation of some Things with Respect to Decimal Fractions, Art. 60.

The Operation for extracting the Roots of affected numeral Equations, with the Abridgments made use of therein by the Author fully explained, Art. 62.

Dr. Halley's Rational Formula for the Resolution of Equations deduced and demonstrated, Art. 65.

Many other Methods for approximating to the Roots of Equations, may be investigated, Art. 66.

When there is any Doubt about the Approximate Value of the Supplement being near enough the Truth, which is found by the former Method: what to be done, Art. 67.

A Rule laid down by Sir Isaac with Respect to that Case, demonstrated, Art. 68.

Dr. Halley's irrational Formula; and Mr. Raphson's Method of resolving Equations, virtually included in our Author's Method, Art. 69.

This Method applicable to all Sorts of numeral Equations, whether the Terms be complete or not: and consequently to the extracting any Roots of Numbers, Art. 70.

Although the supplementary Equation have as many Roots as the original Equation, that yet the least Root is the Supplement: and that the preceding Methods discover this nearly: demonstrated, Art. 71.

Some further Remarks with respect to this Method of resolving Equations, Art. 72.

S E C T. IV.

Explication of the Resolution of affected, specious or literal Equations, contained in Art. 30—34. 57.

In specious Equations involving two unknown Quantities, the one as x is considered as the Absciss, and the other y , as the Ordinate of a Curve:

- Curve: and in their Resolution x is considered as very small; or very great; or differing very little from some given Quantity, *Art. 73.*
- The first Case, in which x is supposed very small, chiefly insisted upon by the Author: with the Manner of proceeding, *Art. 74.*
- Some Remarks serving to explain and demonstrate the Author's Process for extracting the Roots of literal Equations, when the species x is supposed to be very little, *Art. 75.*
- An Abridgment for shortening the Process in extracting the Roots of these Equations, mentioned by Sir Isaac, *Art. 31.* explained, *Art. 76.*
- The Process of the Extraction exhibited, according to the preceding Rules and Remarks, *Page 406.*
- The same further illustrated by another Example, *Art. 79.*
- Explication of the Author's Method of extracting the Roots of literal Equations, when the Species x is supposed to be very great, *Art. 80.*
- The Process of the Extraction exhibited, *Art. 81.*
- An Explication of the Steps of that Process: with a Rule for determining to what Length the Division, for obtaining the last Terms of the Series, may be continued, *Art. 82.*
- How the Value of y may be found otherwise, in the Case of x being very great, *Art. 83.*
- The Value of the Ordinate y being extracted, how the Area of the Curve, and it's other Symptoms are determined, *Art. 84.*
- How the Root y is extracted, when it is known or supposed that it differs but little from some given Quantity, *Art. 85.*
- How the three different Methods of extracting the Root y , may be reduced to one, *Art. 86.*

S E C T. V.

Of the various Methods of extracting and expressing the Roots of affected literal Equations, including two unknown Quantities, by Means of infinite Series's: in which the Author's 35, 42—46 Articles are explained, Page 416.

- Various Series's capable of being deduced from the same Equation, *Art. 87.*
- Some Lemmas, serving to demonstrate the Methods for discovering the first Terms of converging Series's, *Art. 88.*
- Method 1. For finding the first Terms of converging Series's; to be applied when there are two or more Terms of the proposed Equation, which involve not the Quantity x , and one or more of which involve the Quantity y , *Page 423.*

Method 2. By which the first Terms of all the converging Series's may be discovered, which are the Roots of the proposed Equation, either when x is very small, or very great, *Page 424.*

Method 3. By which, for the most Part, the first Terms of two converging Series's may be discovered, one of them for the Case of x being very small; the other for the Case of x being very great, *Page 428.*

Method 4. By the Help of a rectangular Parallelogram: by Means of which the first Terms of all the converging Series's may be found, which are Roots of the given Equation, whether x be very small; or very great, *Page 431.*

Of the different Methods for finding the subsequent Terms of the Quotient or converging Series for the Value of y , *Page 436.*

The first Method, by Means of the Parallelogram, *Art. 115.*

In what Manner certain Equations must be prepared, *Art. 117.*

Another Method for discovering the subsequent Terms, being that made use of by Sir *Isaac* in this Treatise: which is here fully explained and demonstrated, *Art. 120.*

Prob. To find whether the Exponents of the Powers of x , in the Terms of the converging Series, go on in an arithmetical Progression, or not: and if so, what the common Difference of that Progression is, *Page 442.*

A third Method for extracting the Roots of affected Equations, including two unknown Quantities, by converging Series: *viz.* by the Assumption of a Series of a proper Form with indeterminate Coefficients, *Art. 126.*

The Resolution of Equations by infinite Series, includes *virtually* in it, the Reduction of complex Fractions and Radicals into infinite Series, *Art. 134.*

The preceding Doctrine applied to the *Reversion of Series*, *Art. 135.*

How Equations *infinitely* affected are resolved thereby, *Art. 137.*

How the famous Analyst Mr. *Abraham de Moivre's* Theorem for resolving an infinite Equation; together with other still more general Theorems, may be investigated by Help of the foregoing Doctrine, *Art. 138.*

The Terms of any Power or Root whatsoever of an infinite universal Series, distinctly exhibited, to the Length of seven Terms, *Art. 139.*

What to be done when the Root of the fictitious Equation by Means of which the first Term of the converging Series ought to be found, is absurd; or wholly unknown, *Art. 140.*

What to be done when the fictitious Equation from whence the first Term of the converging Series ought to be derived, hath no real or possible Root, *Art. 142.*

That

That the Roots of Equations including two unknown variable Quantities, may be extracted and expressed by infinite Ways, *Art.* 145.
 The Application of the Extraction of Roots of affected literal Equations to the Quadrature of Curves, illustrated by Example, *Art.* 146.
 Some Directions with respect to the finding of the Curvilinear Area adjacent to any given Part of the Base or Absciss: and the changing the Beginning of the Absciss, *Art.* 148.
 Some Advantages in taking the Beginning of the Absciss exactly in the Middle of the given Part of it, *Art.* 150.
 Some Observations for facilitating the Business of Quadratures, *Art.* 152.
 An infinite Series for the Root y , of an affected Equation, being once investigated, by supposing x to be indefinitely small or great; will express the Root, whatever be the Value of x . *Art.* 153.

S E C T. VI.

The Investigation and Demonstration of the Binomial Theorem, Page 470.

Case 1. When the Exponent of the Power is a positive Integer, *Art.* 155:

Case 2. When it is a positive Fraction, *Art.* 158.

Case 3. When the Exponent is any negative Number, integral or fractional, *Art.* 160.

This Theorem more easily demonstrated by the Help of Fluxions: but in order to this the Rule for finding the Fluxion of any Power or Root, must be demonstrated from other Principles: which is here done, *Art.* 161.

Demonstration of it upon that Foundation, *Art.* 163.

What further remains, with respect to the Application of this Analysis to other Subjects, sufficiently explained, *Art.* 165.

General Scholium, by way of Conclusion to the whole, *Page 478.*

Sir

Sir *ISAAC NEWTON*'s
T R E A T I S E
 O F T H E
Quadrature of C U R V E S.

INTRODUCTION to the Quadrature of Curves.

I. **I** Consider mathematical Quantities in this Place not as consisting of very small Parts; but as describ'd by a continued Motion. Lines are describ'd, and thereby generated not by the Apposition of Parts, but by the continued Motion of Points; Superficies's by the Motion of Lines; Solids by the Motion of Superficies's; Angles by the Rotation of the Sides; Portions of Time by a continual Flux: and so in other Quantities. These Geneses really take Place in the Nature of Things, and are daily seen in the Motion of Bodies. And after this Manner the Ancients, by drawing moveable right Lines along immoveable right Lines, taught the Genesis of Rectangles.

2. Therefore considering that Quantities, which increase in equal Times, and by increasing are generated, become greater or less according to the greater or less Velocity with which they increase and are generated; I sought a Method of determining Quantities from the Velocities of the Motions or Increments, with which they are generated; and calling these Velocities of the Motions or Increments *Fluxions*, and the generated Quantities *Fluents*, I fell by degrees upon the Method of Fluxions, which I have made use of here in the Quadrature of Curves, in the Years 1665 and 1666.

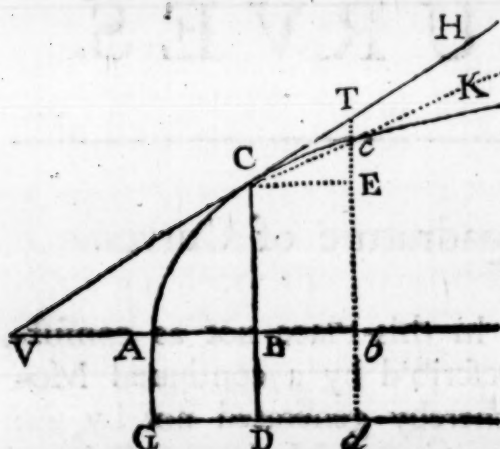
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3. Fluxions

3. Fluxions are very nearly as the Augments of the Fluents generated in equal but very small Particles of Time, and, to speak accurately, they are in the *first Ratio* of the nascent Augments; but they may be expounded by any Lines which are proportional to them.

4. Thus if the Area's ABC, ABDG be described by the Ordinates BC, BD moving along the Base AB with an uniform Motion, the Fluxions of these Area's shall be to one another as the describing Ordinates BC and BD, and may be expounded by these Ordinates, because that these Ordinates are as the nascent Augments of the Area's.

5. Let the Ordinate BC advance from it's Place into any new Place *bc*. Complete the Parallelogram BCE*b*, and draw the right Line VTH touching the Curve in C, and meeting the two Lines *bc* and BA produc'd in T and V: and B*b*, Ec and C*c* will be the Augments



now generated of the Absciss AB, the Ordinate BC and the Curve Line AC*c*; and the Sides of the Triangle CET are in the *first Ratio* of these Augments considered as nascent, therefore the Fluxions of AB, BC and AC are as the Sides CE, ET and CT of that Triangle CET, and may be expounded by these same Sides, or, which is the same thing, by the Sides of the Triangle VBC, which is similar to the Triangle CET.

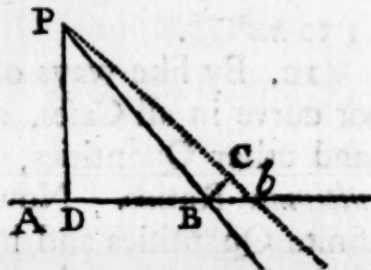
6. It comes to the same Purpose to take the Fluxions in the *ultimate Ratio* of the evanescent Parts. Draw the right Line C*c*, and produce it to K. Let the Ordinate *bc* return into it's former Place BC, and when the Points C and *c* coalesce, the right Line CK will coincide with the Tangent CH, and the evanescent Triangle CE*c* in it's ultimate Form will become similar to the Triangle CET, and it's evanescent Sides CE, Ec and C*c* will be *ultimately* among themselves as the Sides CE, ET and CT of the other Triangle CET, are, and therefore the Fluxions of the Lines AB, BC and AC are in this same Ratio. If the Points C and *c* are distant from one another by any small Distance, the right Line CK will likewise be distant from the Tangent CH by a small Distance. That the right Line CK may coincide with the Tangent CH, and the ultimate Ratios of the Lines CE, Ec and C*c* may be found, the Points C and *c* ought to coalesce and exactly coincide. The very smallest Errors in mathematical Matters are not to be neglected.

7. By

7. By the like way of reasoning, if a Circle describ'd with the Center B and Radius BC be drawn at right Angles along the Absciss AB, with an uniform Motion, the Fluxion of the generated Solid ABC will be as that generating Circle, and the Fluxion of it's Superficies will be as the Perimeter of that Circle and the Fluxion of the Curve Line AC jointly. For in whatever Time the Solid ABC is generated by drawing that Circle along the Length of the Absciss, in the same Time it's Superficies is generated by drawing the Perimeter of that Circle along the Length of the Curve AC. You may likewise take the following Examples of this Method.

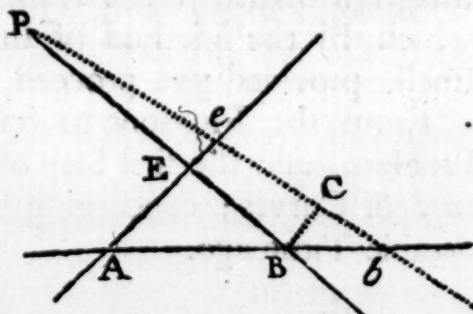
8. Let the right Line PB, revolving about the given Pole P, cut another right Line AB given in Position: it is required to find the Proportion of the Fluxions of these right Lines AB and PB.

Let the Line PB move forward from it's Place PB into the new Place P*b*. In P*b* take PC equal to PB, and draw PD to AB in such manner that the Angle *b*PD may be equal to the Angle *b*BC; and because the Triangles *b*BC, *b*PD are similar, the Augment B*b* will be to the Augment C*b* as P*b* to D*b*. Now let P*b* return into it's former Place PB, that these Augments may vanish, then the ultimate Ratio of these evanescent Augments, that is the ultimate Ratio of P*b* to D*b*, shall be the same with that of PB to DB, PDB being then a right Angle, and therefore the Fluxion of AB is to the Fluxion of PB in that same Ratio.



9. Let the right Line PB, revolving about the given Pole P, cut other two right Lines given in Position, viz. AB and AE in B and E: the Proportion of the Fluxions of these right Lines AB and AE is sought.

Let the revolving right Line PB move forward from it's Place PB into the new Place P*b*, so as to cut the Lines AB, AE in the Points *b* and *e*: and draw BC parallel to AE meeting P*b* in C, and it will be B*b* : BC :: Ab : Ae, and BC : Ee :: PB : PE, and by joining the Ratios, B*b* : Ee :: Ab × PB : Ae × PE. Now let P*b* return into it's former Place PB, and the evanescent Augment B*b* will be to the evanescent Augment Ee as AB × PB to AE × PE; and therefore the Fluxion of the right Line AB is to the Fluxion of the right Line AE in the same Ratio.



10. Hence if the revolving right Line PB cut any curve Lines given in Position in the Points B and E, and the right Lines AB, AE now becoming moveable, touch these Curves in the Points of Section B and E: the Fluxion of the Curve, which the right Line AB touches, shall be to the Fluxion of the Curve, which the right Line AE touches, as $AB \times PB$ to $AE \times PE$. The same thing would happen if the right Line PB perpetually touch'd any Curve given in Position in the moveable Point P.

11. Let the Quantity x flow uniformly, and let it be proposed to find the Fluxion of x^n .

In the same Time that the Quantity x , by flowing, becomes $x + o$, the Quantity x^n will become $x^n + o^n$, that is, by the Method of infinite Series's, $x^n + nox^{n-1} + \frac{n^2 - n}{2} oox^{n-2} + \&c.$ And the Augments o and $no x^{n-1} + \frac{n^2 - n}{2} oox^{n-2} + \&c.$ are to one another as 1 and $nx^{n-1} + \frac{n^2 - n}{2} ox^{n-2} + \&c.$

Now let these Augments vanish, and their ultimate Ratio will be 1 to nx^{n-1} .

12. By like ways of reasoning, the Fluxions of Lines, whether right or curve in all Cases, as likewise the Fluxions of Superficies's, Angles and other Quantities, may be collected by the Method of *prime* and *ultimate* Ratios. Now to institute an Analysis after this manner in finite Quantities and investigate the *prime* or *ultimate* Ratios of these finite Quantities when in their nascent or evanescent State, is consonant to the Geometry of the Ancients: and I was willing to show that, in the Method of Fluxions, there is no necessity of introducing Figures infinitely small into Geometry. Yet the Analysis may be performed in any kind of Figures, whether finite or infinitely small, which are imagin'd similar to the evanescent Figures; as likewise in these Figures, which, by the Method of Indivisibles, use to be reckoned as infinitely small, provided you proceed with due Caution.

From the Fluxions to find the Fluents, is a much more difficult Problem, and the first Step of the Solution is equivalent to the Quadrature of Curves; concerning which I wrote what follows some considerable Time ago.

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OF THE

QUADRATURE of CURVES.

13. **I**N what follows I consider indeterminate Quantities as increasing or decreasing by a continued Motion, that is, as flowing forwards, or backwards, and I design them by the Letters z, y, x, v , and their Fluxions or Celerities of increasing I denote by the same Letters pointed $\dot{z}, \dot{y}, \dot{x}, \dot{v}$. There are likewise Fluxions or Mutations more or less swift of these Fluxions, which may be call'd the second Fluxions of the same Quantities z, y, x, v , and may be thus design'd $\ddot{z}, \ddot{y}, \ddot{x}, \ddot{v}$: and the first Fluxions of these last, or the third Fluxions of z, y, x, v are thus denoted $\dot{\dot{z}}, \dot{\dot{y}}, \dot{\dot{x}}, \dot{\dot{v}}$: and the fourth Fluxions thus $\ddot{\dot{z}}, \ddot{\dot{y}}, \ddot{\dot{x}}, \ddot{\dot{v}}$. And after the same manner that $\dot{z}, \dot{y}, \dot{x}, \dot{v}$ are the Fluxions of the Quantities z, y, x, v , and these the Fluxions of the Quantities $\dot{z}, \dot{y}, \dot{x}, \dot{v}$; and these last the Fluxions of the Quantities z, y, x, v : so the Quantities z, y, x, v may be considered as the Fluxions of others, which I shall design thus $\dot{z}, \dot{y}, \dot{x}, \dot{v}$; and these as the Fluxions of others $\ddot{z}, \ddot{y}, \ddot{x}, \ddot{v}$; and these last still as the Fluxions of others $\ddot{\dot{z}}, \ddot{\dot{y}}, \ddot{\dot{x}}, \ddot{\dot{v}}$.

Therefore $z, \dot{z}, z, \dot{\dot{z}}, \ddot{z}, \ddot{\dot{z}}, \ddot{\dot{\dot{z}}}, \&c.$ design a Series of Quantities whereof every one that follows is the Fluxion of the one immediately preceding, and every one that goes before, is a flowing Quantity having that which immediately succeeds, for it's Fluxion. The like is the Series $\sqrt{az - zz}, \sqrt{az - zz}, \sqrt{az - zz}, \sqrt{az - zz}, \sqrt{az - zz}, \sqrt{az - zz}$; as likewise the Series $\frac{az + zz}{a - z}, \frac{az + zz}{a - z}, \frac{az + zz}{a - z}, \frac{az + zz}{a - z}, \frac{az + zz}{a - z}, \frac{az + zz}{a - z}, \&c.$

14. And it is to be remarked that any preceding Quantity in these Series's is as the Area of a Curvilinear Figure of which the succeeding is the rectangular Ordinate, and the Absciss is z : as $\sqrt{az - zz}$ the Area of

The Quadrature of CURVES.

of a Curve, whose Ordinate is $\sqrt{az - zz}$, and Absciss z . The Design of all these things will appear in the following Propopositions.

PROP. I. PROB. I.

15. *An Equation being given involving any Number of flowing Quantities, to find the Fluxions.*

SOLUTION.

Let every Term of the Equation be multiplied by the Index of the Power* of every flowing Quantity that it involves, and in every Multiplication change the Side or Root of the Power into it's Fluxion, and the Aggregate of all the Products with their proper Signs, will be the new Equation.

EXPLICATION.

16. Let $a, b, c, d, \&c.$ be determinate and invariable Quantities, and let any Equation be proposed involving the flowing Quantities z, y, x , &c. as $x^3 - xy^2 + a^2z - b^3 = 0$. Let the Terms be first multiplied by the Indexes of the Powers of x , and in every Multiplication for the Root, or x of one Dimension write $\dot{x}$, and the Sum of the Factors will be $3\dot{x}x^2 - \dot{x}y^2$. Do the same in y , and there arises $-2xy\dot{y}$. Do the same in z , and there arises $aaz\dot{z}$. Let the Sum of these Products be put equal to nothing, and you'll have the Equation $3\dot{x}x^2 - \dot{x}y^2 - 2xy\dot{y} + aaz\dot{z} = 0$. I say the Relation of the Fluxions is defin'd by this Equation.

DEMONSTRATION.

17. For let o be a very small Quantity, and let $o\dot{z}, o\dot{y}, o\dot{x}$ be the Moments, that is the momentaneous synchronal Increments of the Quantities z, y, x . And if the flowing Quantities are just now z, y, x , then after a Moment of Time, being increas'd by their Increments $o\dot{z}, o\dot{y}, o\dot{x}$, these Quantities shall become $z + o\dot{z}, y + o\dot{y}, x + o\dot{x}$: which being wrote in the first Equation for z, y and x , give this

* The Word translated here *Power* is *Dignitas*, Dignity, by which must be understood not only perfect, but also imperfect Powers or surd Roots, which are express'd in the Manner of perfect Powers, as is well known, by fractional Indexes. In which Sense $x^{\frac{1}{2}}, x^{\frac{2}{3}}, \&c.$ are Powers; $\frac{1}{2}$ and $\frac{2}{3}$ their Indexes, and x the Side or Root. I use the Word Power, because Dignity is seldom us'd in *English* in this Sense.

Equation

Equation $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3 - xy^2 - \dot{x}\dot{y}^2 - 2x\dot{y}\dot{y} - 2\dot{x}\dot{y}\dot{y} - x\dot{y}\dot{y} - \dot{x}\dot{y}\dot{y} + a^2\dot{z} + a^2\dot{z} - b^3 = 0$.

Subtract the former Equation from the latter, divide the remaining Equation by $\dot{x}$, and it will be $3x^2 + 3x\dot{x} + \dot{x}^2 - \dot{y}^2 - 2x\dot{y} - 2\dot{x}\dot{y} - x\dot{y} - \dot{x}\dot{y} + a^2\dot{z} = 0$. Let the Quantity $\dot{x}$ be diminished infinitely, and neglecting the Terms which vanish, there will remain $3x^2 - \dot{y}^2 - 2x\dot{y} + a^2\dot{z} = 0$. Q. E. D.

A fuller Explication.

18. After the same manner if the Equation were $x^3 - xy^2 + aa\sqrt{ax - y^2} - b^3 = 0$, thence would be produced $3x^2\dot{x} - \dot{x}\dot{y}^2 - 2x\dot{y}\dot{y} + aa\sqrt{ax - y^2} = 0$. Where if you would take away the Fluxion $\sqrt{ax - y^2}$, put $\sqrt{ax - y^2} = z$, and it will be $ax - y^2 = z^2$, and by this Proposition $a\dot{x} - 2\dot{y}\dot{y} = 2\dot{z}z$, or $\frac{a\dot{x} - 2\dot{y}\dot{y}}{2z} = \dot{z}$, that is $\frac{a\dot{x} - 2\dot{y}\dot{y}}{2\sqrt{ax - y^2}} = \dot{z}$. And thence $3x^2\dot{x} - \dot{x}\dot{y}^2 - 2x\dot{y}\dot{y} + \frac{a^3\dot{x} - 2a^2\dot{y}\dot{y}}{2\sqrt{ax - y^2}} = 0$.

19. And by repeating the Operation, you proceed to second, third and subsequent Fluxions. Let $zy^3 - z^4 + a^4 = 0$ be an Equation propos'd, and by the first Operation it becomes $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$; by the second $\dot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2\dot{y} - 4\dot{z}\dot{z}^3 - 12\dot{z}^2\dot{z}^2 = 0$; by the third, $\dot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 18\dot{z}\ddot{y}y^2 + 3z\ddot{y}\dot{y}^2 + 18z\dot{y}\ddot{y}\dot{y} + 6z\dot{y}^3 - 4\dot{z}\dot{z}^3 - 36\dot{z}\dot{z}^2\dot{z}^2 - 24\dot{z}^3\dot{z} = 0$.

20. But when one proceeds thus to second, third and following Fluxions, it is proper to consider some Quantity as flowing uniformly, and for it's first Fluxion to write Unity, for the second and subsequent ones, nothing. Let there be given the Equation $zy^3 - z^4 + a^4 = 0$, as above; and let z flow uniformly, and let it's Fluxion be Unity: then by the first Operation it shall be $y^3 + 3z\dot{y}y^2 - 4z^3 = 0$; by the second $6\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2\dot{y} - 12z^2 = 0$; by the third $9\ddot{y}y^2 + 18\dot{y}\ddot{y}\dot{y} + 3z\ddot{y}\dot{y}^2 + 18z\dot{y}\ddot{y}\dot{y} + 6z\dot{y}^3 - 24z = 0$.

But in Equations of this Kind it must be conceived that the Fluxions in all the Terms are of the same Order, *i. e.* either all of the first Order $\dot{y}$, $\dot{z}$; or all of the second $\ddot{y}$, $\ddot{z}$, $\dot{y}\dot{z}$, $\dot{z}^2$; or all of the third $\ddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, &c. And where the Case is otherwise the Order is to be completed by means of the Fluxions of a Quantity that flows uniformly, which Fluxions are understood. Thus the last Equation,

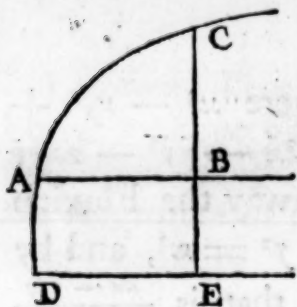
The Quadrature of CURVES.

tion, by completing the third Order, becomes $9zyy^2 + 18zy^2y + 3zyy^2 + 18zyyy + 6zy^3 - 24z\dot{z}^3 = 0$.

PROP. II. PROB. II.

22. To find such Curves as can be squared.

Let ABC be the Figure to be found, BC the rectanglar Ordinate, and AB the Absciss. Produce CB to E, so that $BE = 1$, and complete the Parallelogram ABED: and the Fluxions of the Areas ABC, ABED shall be as BC and BE. Assume therefore any Equation, by which the Relation of the Areas may be defined, and thence the Relation of the Ordinates BC and BE will be given by the first Proposition.



Q. E. I.

The two following Propositions afford Examples of this.

PROP. III. THEOR. I.

23. If for the Absciss AB and Area AE or $AB \times 1$ you write z promiscuously, and if for $e + fz^n + gz^{2n} + bz^{3n} + \&c.$ you write R : let the Area of the Curve be $z^\theta R^\lambda$, the Ordinate BC shall be equal to

$$\theta e^{\frac{\theta}{\lambda\eta}} \times f z^{\frac{\theta}{\lambda\eta}} + 2\lambda\eta \times g z^{\frac{\theta}{\lambda\eta}} + 3\lambda\eta \times b z^{\frac{\theta}{\lambda\eta}} + \&c. \times z^{\theta-1} R^{\lambda-1}.$$

DEMONSTRATION.

24. For let $z^\theta R^\lambda = v$, it will be (by Prop. I.) $\theta \dot{z} z^{\theta-1} R^\lambda + \lambda z^\theta \dot{R} R^{\lambda-1} = \dot{v}$. For R^λ in the first Term of the Equation and z^θ in the second write $\dot{R} R^{\lambda-1}$ and $\dot{z} z^{\theta-1}$, and it will become $\theta \dot{z} R + \lambda z \dot{R} \times z^{\theta-1} R^{\lambda-1} = \dot{v}$. But it was $R = e + fz^n + gz^{2n} + bz^{3n} + \&c.$: and thence (by Prop. I.) it becomes $\dot{R} = \eta f \dot{z} z^{n-1} + 2\eta g \dot{z} z^{2n-1} + 3\eta b \dot{z} z^{3n-1} + \&c.$ which being substituted, and BE or 1 wrote for z , it becomes

$$\theta e^{\frac{\theta}{\lambda\eta}} \times f z^{\frac{\theta}{\lambda\eta}} + 2\lambda\eta \times g z^{\frac{\theta}{\lambda\eta}} + 3\lambda\eta \times b z^{\frac{\theta}{\lambda\eta}} + \&c. \times z^{\theta-1} R^{\lambda-1} = \dot{v} = BC.$$

Q. E. D.

PROP.

PROP. IV. THEOR. II.

25. If AB the Absciss of a Curve be x , and if for $e + fx^n + gx^{2n} + \&c.$ you write R; and for $k + lx^n + mx^{2n} + \&c.$ you write S: let the Area of the Curve be $x^\theta R^\lambda S^\mu$, the Ordinate BC will be equal to

$$\left. \begin{array}{l} \theta e k \frac{+\theta}{+\lambda\eta} \times f k x^{\eta} \quad \frac{+\theta}{+2\lambda\eta} \times g k x^{2\eta} \quad - - - * - - - * - - - \\ \frac{+\theta}{+\mu\eta} \times e l x^{\eta} \quad \frac{+\theta}{+\lambda\eta} \times f l x^{2\eta} \quad \frac{+\theta}{+2\lambda\eta} \times g l x^{3\eta} \quad - - - * - - - \\ \frac{+\theta}{+2\mu\eta} \times e m x^{2\eta} \quad \frac{+\theta}{+\lambda\eta} \times f m x^{3\eta} \quad \frac{+\theta}{+2\lambda\eta} \times g m x^{4\eta} \end{array} \right\} \times \left\{ \begin{array}{l} z^{\theta-1} \\ R^{\lambda-1} \\ S^{\mu-1} \end{array} \right.$$

This Proposition is demonstrated after the Manner of the preceding.

PROP. V. THEOR. III.

26. If the Absciss AB of a Curve be x , and for $e + fx^n + gx^{2n} + bx^{3n} + \&c.$ be wrote R: Let the Ordinate be $x^{\theta-1} R^{\lambda-1} \times a + bx^n + cx^{2n} + dx^{3n} + \&c.$ and put $\frac{\theta}{n} = r, r + \lambda = s, s + \lambda = t, t + \lambda = v, \&c.$ The Area shall be equal to

$$\begin{aligned}
 z^0 R^1 \text{ into } & + \frac{\frac{1}{n}a}{re} \\
 & + \frac{\frac{1}{n}b - sfA}{\overline{r+1} \times e} z^\eta \\
 & + \frac{\frac{1}{n}c - \overline{s+1} \times fB - tgA}{\overline{r+2} \times e} z^{2\eta} \\
 & + \frac{\frac{1}{n}d - \overline{s+2} \times fC - \overline{t+1} \times gB - ubA}{\overline{r+3} \times e} z^{3\eta} \\
 & + \frac{\overline{s+3} \times fD - \overline{t+2} \times gC - \overline{u+1} \times bB}{\overline{r+4} \times e} z^{4\eta} \\
 & + \&c.
 \end{aligned}$$

Where

Now put $\frac{\theta}{\eta} = r$, $r + \lambda = s$, $s + \lambda = t$, &c. and in the Area $x^\theta R^\lambda \times A + Bx^\eta + Cx^{2\eta} + Dx^{3\eta} + \&c.$ write the Values of A, B, C, &c. just now found, and the Series proposed will come out. Q.E.D.

28. And it must be observed that every Ordinate is resolved into a Series two different ways. For the Index η may either be affirmative or negative.

Let the Ordinate $\frac{3k - lx}{xx\sqrt{kx - lx^2 + mx^3}}$ be proposed: it may be wrote either thus $x^{-\frac{1}{2}} \times \frac{3k - lx}{\sqrt{kx - lx^2 + mx^3}}$

or thus $x^{-2} \times \frac{-l + 3kx}{\sqrt{kx - lx^2 + mx^3}}$

In the former Case you have $a = 3k$, $b = 0$, $c = -l$: $e = k$, $f = 0$, $g = -l$, $h = m$: $\lambda = \frac{1}{2}$, $\eta = 1$, $\theta - 1 = -\frac{1}{2}$, $\theta = -\frac{1}{2} = r$, $s = -1$, $t = -\frac{1}{2}$, $v = 0$.

In the latter it is $a = -l$, $b = 0$, $c = 3k$: $e = m$, $f = -l$, $g = 0$, $h = k$: $\lambda = \frac{1}{2}$, $\eta = -1$, $\theta - 1 = -2$, $\theta = -1$, $r = 1$, $s = 1\frac{1}{2}$, $t = 2$, $v = 2\frac{1}{2}$.

Both Cases must be tried. And if either of the Series's, by means of the Terms at length failing, break off and terminate, the Area of the Curve will be had in finite Terms. Thus in the first Case of this Example, by writing in the Series the Values of $a, b, c, e, f, g, h, \lambda, \theta, r, s, t, v$, all the Terms after the first vanish in *inf.* and the Area of the Curve comes out $-2\sqrt{\frac{k - lx^2 + mx^3}{x^3}}$, and this Area, because of the negative Sign, is adjacent to the Absciss produced beyond the Ordinate. For every affirmative Area is adjacent both to Absciss and Ordinate; but a negative Area falls upon the opposite Side of the Ordinate, and is adjacent to the Absciss produced, the Sign of the Ordinate being understood to remain. By this means one of the Series's, and sometimes both of them, always terminates and becomes finite, if the Curve can be squared geometrically. But if the Curve don't admit of being thus squared, both Series's will be continued infinitely, and one of them will converge and give the Area by Approximation, except when r (because of the Area being infinite) is either nothing or a negative Integer; or when $\frac{z^\eta}{e}$ is equal to Unity. If $\frac{z^\eta}{e}$ be less than Unity, that

Series will converge in which the Index η is affirmative: But if $\frac{z^\eta}{e}$ be greater than Unity, the other Series will converge. If in the one Case the Area be adjacent to the Absciss drawn as far as the Ordinate, in the other it is adjacent to the Absciss produced beyond the Ordinate.

29. Besides you are to observe, that if the Ordinate be a Product contain'd under a rational Factor Q and an irreducible surd Factor R^π , and the Side R of the surd Factor don't divide the rational Factor Q ; it will be $\lambda - 1 = \pi$ and $R^{\lambda-1} = R^\pi$. But if the Side R of the surd Factor divide the rational Factor once, then it will be $\lambda - 1 = \pi + 1$, and $R^{\lambda-1} = R^{\pi+1}$: if it divide it twice, it will be $\lambda - 1 = \pi + 2$, and $R^{\lambda-1} = R^{\pi+2}$: if thrice, then $\lambda - 1 = \pi + 3$, and $R^{\lambda-1} = R^{\pi+3}$: and so on.

30. If the Ordinate be an irreducible rational Fraction having it's Denominator made up of two or more Terms: the Denominator must be resolved into all it's *prime* Divisors. And if there be any Divisor that has no equal, the Curve cannot be squared: but if there be two or more equal Divisors, one of them is to be rejected, and if there be yet other two or more equal to each other and unequal to the former, one of them is likewise to be rejected; and so you are to do in all the other equal Divisors, if more still remain: afterwards the Divisor which is left; or the Product contain'd under all the Divisors which are left (if there are more of them) must be put for R , and the Reciprocal of it's Square R^{-2} for $R^{\lambda-1}$, unless when that Product is a Square, or a Cube, or a Biquadrate, &c. in which Case it's Side or Root must be put for R , and the Index of the Power, *viz.* 2, or 3, or 4, taken negatively, for λ : and the Ordinate is to be reduced to the Denominator R^2 , or R^3 , or R^4 , or R^5 , &c.

Thus if the Ordinate were $\frac{z^5 + z^4 - 8z^3}{z^5 + z^4 - 5z^3 - z^2 + 8z - 4}$; because this Fraction is irreducible, and the Divisors of the Denominator have all of them Equals; for they are $z - 1$, $z - 1$, $z - 1$; and $z + 2$, $z + 2$, I reject one Divisor of each Magnitude, and the Product of the remaining Divisors $z - 1$, $z - 1$, $z + 2$, *viz.* $z^3 - 3z + 2$, I put for R , and the Reciprocal of it's Square $\frac{1}{R^2}$, or R^{-2} , for $R^{\lambda-1}$. Then I reduce the Ordinate to the Denominator R^2 or $R^{1-\lambda}$, and it becomes $\frac{z^5 - 9z^4 + 8z^3}{(z^3 - 3z + 2)^2}$, i. e. $z^3 \times \frac{8 - 9z + z^3}{z^3 - 3z + 2} \times \frac{1}{z^3 - 3z + 2}$. And thence it is $a = 8$, $b = -9$, $c = 0$, $d = 1$, &c. $e = 2$, $f = -3$, $g = 0$, $h = 1$, $\lambda - 1 = -2$, $\lambda = -1$, $\eta = 1$, $\theta - 1 = 3$, $\theta = 4 = r$, $s = 3$, $t = 2$, $v = 1$. And these Values being inserted in the Series, the Area comes out $\frac{z^4}{z^3 - 3z + 2}$, all the Terms of the Series throughout, after the first, vanishing.

31. Finally,

31. Finally, if the Ordinate be an irreducible Fraction, and it's Denominator be a Product contained under a rational Factor Q and an irreducible surd Factor R^π , you must find all the *prime* Divisors of the Side R, and reject one Divisor of each Magnitude, and then multiply the rational Factor Q by the remaining Divisors, if there be any such; and if the Product thence arising be equal to the Side R, or any Power of it whose Index is a whole Number, let that Index be m , and it will be $\lambda - 1 = -\pi - m$ and $R^{\lambda-1} = R^{-\pi-m}$.

As if the Ordinate was $\frac{3q^5 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{q^2 - x^2 \times q^3 + q^2x - qx^2 - x^3}^{\frac{1}{3}}$, because R the Side of the surd Factor or $q^3 + q^2x - qx^2 - x^3$ has these Divisors $q + x$, $q + x$, $q - x$, which are of two different Magnitudes, I reject one Divisor of each Magnitude, and I multiply the rational Factor $q^2 - x^2$ by the Divisor which remains $q + x$. And because the Product $q^3 + q^2x - qx^2 - x^3$ is equal to the Side R, I put $m = 1$, whence, since $\pi = \frac{1}{3}$, it becomes $\lambda - 1 = -\frac{4}{3}$. Therefore I reduce the Ordinate to the Denominator $R^{-\frac{4}{3}}$, and it becomes

$x^0 \times 3q^6 + 2q^5x + 8q^4x^2 + 8q^3x^3 - 7q^2x^4 - 6qx^5 \times q^3 + q^2x - qx^2 - x^3)^{-\frac{4}{3}}$
Whence $a = 3q^6$, $b = 2q^5$, &c. $e = q^3$, $f = q^2$, &c. $\theta = 1 = 0$, $\theta = 1 = \eta$, $\lambda = -\frac{4}{3}$, $r = 1$, $s = \frac{2}{3}$, $t = \frac{1}{3}$, $v = 0$. And these being wrote in the Series, the Area comes out $\frac{3q^3x + 3qx^3}{q^3 + q^2x - qx^2 - x^3}^{\frac{1}{3}}$, all the Terms through the whole Series after the third vanishing.

PROP. VI. THEOR. IV.

32. If the Absciss AB of a Curve be x , and R be put for $e + fx^n + gx^{2n} + bx^{3n} + \&c.$ and S for $k + lx^n + mx^{2n} + nx^{3n} + \&c.$ and if the Ordinate be $x^{\theta-1} R^{\lambda-1} S^{\mu-1}$ into $a + bx^n + cx^{2n} + dx^{3n} + \&c.$; and the Rectangles of the Terms $e, f, g, b, \&c.$ and $k, l, m, n, \&c.$ be

ek	fk	gk	bk	$\&c.$
el	fl	gl	bl	$\&c.$
em	fm	gm	bm	$\&c.$
en	fn	gn	bn	$\&c.$

* In the former Impressions there was an Error in the Expression of this Ordinate, which stood thus $\frac{3q^5 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{q^2 - x^2 \times q^3 + q^2x - qx^2 - x^3}^{\frac{1}{3}}$: as is easily collected from what follows. See the Notes Art. 136.

† Here also was an Error in the former Impressions, the Numerator of the Fraction being according to them $3q^2x + 3x^3$. See the Notes.

I

And

The Quadrature of CURVES.

And if the numeral Coefficients of these Rectangles respectively be

$$\frac{\theta}{\eta} = r. \quad r + \lambda = s. \quad s + \lambda = t. \quad t + \lambda = v, \text{ \&c.}$$

$$r + \mu = s'. \quad s + \mu = t'. \quad t + \mu = v'. \quad v + \mu = w, \text{ \&c.}$$

$$s' + \mu = t''. \quad t' + \mu = v''. \quad v' + \mu = w''. \quad w + \mu = x, \text{ \&c.}$$

$$t'' + \mu = v'''. \quad v'' + \mu = w'''. \quad w'' + \mu = x''. \quad x + \mu = y, \text{ \&c.}$$

The Area of the Curve will be

$$\begin{aligned} z^\theta R^\lambda S^\mu \text{ into } & + \frac{\frac{1}{\eta} a}{r + \lambda \times ek} \\ & + \frac{\frac{1}{\eta} b - s f k \quad A}{r + 1 \times ek} z^\eta \\ & + \frac{\frac{1}{\eta} c - s' + 1 \times f k \quad - t g k \quad A}{r + 2 \times ek} z^{2\eta} \\ & + \frac{\frac{1}{\eta} d - s'' + 2 \times f k \quad - t' + 1 \times g k \quad - v h k \quad A}{r + 3 \times ek} z^{3\eta} \\ & + \frac{- s' + 2 \times e l \quad - t'' + 1 \times f l \quad - v' g l \quad B}{r + 3 \times ek} z^{3\eta} \\ & + \frac{- s'' + 2 \times e l \quad - t'' + 1 \times f l \quad - v'' g l \quad B}{r + 3 \times ek} z^{3\eta} \\ & + \text{ \&c.} \end{aligned}$$

Where A denotes the given Coefficient of the first Term $\frac{1}{\eta} a$ with it's Sign $+$ or $-$, B the given Coefficient of the second Term, C the given Coefficient of the third : and so on. And one or more of the Terms $a, b, c, \text{ \&c. } e, f, g, \text{ \&c. } k, l, m, \text{ \&c.}$ may be wanting.

The Proposition is demonstrated after the Manner of the preceding, and what Things are there remarked, likewise obtain here. Moreover the Series of such Propositions goes on infinitely, and the Progression of the Series is manifest.

P R O P. VII. T H E O R. V.

33. If for $e + f z^\eta + g z^{2\eta} + \text{ \&c.}$ be wrote R as above, and in the Ordinate of any Curve $z^{\theta \pm \eta \sigma} R^{\lambda \pm \tau}$, the given Quantities $\theta, \eta, \lambda, e, f, g, \text{ \&c.}$ remain the same; and for σ and τ be wrote successively any integral Numbers :

Numbers: and if there be given the Area of one of these Curves which are design'd by innumerable Ordinates thus arising, when the Ordinates are Binomials under the Vinculum of the Root; or if the Areas of two of the Curves be given when the Ordinates are Trinomials under the Vinculum of the Root; or the Areas of three of the Curves, when the Ordinates are Quadrinomials under the Vinculum of the Root, and so on *in infinitum*: I say the Areas of all these Curves shall be given.

34. I reckon here for Names all the Terms under the Vinculum of the Root as well them that are deficient, as them that are full, whose Indexes of their Powers are in an arithmetical Progression. Thus the Ordinate $\sqrt{a^4 - ax^3 + x^4}$, because of two deficient Terms betwixt a^4 and $-ax^3$ ought to be esteemed a Quinquinomial. But $\sqrt{a^4 - x^4}$ is a Binomial and $\sqrt{a^4 + x^4 - \frac{x^8}{a^4}}$ a Trinomial, since the Progression goes on now by greater Differences. Now the Proposition is thus demonstrated.

C A S E I.

35. Let the Ordinates of two Curves be $pz^{\theta-1}R^{\lambda-1}$ and $qz^{\theta+2\eta-1}R^{\lambda-1}$ and their Areas pA and qB , R being a Trinomial $e + fz^{\eta} + gz^{2\eta}$. And since by Prop. 3. $z^{\theta}R^{\lambda}$ is the Area of a Curve, whose Ordinate is $\theta e + \theta + \lambda\eta \times fz^{\eta} + \theta + 2\lambda\eta \times gz^{2\eta}$ into $z^{\theta-1}R^{\lambda-1}$, subtract the former Ordinates and Areas from the latter Ordinate and Area, and there remains $\theta e - p + \frac{\theta}{\lambda\eta} \times f - q \times z^{\eta} + \frac{\theta}{2\lambda\eta} \times gz^{2\eta} \times z^{\theta-1}R^{\lambda-1}$ the Ordinate of a new Curve, and $z^{\theta}R^{\lambda} - pA - qB$ it's Area. Put $\theta e = p$, and $\theta f + \lambda\eta f = q$, and the Ordinate becomes $\theta + 2\lambda\eta \times gz^{2\eta} \times z^{\theta-1}R^{\lambda-1}$, and the Area $z^{\theta}R^{\lambda} - \theta eA - \theta fB - \lambda\eta fB$. Divide both by $\theta g + 2\lambda\eta g$, and call the Area thence arising C , and assuming any Quantity r , rC will be the Area of a Curve whose Ordinate is $rz^{\theta+2\eta-1}R^{\lambda-1}$. And by the same Method we have found the Area rC corresponding to the Ordinate $rz^{\theta+2\eta-1}R^{\lambda-1}$ from the Areas pA and qB , you may from the Areas qB and rC find a fourth Area, *viz.* sD corresponding to the Ordinate $sz^{\theta+3\eta-1}R^{\lambda-1}$, and so on *in inf.* And the like is the Rate of the Progression which proceeds the contrary way from the Areas B and A . If any of the Terms θ , $\theta + \lambda\eta$, and $\theta + 2\lambda\eta$ be wanting and break off the Series, let the Area pA be assumed in the Beginning of the one Progression, and the Area qB in the Beginning of the other, and from these two Areas, all the Areas will be given in both Progressions. And contrarily, from any other two Areas

Areas assumed you can return by Analysis to the Areas A and B, so that from two Areas given all the rest are given. Q. E. O.

This is the Case of Curves in which θ the Index of z is increased or diminished by the perpetual Addition or Subtraction of the Quantity η . The other Case respects Curves where the Index λ is increased or diminished by Units.

CASE II.

36. If the Ordinates $px^{\theta-1}R^\lambda$ and $qx^{\theta+\eta-1}R^\lambda$, to which the Areas pA and qB may now be supposed to answer, be multiplied by R or $e + fz^\eta + gz^{2\eta}$, and then again be divided by R , they become $pe + pfz^\eta + pgz^{2\eta} \times x^{\theta-1}R^{\lambda-1}$, and $qez^\eta + qfz^{2\eta} + qgz^{3\eta} \times x^{\theta-1}R^{\lambda-1}$. And (by Prop. 3.) $ax^\theta R^\lambda$ is the Area of a Curve, whose Ordinate is $\theta ae + \theta + \lambda\eta \times afz^\eta + \theta + 2\lambda\eta \times agz^{2\eta} \times x^{\theta-1}R^{\lambda-1}$, and $bx^{\theta+\eta}R^\lambda$ the Area of a Curve whose Ordinate is $\theta + \eta \times bez^\eta + \theta + \eta + \lambda\eta \times bfz^{2\eta} + \theta + \eta + 2\lambda\eta \times bgz^{3\eta} \times x^{\theta-1}R^{\lambda-1}$.

And the Sum of these four Areas is $pA + qB + ax^\theta R^\lambda + bx^{\theta+\eta}R^\lambda$ and the Sum of the corresponding Ordinates

$$\begin{array}{r} \theta ae + \theta + \lambda\eta \times afz^\eta + \theta + 2\lambda\eta \times agz^{2\eta} + \theta + \eta + 2\lambda\eta \times bgz^{3\eta} \times x^{\theta-1}R^{\lambda-1} \\ + pe + \theta + \eta \times be + \theta + \eta + \lambda\eta \times bf + 1 \times qg \\ + 1 \times pf + 1 \times pg \\ + 1 \times qe + 1 \times qf. \end{array}$$

If the first Term, the third and fourth be put equal to nothing separately, by the first it will be $\theta ae + pe = 0$, or $-\theta a = p$, by the fourth $-\theta b - \eta b - 2\lambda\eta b = q$, and by the third (exterminating p and q) $\frac{2ag}{f} = b$. Whence the second becomes $\frac{\lambda\eta aff - 4\lambda\eta age}{f}$, and therefore the Sum of the four Ordinates is $\frac{\lambda\eta af^2 - 4\lambda\eta age}{f} x^{\theta+\eta-1}R^{\lambda-1}$, and the Sum of as many corresponding Areas is $ax^\theta R^\lambda + \frac{2ag}{f} x^{\theta+\eta}R^\lambda - \theta aA + \frac{2\theta + 2\eta + 4\lambda\eta}{f} agB$. Divide these Sums by $\frac{\lambda\eta af^2 - 4\lambda\eta age}{f}$, and if the last Quote be call'd D, then shall D be the Area of a Curve whose Ordinate is the first Quote $x^{\theta+\eta-1}R^{\lambda-1}$.

And by the same Method, if you put all the Terms of the Ordinate save the first equal each to nothing, you may find the Area of a Curve whose Ordinate is $x^{\theta-1}R^{\lambda-1}$. Let that Area be called C, and by the same Means that the Areas C and D have been found from the Areas A and

A and B, you may from these Areas C and D find other two Areas E and F answering to the Ordinates $x^{\theta-1}R^{\lambda-2}$ and $x^{\theta+\eta-1}R^{\lambda-2}$, and so on continually. And by a contrary Analysis you may return from the Areas E and F to the Areas C and D, and thence to the Areas A and B, and likewise to the other Areas which follow in the Progression. Therefore if the Index λ be increas'd or diminish'd by the continual Addition or Subtraction of Units, and of the Areas which answer to the Ordinates thus arising two of the most simple be given; all the others *in inf.* are given.

C A S E III.

37. And by joining these two Cases together, if both the Index θ be any how increas'd or diminish'd by the continual Addition or Subtraction of η , and the Index λ by the continual Addition or Subtraction of Unity, the Areas corresponding to the several Ordinates thus arising, shall be given. Q. E. O.

C A S E IV.

38. And by a like way of reasoning, if the Ordinate consist of a Quadrinomial under the radical Sign, and there be given three of the Areas, or if it consist of a Quinquinomial, and there be given four of the Areas, and so on: all the Areas shall be given which can be produced by the Addition or Subtraction of the Number η to the Index θ , or of Unity to the Index λ . And the like is the Reasoning with respect to the Curves, whose Ordinates are made up of the Binomials, and one Area of those which are not geometrically quadrable is given. Q. E. O.

P R O P. VIII. T H E O R. VI.

39. If for $e + fx^{\eta} + gx^{2\eta} + \&c.$ and $k + lx^{\eta} + mx^{2\eta} + \&c.$ be wrote R and S as above, and in the Ordinate of any Curve $x^{\theta \pm \eta \sigma} R^{\lambda \pm \tau} S^{\mu \pm \nu}$, the given Quantities $\theta, \eta, \lambda, \mu, e, f, g, k, l, m, \&c.$ remain the same; and for σ, τ and ν be wrote any whole Numbers successively: and if the Areas of two of the Curves be given which are design'd by the Ordinates thus arising, when the Quantities R and S are Binomials, or if the Areas of three of the Curves be given, when R and S together consist of five Nomes or Terms, or the Areas of four of the Curves when R and S both taken together consist of six Nomes or Terms, and so on *in infinitum*: I say the Areas of all the Curves shall be given. It is demonstrated after the Manner of the preceding Proposition.

D

P R O P.

PROP. IX. THEOR. VII.

40. The Areas of those Curves are equal among themselves, whose Ordinates are reciprocally as the Fluxions of their Abscisses.

For the Rectangles contain'd under the Ordinates, and the Fluxions of the Abscisses will be equal, and the Fluxions of the Areas are as these Rectangles.

COROL. I.

41. If there be assum'd any Relation betwixt the Abscisses of two Curves, and thence by Prop. I. you seek the Relation of the Fluxions of the Abscisses, and the Ordinates be made reciprocally proportional to these Fluxions, you may find innumerable Curves, whose Areas are equal among themselves.

COROL. II.

42. For every Curve whose Ordinate is $z^{\theta-1} \times e + fz^n + gz^{2n} + \&c.$, by assuming any Quantity for v , and putting $\frac{v}{z} = s$ and $z' = x$, passes into another Curve equal to itself whose Ordinate is $\frac{v}{x} \times e + fx' + gx^{2'} + \&c.$

COROL. III.

43. And every Curve whose Ordinate is $z^{\theta-1} \times a + bz^n + cz^{2n} + \&c.$ $\times e + fz^n + gz^{2n} + \&c.$, if you assume any Quantity for v , and put $\frac{v}{z} = s$, and $z' = x$, passes into another Curve equal to itself, whose Ordinate is $\frac{v}{x} \times a + bx' + cx^{2'} + \&c.$ $\times e + fx' + gx^{2'} + \&c.$

COROL. IV.

44. And every Curve whose Ordinate is $z^{\theta-1} \times a + bz^n + cz^{2n} + \&c.$ $\times e + fz^n + gz^{2n} + \&c.$ $\times k + lz^n + mx^{2n} + \&c.$ by assuming any Quantity for v , and putting $\frac{v}{z} = s$, and $z' = x$, passes into another Curve equal to it, whose Ordinate is

$$\frac{v}{x} \times a + bx' + cx^{2'} + \&c. \times e + fx' + gx^{2'} + \&c. \times k + lx' + mx^{2'} + \&c.$$

COROL.

COROL. V.

45. And every Curve whose Ordinate is $z^{\theta-1} \times \sqrt[\lambda]{e + fz^n + gx^{2n} + \mathcal{E}c.}^\lambda$, by putting $\frac{1}{z} = x$, passes into another Curve equal to itself, whose Ordinate is $\frac{1}{x^{\theta+1}} \times \sqrt[\lambda]{e + fx^{-n} + gx^{-2n} + \mathcal{E}c.}^\lambda$: that is $\frac{1}{x^{\theta+1+\eta\lambda}} \times \sqrt[\lambda]{f + ex^\eta}^\lambda$, if the Quantity under the Vinculum of the Root be a Binomial; or $\frac{1}{x^{\theta+1+2\eta\lambda}} \times \sqrt[\lambda]{g + fx^\eta + ex^{2\eta}}^\lambda$, if it be a Trinomial: and so for others.

COROL. VI.

46. And every Curve whose Ordinate is $z^{\theta-1} \times \sqrt[\lambda]{e + fz^n + gx^{2n} + \mathcal{E}c.}^\lambda \times \sqrt[\mu]{k + lx^n + mx^{2n} + \mathcal{E}c.}^\mu$, by putting $\frac{1}{z} = x$, passes into another equal Curve, whose Ordinate is $\frac{1}{x^{\theta+1}} \times \sqrt[\lambda]{e + fx^{-n} + gx^{-2n} + \mathcal{E}c.}^\lambda \times \sqrt[\mu]{k + lx^{-n} + mx^{-2n} + \mathcal{E}c.}^\mu$, that is $\frac{1}{x^{\theta+1+\eta\lambda+\eta\mu}} \times \sqrt[\lambda]{f + ex^\eta}^\lambda \times \sqrt[\mu]{l + kx^\eta}^\mu$, if the Quantities under the Vinculums of the Roots be Binomials, or $\frac{1}{x^{\theta+1+2\eta\lambda+2\eta\mu}} \times \sqrt[\lambda]{g + fx^\eta + ex^{2\eta}}^\lambda \times \sqrt[\mu]{l + kx^\eta}^\mu$ if the Quantity under the Vinculum of the first Root be a Trinomial, and that under the second a Binomial: and so for others.

47. And observe, that the two equal Areas in these two last Corollaries, lye towards contrary Parts of the Ordinates. If the Area in the one Curve be adjacent to the Absciss, the Area equal to it in the other Curve, is adjacent to the Absciss produced.

COROL. VII.

48. If the Relation betwixt the Ordinate y of any Curve and the Absciss z be defin'd by any affected Equation of this Form,

$$y^a \times e + fy^n z^b + gy^{2n} z^{2b} + by^{3n} z^{3b} + \mathcal{E}c. = z^\beta \times k + ly^n z^\gamma + my^{2n} z^{2\gamma} + \mathcal{E}c.$$

this Figure, by affuming $s = \frac{\gamma-\beta}{\eta}$, $x = \frac{1}{z}$ and $\lambda = \frac{\eta-\beta}{\alpha\delta+\beta\eta}$, passes into another equal to itself, whose Absciss x , from the Ordinate v given, is determined by an Equation not affected, viz.

$$\frac{1}{s^\alpha} \times \sqrt[\lambda]{e + fv^\alpha + gv^{2\alpha} + \mathcal{E}c.}^\lambda \times \sqrt[\lambda]{k + lv^\gamma + mv^{2\gamma} + \mathcal{E}c.}^\lambda = x.$$

COROL. VIII.

49. If the Relation betwixt y the Ordinate of any Curve, and z it's Absciss be defined by any affected Equation of this Form $y^n \times$

$$\overline{e + fy^n z + gy^{2n} z^{2\delta} + \mathcal{E}c.} = \overline{z^\beta \times k + ly^n z^\delta + my^{2n} z^{2\delta} + \mathcal{E}c.} \\ + \overline{z^\gamma \times p + qy^n z^\delta + ry^{2n} z^{2\delta} + \mathcal{E}c.}$$

This Figure, by taking $s = \frac{n-\delta}{n}$, $x = \frac{1}{s} z$, $\mu = \frac{\alpha\delta + \beta n}{n-\delta}$ and $v = \frac{\alpha\delta + \gamma n}{n-\delta}$, passes into another equal to itself, whose Absciss x is determined from the Ordinate v given, by this Equation less affected $v^n \times$

$$\overline{e + fv^n + gv^{2n} + \mathcal{E}c.} = \overline{s^\mu x^\mu \times k + lv^n + mv^{2n} + \mathcal{E}c.} \\ + \overline{s^\gamma x^\gamma \times p + qv^n + rv^{2n} + \mathcal{E}c.}$$

COROL. IX.

50. Every Curve whose Ordinate is

$$\overline{\pi z^{\theta-1} \times ve + v + \eta fz^n + v + 2\eta gz^{2n} + \mathcal{E}c.} \times \overline{e + fz^n + gz^{2n} + \mathcal{E}c.}^{\lambda-1} \\ \times \overline{a + b \times ex^\tau + fx'^{\tau+n} + gx'^{\tau+2n} + \mathcal{E}c.}^{\tau}$$

if $\theta = \lambda\tau$, and you assume $x = \overline{ex^\tau + fx'^{\tau+n} + gx'^{\tau+2n} + \mathcal{E}c.}^{\tau}$, $\sigma = \frac{\tau}{\lambda}$ and $\vartheta = \frac{\lambda-\pi}{\tau}$, passes into another equal to itself, whose Ordinate, is $x^\vartheta \times a + bx^\sigma$. And you may observe that the former Ordinate, in this Corollary, becomes more simple by putting $\lambda = 1$, or by putting $\tau = 1$, and so ordering it that the Root of the Power may be extracted whose Index is ω ; or yet by putting $\omega = -1$, and $\lambda = 1 = \tau = \sigma = \pi$, to pass other Curves.

COROL. X.

51. For $\overline{ex^\tau + fx'^{\tau+n} + gx'^{\tau+2n} + \mathcal{E}c.}$, $\overline{ve x^{\tau-1} + v + \eta fz'^{\tau+n-1} + v + 2\eta gz'^{\tau+2n-1} + \mathcal{E}c.}$, $\overline{k + lx^n + mx^{2n} + \mathcal{E}c.}$, and $\overline{\eta lx^{n-1} + 2\eta mx^{2n-1} + \mathcal{E}c.}$ write R , r , S and s respectively, and every Curve whose Ordinate is $\pi Sr + \phi R s \times R^{\lambda-1} S^{\mu-1} \times \overline{a S^v + b R^\tau}^\omega$, if it be $\frac{\mu+\nu\omega}{\lambda} = \frac{-v}{\tau} = \frac{\phi}{\pi}$, $\frac{\tau}{\pi} = \sigma$, $\frac{\lambda-\pi}{\pi} = \vartheta$, and $R^\pi S^\vartheta = x$, passes into another equal to itself, whose Ordinate

* There is in this Place an Error in the printed Copies, which I have corrected. See the Demonstration in the Notes, Art. 332-335.

is $x^{\theta} \times a + bx^{\tau}$. And observe that the former Ordinate becomes more simple, by putting Unities for τ , ν , and λ or μ , and so ordering it that the Root of that Power may be extracted whose Index is ω , or by putting $\omega = -1$ or $\mu = \omega$.

PROP. X. PROB. III.

52. To find the most simple Figures with which any Curve may be geometrically compared, whose Ordinate y , from the Absciss x given, is determined by an Equation not affected.

CASE I.

53. Let the Ordinate be $ax^{\theta-1}$, and the Area shall be $\frac{1}{\theta}ax^{\theta}$, as is easily collected from Prop. 5. by putting $b = 0 = c = d = f = g = h$ and $e = 1$.

CASE II.

54. Let the Ordinate be $ax^{\theta-1} \times e + fx^{\tau} + gx^{2\tau} + \mathcal{E}c$, and if the Figure can be geometrically compared with rectilinear Figures, it will be squared by Prop. 5, by putting $b = 0 = c = d$. But if otherwise, you must change it into another Curve equal to it, whose Ordinate is $\frac{a}{\tau}x^{\frac{\theta-\tau}{\tau}} \times e + fx + gx^2 + \mathcal{E}c$, by Cor. 2. Prop. 9. Afterwards if you reject the Units from the Indexes of the Powers, viz. $\frac{\theta-\tau}{\tau}$ and $\lambda - 1$ by the seventh Prop. till these Powers become as low as possible, you shall have arrived at the most simple Figures, which can be discovered in this way. Then every one of them gives another which is sometimes more simple by Cor. 5. Prop. 9. And from these compared together by Prop. 3. and Cor. 9. and 10. Prop. 9, more simple Figures as yet sometimes discover themselves. Finally, from the Assumption of the most simple Figures the Area sought may be computed by coming back through the same Steps.

CASE III.

55. Let the Ordinate be $x^{\theta-1} \times a + bx^{\tau} + cx^{2\tau} + \mathcal{E}c \times e + fx^{\tau} + gx^{2\tau} + \mathcal{E}c$, and this Figure, if it can be squared, will be squared by Prop. 5. If not, it's Ordinate must be distinguished into the Parts $x^{\theta-1} \times a \times e + fx^{\tau} + gx^{2\tau} + \mathcal{E}c$, $x^{\theta-1} \times bx^{\tau} \times e + fx^{\tau} + gx^{2\tau} + \mathcal{E}c$, $\mathcal{E}c$. and then the most simple Figures must be found, with which the Figures answering to these Parts can

The Quadrature of CURVES.

can be compared, by Case 2. For the Areas of the Figures answering to these Parts, join'd together with their proper Signs + and - make up the whole Area required.

CASE IV.

56. Let the Ordinate be $x^{p-1} \times a + bz^n + cz^{2n} + \mathcal{E}c. \times \sqrt[e + fz^n + gz^{2n} + \mathcal{E}c.]^{p-1} \times k + lx^n + mx^{2n} + \mathcal{E}c.]^{q-1}$; and if the Curve can be squared, it will be squared by Prop. 6. If otherwise, you must convert it into a more simple one by Cor. 4. Prop. 9. and thence you may compare it with the most simple Figures by Prop. 8. and Cor. 6, 9 and 10. Prop. 9. as is done in Cases second and third.

CASE V.

57. If the Ordinate consist of different Parts, all the Parts must be accounted as Ordinates of so many Curves; and these Curves, as many of them as can be squared, are to be squared singly, and their Ordinates to be taken away from the whole Ordinate. Then the Curve, which the remaining Part of the Ordinate designs, must be compared separately (as in the second, third and fourth Cases) with the most simple Figures it can be compared with. And the Sum of all the Areas must be accounted the Area of the Curve propos'd.

COROL. I.

58. Hence also, every Curve whose Ordinate is the affected square Root of it's own Equation, may be compared with the most simple Figures, either rectilinear, or curvilinear. For that Root always consists of two Parts, which considered separately, are not the affected Roots of Equations.

Let the Equation $a^2y^2 + x^2y^2 = 2a^3y + 2xz^3y - x^4$ be propos'd; and the Root being extracted will be $y = \frac{a^3 + x^3 + a\sqrt{a^4 + 2ax^3 - x^4}}{aa + xz}$:

whose rational Part $\frac{a^3 + x^3}{a^2 + x^2}$, and irrational Part $\frac{a\sqrt{a^4 + 2ax^3 - x^4}}{a^2 + x^2}$ are the Ordinates of Curves, which, according to this Proposition, may either be squared; or else compared with the most simple Figures, with which they admit of a geometrical Comparison.

COROL.

COROL. II.

59. And every Curve whose Ordinate is defin'd by any affected Equation, which, by Cor. 7. Prop. 9, passes into an Equation not affected, is either squared by this Proposition, if it can be squared; or is compared with the most simple Figures with which it may be compared. And by this means every Curve is squared whose Equation consists of three Terms. For that Equation, if it be affected, is changed into one not affected by Cor. 7. Prop. 9: and then, by Cor. 2^d and 5th, Prop. 9, by passing into the most simple Curve, it either gives the Quadrature of the Figure, if it admit of one; or gives the most simple Figure, with which it may be compared.

COROL. III.

60. And every Curve, whose Ordinate is defined by any affected Equation, which, by Cor. 8. Prop. 9, passes into an affected quadratic Equation, is either squared by this Prop. and Cor. 1, if it can be squared; or is compared with the most simple Figures with which it admits of a geometrical Comparison.

SCHOLIUM.

61. When Figures are to be squared, it would be too troublesome a Work to have recourse always to these general Rules: it's much better once to square the more simple and useful Kind of Figures, and record the Quadratures in a Table; afterwards consult the Table, as often as you have any such Curve to square. And the two following Tables are of this Kind: in which x denotes the Absciss, y the rectangular Ordinate, and z the Area of the Curve to be squared; and d, e, f, g, h, η are given Quantities with their proper Signs $+$ and $-$.

A TABLE

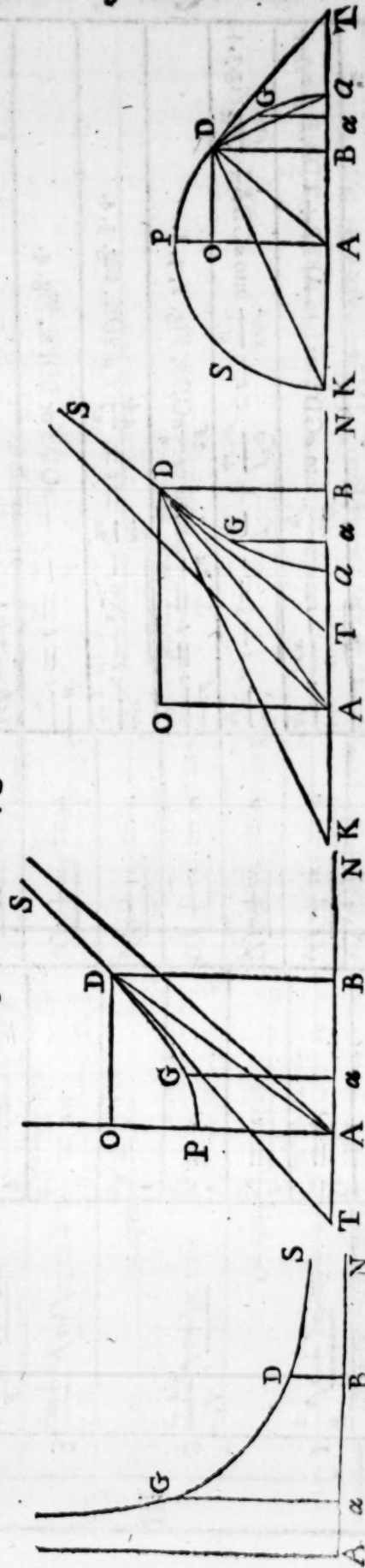
A T A B L E		
Of the more simple kind of Curves which may be squared.		
Forms of Curves.		Areas of the Curves.
I	$dx^{\eta-1} = y$	$\frac{d}{\eta} x^{\eta} = t.$
II	$\frac{dx^{\eta-1}}{e^2 + 2efx^{\eta} + f^2x^{2\eta}} = y$	$\frac{dx^{\eta}}{\eta e^2 + \eta f x^{\eta}} = t. \text{ Or } \frac{-d}{\eta f + \eta f^2 x^{\eta}} = t$
III	1 $dx^{\eta-1} \sqrt{e + fx^{\eta}} = y$	$\frac{2d}{3\eta f} R^3 = t. \text{ Where } R = \sqrt{e + fx^{\eta}}$
	2 $dx^{2\eta-1} \sqrt{e + fx^{\eta}} = y$	$\frac{-4e + 6fx^{\eta}}{15\eta f^2} dR^3 = t$
	3 $dx^{3\eta-1} \sqrt{e + fx^{\eta}} = y$	$\frac{16e^2 - 24efx^{\eta} + 30f^2x^{2\eta}}{105\eta f^3} dR^3 = t$
	4 $dx^{4\eta-1} \sqrt{e + fx^{\eta}} = y$	$\frac{-96e^3 + 144e^2fx^{\eta} - 180ef^2x^{2\eta} + 210f^3x^{3\eta}}{945\eta^4} dR^3 = t$
IV	1 $\frac{dx^{\eta-1}}{\sqrt{e + fx^{\eta}}} = y$	$\frac{2d}{\eta f} R = t.$
	2 $\frac{dx^{2\eta-1}}{\sqrt{e + fx^{\eta}}} = y$	$\frac{-4e + 2fx^{\eta}}{3\eta f^2} dR = t$
	3 $\frac{dx^{3\eta-1}}{\sqrt{e + fx^{\eta}}} = y$	$\frac{16e^2 - 8efx^{\eta} + 6f^2x^{2\eta}}{15\eta f^3} dR = t$
	4 $\frac{dx^{4\eta-1}}{\sqrt{e + fx^{\eta}}} = y$	$\frac{-96e^3 + 48e^2fx^{\eta} - 36ef^2x^{2\eta} + 30f^3x^{3\eta}}{195\eta^4} dR = t$

A T A B L E

A T A B L E

Of the more simple kind of Curves which may be compared with the Ellipsis and Hyperbola.

Let aGD , or PGD , or GDS be a Conic Section whose Area is required to the Quadrature of the Curve proposed, and let it's Centre be A , Axis Ka , Vertex a , half Conjugate Axis AP , the given Beginning of the Absciss A , or a , the Absciss AB , or aB , or $aB = x$, the Rectangular Ordinate $BD = v$, and the Area $ABDP$, or $aBDG$, or $aBDG = s$, aG being an Ordinate at the Point a . Let KD , AD , aD be joined; draw the Tangent DT meeting the Absciss AB in T ; and let the Parallelogram $ABDO$ be completed. And if perchance the Areas of two Conic Sections be required to the Quadrature of the Curve proposed, call the Absciss of the latter ξ , the Ordinate τ , and the Area σ . And let $\div$ denote the Difference of two Quantities, when it is uncertain whether the latter should be subtracted from the former, or the former from the latter. And in the sixth Form write p for $\sqrt{ff} - 4eg$.



Forms of the Curves.	Conic Section, it's Absciss.	Conic Section, it's Ordinate.	Areas of the Curves.
1 $\frac{dx^{2n-1}}{\sqrt{e+fx^n}} = y$	$x^n = x$	$\frac{d}{e+fx} = v$	$\frac{1}{n} s = t = aGDB$, Fig. 1.
2 $\frac{dx^{2n-1}}{\sqrt{e+fx^n}} = y$	$x^n = x$	$\frac{d}{e+fx} = v$	$\frac{d}{2nf} x^{2n} - \frac{e}{nf^2} x^2 + \frac{e^2}{nf^3} = t$
3 $\frac{dx^{2n-1}}{\sqrt{e+fx^n}} = y$	$x^n = x$	$\frac{d}{e+fx} = v$	

Forms of Curves.	Abciss.	Conic Section, it's Ordinate.	Areas of the Curves.
I II	$\sqrt{\frac{d}{e+fz^n}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f}x^2} = v$	$\frac{2xv}{\eta} \div 4f = t = \frac{4}{\eta} \text{ADGa, Fig. 3, 4.}$
	$\sqrt{\frac{d}{e+fz^n}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f}x^2} = v$	$\frac{2d}{\eta f} \frac{1}{2} \eta + \frac{4es - 2exv}{\eta f} = t$
	$\sqrt{\frac{d}{e+fz^n}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f}x^2} = v$	$\frac{2d}{3\eta f} \frac{3}{2} \eta - \frac{2de}{\eta f^2} \frac{1}{2} \eta + \frac{2e^2xv - 4e^2s}{\eta f^2} = t$
I III	$\frac{1}{z^n} = x^2$	$\sqrt{f+ex^2} = v$	$\frac{4de}{\eta f} \times \frac{v^3}{2ex} - s = t = \frac{4de}{\eta f}$ into aGDT or into APDB ÷ TDB, Fig. 2, 3, 4.
	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{8de^2}{\eta f^2} \times s - \frac{1}{2}xv - \frac{fv}{4e} + \frac{f^2v}{4e^2x} = t = \frac{8de^2}{\eta f^2}$ into aGDA + $\frac{f^2v}{4e^2x}$ Fig. 3, 4.
	$\frac{1}{z^n} = x^2$	$\sqrt{f+ex^2} = v$	$\frac{-2d}{\eta} s = t = \frac{2d}{\eta} \text{APDB or } \frac{2d}{\eta} \text{aGDB, Fig. 2, 3, 4.}$
I IV	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{4de}{\eta f} \times s - \frac{1}{2}xv - \frac{fv}{2e} = t = \frac{4de}{\eta f} \times \text{aGDK, Fig. 3, 4.}$
	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{-d}{\eta} s = t = \frac{d}{\eta} \times \text{aGDB or BDPK, Fig. 4.}$
	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{3dfv - 2dv^3}{6ne} = t$
I IV	$\frac{1}{z^n} = x^2$	$\sqrt{f+ex^2} = v$	$\frac{4d}{\eta f} \times \frac{1}{2}xv \div s = t = \frac{4d}{\eta f}$ into PAD or into aGDA, Fig. 2, 3, 4.
	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{8de}{\eta f^2} \times s - \frac{1}{2}xv - \frac{fv}{4e} = t = \frac{8de}{\eta f^2}$ into aGDA, Fig. 3, 4.
	$\frac{1}{z^n} = x^2$	$\sqrt{f+ex^2} = v$	$\frac{2d}{\eta} \times s - xv = t = \frac{2d}{\eta}$ into POD or into AODGa, Fig. 2, 3, 4.
I IV	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{4d}{\eta f} \times \frac{1}{2}xv \div s = t = \frac{4d}{\eta f}$ into aDGA, Fig. 3, 4.
	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{d}{\eta e} \times 3s \div 2xv = t = \frac{d}{\eta e}$ into 3aDGA ÷ aADB, Fig. 3, 4.
	$\frac{1}{z^n} = x$	$\sqrt{fx+ex^2} = v$	$\frac{10dfxv - 15dfv - 2dex^2v}{6ne^2} = t$

Forms of the Curves.	Absciss.	Conic Section, it's Ordinate.	Areas of the Curves.
V	1 $\frac{dx^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{xv-2s}{n}=t$
	2 $\frac{dx^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{2s-xv}{n}=t$
	Or thus		
	2 $\frac{dx^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{d\sigma+2fs-fxv}{2ng}=t$
VI	1 $\frac{dx^{\frac{1}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{2xv-4s-2\xi v+4\sigma}{np}=t$
	2 $\frac{dx^{\frac{1}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{4s-2xv-4\sigma+2\xi v}{np}=t$
VII	1 $\frac{dx^{\frac{1}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{4de\xi v+2defv-2dfgxv+4degv-8de\sigma+4dfgs}{4ng-nf^2}=t$
	2 $\frac{dx^{\frac{1}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{d}{n}=t=\frac{d}{n}\times \text{a GDB, Fig. 2, 3, 4.}$
	3 $\frac{dx^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{d}{3ng}=\frac{df}{2ng}=\frac{df}{2ng}=\frac{df}{2ng}$
	4 $\frac{dx^{3n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\frac{6deg-5df}{24ng^2}v+\frac{5df^2-4deg}{16ng^2}=\frac{df}{2ng}$

Forms of the Curves.	Abciss.	Conic Section, its Ordinate.	Areas of the Curves.
VIII	1 $\frac{dx^{n-1}}{\sqrt{e+fx^2+gx^2n}} = y^{2n} = x$	$\sqrt{e+fx+gx^2} = v$	$\frac{8dgs - 4dgsxv - 2dfv}{4neg - \eta f^2} = t = \frac{8dg}{4neg - \eta f^2} \times aGDB + \triangle DBA, \text{Fig. 2, 4}$
	2 $\frac{dx^{2n-1}}{\sqrt{e+fx^2+gx^2n}} = y^{2n} = x$	$\sqrt{e+fx+gx^2} = v$	$\frac{-4dfv + 2dfxv + 4dgv}{4neg - \eta f^2} = t$
	3 $\frac{dx^{3n-1}}{\sqrt{e+fx^2+gx^2n}} = y^{2n} = x$	$\sqrt{e+fx+gx^2} = v$	$\frac{3df^2 - 2df^2xv - 2dfv}{-4deg + 4deg} = t$
	4 $\frac{dx^{4n-1}}{\sqrt{e+fx^2+gx^2n}} = y^{2n} = x$	$\sqrt{e+fx+gx^2} = v$	$\frac{30ddeg + 8deg^2x^2v + 10df^2xv + 10deg^2xv - 28deg^2xv - 16deg^2xv}{-15df^2 - 2df^2g - 28deg^2xv - 16deg^2xv} = t$
IX	1 $\frac{dx^{n-1}}{g+bz^n} \sqrt{e+fx^2} = y$	$\sqrt{\frac{df}{b} + \frac{eb - fg}{b}x^2} = v$	$\frac{4fg - 2fgxv + 2df}{-4eb + 2eb} = t$
	2 $\frac{dx^{2n-1}}{g+bz^n} \sqrt{e+fx^2} = y$	$\sqrt{\frac{df}{b} + \frac{eb - fg}{b}x^2} = v$	$\frac{4egb - 2egbxv + \frac{2db}{x^2} - 2dfg}{\eta fb^2} = t$
X	1 $\frac{dx^{n-1}}{g+bz^n} \sqrt{e+fx^2} = y$	$\sqrt{\frac{df}{b} + \frac{eb - fg}{b}x^2} = v$	$\frac{2xv - 4}{\eta f} = t = \frac{4}{\eta f} \triangle DGA, \text{Fig. 3, 4}$
	2 $\frac{dx^{2n-1}}{g+bz^n} \sqrt{e+fx^2} = y$	$\sqrt{\frac{df}{b} + \frac{eb - fg}{b}x^2} = v$	$\frac{4gs - 2gxv + 2d}{\eta fb} = t$
XI	1 $\frac{dx^{n-1}}{g+bz^n} \sqrt{e+fx^2} = y$	$\sqrt{\frac{eb - fg}{b} + \frac{f}{b}x^2} = v$	$\frac{2dxv^3 - 4dxv - 4dfv - 4deg}{\eta fg - neb} = t$
	2 $\frac{dx^{n-1}}{g+bz^n} \sqrt{e+fx^2} = y$	$\sqrt{\frac{eb - fg}{b} + \frac{f}{b}x^2} = v$	$\frac{2d}{\eta b} = t$
	3 $\frac{dx^{2n-1}}{g+bz^n} \sqrt{e+fx^2} = y$	$\sqrt{\frac{eb - fg}{b} + \frac{f}{b}x^2} = v$	$\frac{dbxv^3 - 3dfg}{2\eta fb^2} = t$

62. In these Tables the Series's of Curves of each Form may be continued *in infinitum* towards both Parts. For in the first Table, with respect to the Numerators of the Areas of the third and fourth Forms, the numeral Coefficients of the initial Terms (2, -4, 16, -96, 868, &c.) are found by multiplying the Numbers -2, -4, -6, -8, -10, &c. continually into one another: and the Coefficients of the subsequent Terms are derived from those of the initial Terms, by multiplying them orderly, in the third Form, by $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, $-\frac{9}{8}$, $-\frac{11}{10}$, &c. in the fourth Form, by $-\frac{1}{2}$, $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, $-\frac{9}{10}$, &c.

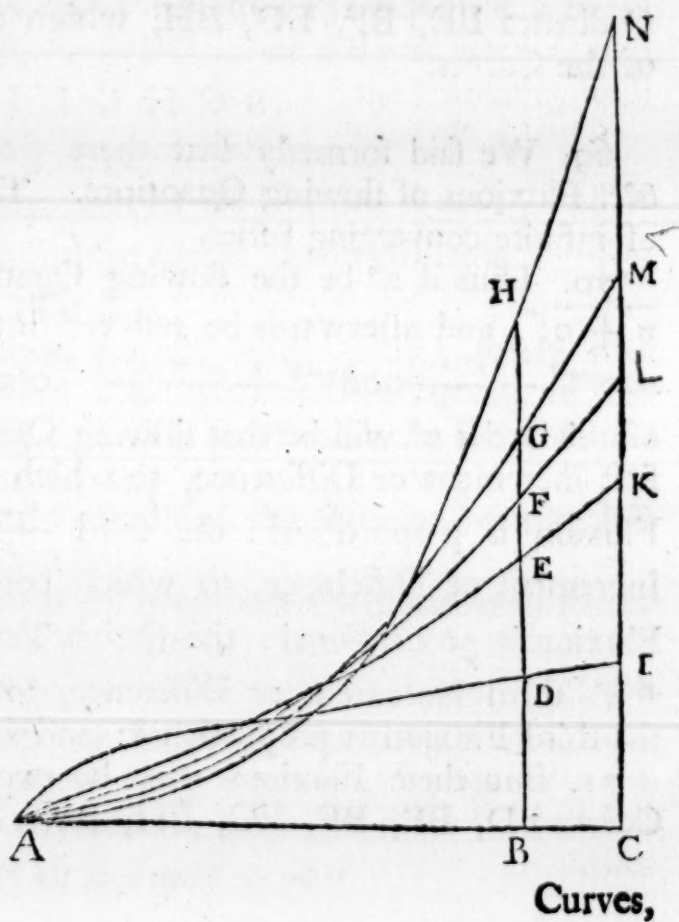
And the Coefficients of the Denominators 3, 15, 105, &c. arise by multiplying the Numbers 1, 3, 5, 7, 9, &c. continually into one another.

63. And in the second Table, the Series's of Curves of the first, second, fifth, sixth, ninth and tenth Forms, by the Help of Division alone; and the other Forms by the Help of the third and fourth Propositions, are drawn out both Ways *in infinitum*.

64. Moreover these Series's, by changing the Sign of the Number n , use to vary. For thus, to take an Example, the Curve $\frac{d}{x}\sqrt{e+fx^n}=y$ becomes $\frac{d}{x^{n+1}}\sqrt{f+ex^n}=y$.

PROP. XI. THEOR. VIII.

65. Let ADIC be any Curve, having the Absciss AB = x , the Ordinate BD = y ; and let AEKC be another Curve, whose Ordinate BE is equal to the Area of the former ADB applied to Unity; and AFLC a third Curve, whose Ordinate BF is equal to Area of the second AEB applied to Unity; and AGM a fourth Curve whose Ordinate BG is equal to the Area of the third Curve AFB applied to Unity; and AHNC a fifth Curve, whose Ordinate BH is equal to AGB the Area of the fourth applied to Unity: and so on *in infinitum*. And let A. B. C. D. E. &c. be the Areas of



The Quadrature of CURVES.

Curves, having the Ordinates $y, zy, z^2y, z^3y, z^4y, \&c.$ and the common Absciss z .

66. Let there be any given Absciss $AC = t$, and let $BC = t - z = x$: and let P, Q, R, S, T be the Areas of Curves having the Ordinates $y, xy, x^2y, x^3y, x^4y, \&c.$ and the common Absciss x .

67. And let all these Areas be terminated at the whole given Absciss AC , and likewise at the Ordinate CI given in Position, and infinitely produced.

And of the Areas mentioned at the Beginning:

$$\text{The 1}^{\text{st}}, ADIC = A = P$$

$$2^{\text{d}}, AEKC = tA - B = Q$$

$$3^{\text{d}}, AFLC = \frac{t^2A - 2tB + C}{2} = \frac{1}{2}R$$

$$4^{\text{th}}, AGMC = \frac{t^3A - 3t^2B + 3tC - D}{6} = \frac{1}{6}S$$

$$5^{\text{th}}, AHNC = \frac{t^4A - 4t^3B + 6t^2C - 4tD + E}{24} = \frac{1}{24}T.$$

C O R O L.

68. Whence if the Curves, whose Ordinates are $y, zy, z^2y, z^3y, \&c.$ or $y, xy, x^2y, x^3y, \&c.$ can be squared; the Curves $ADIC, AEKC, AFLC, AGMC, \&c.$ may likewise be squared; and we shall have the Ordinates BE, BF, BG, BH , which are proportional to the Areas of the Curves.

S C H O L I U M.

69. We said formerly that there were first, second, third, fourth, &c. Fluxions of flowing Quantities. These Fluxions are as the Terms of infinite converging Series.

70. Thus if z^n be the flowing Quantity, and by flowing become $z + 01^n$, and afterwards be resolved into the converging Series $z^n + n0z^{n-1} + \frac{n^2-1}{2}00z^{n-2} + \frac{n^3-3n^2+2n}{6}0^3z^{n-3} + \&c.$ The first Term of this Series z^n will be that flowing Quantity; the second will be the first Increment or Difference, to which consider'd as nascent, it's first Fluxion is proportional: the third $\frac{n^2-1}{2}00z^{n-2}$ will be * it's second Increment or Difference, to which consider'd as nascent the second Fluxion is proportional: the fourth Term $\frac{n^3-3n^2+2n}{6}0^3z^{n-3}$ will be * it's third Increment or Difference, to which considered as nascent the third Fluxion is proportional: and so on *in infinitum*.

71. But these Fluxions may be expounded by the Ordinates of Curves $BD, BE, BF, BG, BH, \&c.$ See the Fig.

* See the Notes upon the Place.

72. Thus

72. Thus if the Ordinate $BE (= \frac{ADB}{1})$ be the flowing Quantity, it's first Fluxion will be as the Ordinate BD .

73. If $BF (= \frac{AEB}{1})$ be the flowing Quantity, it's first Fluxion will be as the Ordinate BE ; and it's second Fluxion as the Ordinate BD .

74. If $BH (= \frac{AGB}{1})$ be the flowing Quantity, it's first, second, third and fourth Fluxions will be as the Ordinates BG , BF , BE and BD respectively.

75. And hence in such Equations as involve only two unknown Quantities, of which one is a Quantity flowing uniformly, and the other any Fluxion of the other flowing Quantity, that other Fluent may be found by the Quadrature of Curves. For let it's Fluxion be expounded by BD , and if it be the first Fluxion, seek the Area $ADB = BE \times 1$; if it be the second Fluxion, seek the Area $AEB = BF \times 1$; if the third, seek the Area $AFB = BG \times 1$; &c. and the Area so found will be the Exponent of the Fluent sought.

76. But moreover in Equations which involve a Fluent and it's first Fluxion without the other Fluent; or two Fluxions of the same Fluent, the first and second, or the second and third, or the third and fourth, &c. without either of the two Fluents: the Fluents may be found by the Quadrature of Curves. Let there be given the Equation $aa\dot{v} = a\dot{v} + v\dot{v}$, while it is $v = BE$, $\dot{v} = BD$, $z = AB$ and $\dot{z} = 1$, and that Equation, by completing the Dimensions of the Fluxions, will become $aa\dot{v} = a\dot{v}z + v\dot{v}z$, or $\frac{aa\dot{v}}{a\dot{v} + v\dot{v}} = \dot{z}$.

77. Now let v flow uniformly, and let it's Fluxion $\dot{v} = 1$; and it will be $\frac{aa}{a\dot{v} + v\dot{v}} = \dot{z}$, and by squaring the Curve whose Ordinate is $\frac{aa}{a\dot{v} + v\dot{v}}$ and Absciss v , you'll have the Fluent z .

78. Moreover let the Equation be $aa\ddot{v} = a\ddot{v} + \dot{v}\dot{v}$; while it is $v = BF$, $\dot{v} = BE$, $\ddot{v} = BD$, and $z = AB$; and by means of the Relation betwixt $\ddot{v}$ and $\dot{v}$ or BD and BE , you'll find the Relation betwixt AB and BE as in the immediately preceding Example. Afterwards by means of this Relation, you'll find the Relation betwixt AB and BF , by squaring the Curve AEB .

79. Such Equations as involve three unknown Quantities may sometimes be reduced to Equations which involve two only: and in these Cases the Fluents will be found from the Fluxions as above. Let the Equation $a - bx^m = cxy^ny + dy^{2n}y\dot{y}$ be proposed: put $y^ny = \dot{v}$, and it will be $a - bx^m = cx\dot{v} + d\dot{v}^2$. This Equation, by squaring the Curve, whose Absciss is x and Ordinate $\dot{v}$, gives the Area v ; and the other

other Equation, viz. $y\dot{y} = v$, by returning back to the Fluents gives $\frac{1}{n+1}y^{n+1} = v$: whence the Fluent y is obtain'd.

80. But besides, in Equations which involve three unknown Quantities, although they can't be reduced to Equations which involve only two, the Fluents will be sometimes found by the Quadrature of Curves. Let the Equation $ax^m + bx^n)^p = rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^s$ be given, while it is $\dot{x} = 1$; and the latter Part $rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^s$, by going back to the Fluents, becomes $ex^ry^s - \frac{f}{n+1}y^{n+1}$, which therefore is as the Area of a Curve whose Absciss is x , and Ordinate $ax^m + bx^n)^p$: and thence the Fluent y is given.

81. Let the Equation $\dot{x} \times ax^m + bx^n)^p = \frac{dy^{n-1}}{\sqrt{e+fy^n}}$ be given, and the Fluent whose Fluxion is $\dot{x} \times ax^m + bx^n)^p$, will be as the Area of a Curve, whose Absciss is x , and it's Ordinate $ax^m + bx^n)^p$. Likewise the Fluent whose Fluxion is $\frac{dy^{n-1}}{\sqrt{e+fy^n}}$ will be as the Area of a Curve whose Absciss is y , and it's Ordinate $\frac{dy^{n-1}}{\sqrt{e+fy^n}}$, that is (by Case 1. Form 4th, Tab. 1.) as the Area $\frac{2d}{nf}\sqrt{e+fy^n}$. Therefore I put $\frac{2d}{nf}\sqrt{e+fy^n}$ equal to the Area of the Curve whose Absciss is x and Ordinate $ax^m + bx^n)^p$, and so the Fluent y will be had.

82. And let it be remarked that every Fluent which is collected from a first Fluxion, may be augmented or diminished by any Quantity which don't flow at all. That which is collected from a second Fluxion may be augmented or diminished by any Quantity whose second Fluxion is nothing. A Fluent collected from a third Fluxion may be increased or diminished by any Quantity that has no third Fluxion. And so on *in inf.*

83. After the Fluents are collected from the Fluxions, if there arise any Doubt about the Truth of the Conclusion, then the Fluxions of the Fluents found are again to be collected, and compared with the Fluxions proposed at the Beginning. For if they come out equal, the Conclusion is right: if otherwise, the Fluents must be so corrected, that their Fluxions may be equal to the Fluxions at first propos'd. For a Fluent may be assumed at pleasure, and the Assumption be corrected, by putting the Fluxion of the assumed Fluent equal to the Fluxion proposed, and comparing the homologous Terms with one another.

And thus, by these Beginnings, we are led to yet more important Truths.

Sir

Sir ISAAC NEWTON's
TREATISE
OF THE
Quadrature of Curves explained.

S E C T. I.

Containing Notes on the Introduction.

1. **T**HE incomparable Sir *Isaac Newton*, having not only enriched Mathematics and Philosophy with many new and useful Discoveries in their various Parts; but also extended the *Limits* of Geometry, by the noble and sublime Invention of *Fluxions*, takes care first of all, in this Introduction to the Quadrature of Curves, to give us just Notions of his Method of Fluxions: because it is a Method entirely new, by which the most remote and useful Properties and Relations of Quantities are discovered, and demonstrated, which were in vain attempted by the geometrical Analysis of the Ancients; or the specious Analysis of the Moderns: and is the Method he made use of in the Doctrine of the Quadrature of Curves contained in this Treatise: as he himself informs us in this Place.

2. The Method of Fluxions is twofold: or rather, it may be reduced to two very general and comprehensive Problems. 1°. From the *Fluents* given to find the *Fluxions*. 2°. From the *Fluxions* given to find the *Fluents*. Which two Problems may be conceived as equivalent to these, 1°. The *Length* of the Space described by a moving
F Body

Body being continually or at all Times given, to find the *Velocity* of the Motion at any Time proposed. 2°. The *Velocity* of the Motion being at all Times given, to find the *Length* of the Space described at any Time proposed. The first of these Problems includes the *direct* Method of Fluxions; the other, the *inverse* Method.

3. Both these Methods of Fluxions are capable of being applied; and accordingly have been applied to a vast Number of useful Purposes, by such as have treated of this Subject: of all which one of the most noble and useful is the Doctrine of the Quadrature of Curves delivered in this Treatise. The Design of which is to shew how infinite Numbers of various Kinds of geometrical Curves may be squared, *i. e.* their Areas exactly determined, by comparing them with right-lined Figures: or, if that cannot be done, it shews what way their Areas may be nearly determined, by an infinite converging Series; and the Curves geometrically compared, either with the Circle, Ellipse and Hyperbola; or else, with other the most simple Curves, with which they admit of a geometrical Comparison. So that by what our Author delivers in this Treatise, and in his Analysis by Equations of an infinite Number of Terms, (which I have subjoined to this Performance) all geometrical Curves, *i. e.* such, the Relation of whose Absciss to the Ordinate is defined by an algebraical Equation of any Degree, can be either exactly squared; or, if that cannot be done, the Area may be expressed by an infinite converging Series, of which a finite Number of Terms shall make such a near Approach to the true Area, as to differ from it by less than any given Difference. And even in *mechanical* Curves, *i. e.* such Curves, whose Property cannot be defined by any *finite* Equation of what Degree soever, he shews how to exhibit approximate Values of their Areas; or to reduce them to geometrical Curves.

* Art. 2.
of this Ex-
plication.

4. But although this be the primary Design of this Treatise of our Author's; yet it may be considered in a more general and extensive View, as containing the Doctrine of Fluxions, both *direct* and *inverse*, whatever the Nature of the flowing Quantities be. For the first Prop. contains the Solution of the first General Problem mentioned above*. And by means of the Quadrature of Curves, the second General Problem, containing the *inverse* Method of Fluxions, may be solved in a great many Cases, although not universally: as we shall have Occasion to shew, when we come to explain our Author's last Scholium.

Art. 3.
of the
Quadrature
of
Curves.

5. After the true Meaning of *Fluents* and *Fluxions* is laid down by our Author, he proceeds next, in this Article, to shew what *Proportion* the Fluxions of flowing Quantities bear to one another. Accordingly

ingly he tells us, " They are very nearly to one another as the *Augments* of the Fluents generated in equal, but very small Times: and " that if we would speak accurately, they are in the *first* Ratio of the " *nascent* Augments, but may be expounded by any Lines, which are " proportional to them." He says afterwards *, " that it comes to * Art. 6. " the same Purpose if the Fluxions be taken in the *last* Ratio of the " *evanescent* Parts." Now, because the Doctrine of *prime* and *ultimate* Ratios has been so much controverted of late ¹, I shall here enquire whether we have any distinct Idea thereof: which I reckon to be the more necessary, because the noble Inventor has laid down this Notion of Fluxions in this Place, as that by which we are to be directed in our Conceptions of them.

6. Before we enter upon the Discussion of this Point, it will be proper to observe, 1^o, That the Fluxions of flowing Quantities, being the *Celerities* with which they are supposed to flow; and by which they are generated: the Fluxion of any variable Quantity, ought never to be considered *absolutely*, or by itself; but *relatively*, or with Relation to the Fluxion of some other flowing Quantity of the same Kind. So that if at any Time, for Brevity's sake, the Fluxion of a Quantity be mentioned absolutely; yet there is always a *supposed* Relation to the Fluxion of some other Quantity with which it is understood to be compared: and ordinarily in the Comparison, one of the flowing Quantities is supposed to flow *equally* and *uniformly*; so that it's Fluxion, being constant and invariable, is considered as a *Standard* or Measure, to which the Fluxions of the other Quantities are referred. 2^o, That if the Fluxion of any flowing Quantity, or it's Celerity of flowing, be continually varying, *i. e.* continually accelerated or retarded, for any Time, the Fluxion in any one Place or at any one Instant of that Time, is different from the Fluxion of that Quantity in any other Place, or at any other Instant of Time. For whatever is continually varying, by Increase or Decrease, by the very Supposition, must be of a different Value in every different Place, or at every different Instant of Time, otherwise it would not be continually varying. To illustrate which in the Case of a falling Body: the Velocity with which it falls in any one Place, or at any one Instant of Time, during it's Fall, is different from the Velocity it has in any other Place, or at any other Instant of Time: and it is all one, whether the Motion be uniformly accelerated, or retarded; or whether it be accelerated or retarded according to any other Law.

¹ See the Analyst published by the Bishop of Cloyne, with an Answer to it by *Philalethes Cantabrigienfis*: likewise the Bishop's Defence, and *Philalethes*'s Reply.

7. Before we prosecute this Affair any further, we shall here borrow from Sir *I. Newton* the following Lemma ¹.

“Quantities, as likewise the Ratios of Quantities, which constantly tend to Equality during any finite Time, and before the End of that Time come nearer to one another than by any given Difference, at last become equal.

“If you deny it; let them be at last unequal, and let their last Difference be *D*. Therefore they cannot come nearer to Equality than by the given Difference *D*: contrary to the Hypothesis.”

8. If it should be alledged by any, that, notwithstanding this Demonstration, the Quantities and Ratios mentioned in the Lemma may differ at last, although that Difference be less than any given or assignable Difference: such would do well to consider, that such a way of reasoning, being admitted, would overturn some of the finest Demonstrations of the most accurate Geometricians among the Ancients themselves, such as *Euclid* and *Archimedes*, whose Works have undergone the strictest Scrutiny and Examination of the best Geometricians, since their Time to this very Day. For Example, How does *Archimedes* demonstrate that a Circle is equal to a rectangular Triangle, having it's Base equal to the Circumference, and it's perpendicular Altitude equal to the Radius of the Circle ²? He does it by shewing that a Circle can neither be greater nor less than such a Triangle. But how does he prove this? By shewing that the Circle is neither greater nor less than the Triangle by any given Space. Again, *Euclid* demonstrates ³ that Circles are to each other as the Squares of their Diameters, by shewing that the Square of the Diameter of the one Circle is to the Square of the Diameter of the other, neither as the first Circle is to a Space greater, nor yet to a Space less than the other Circle. But let us see what he means by a Space greater or less than the other Circle. Why, he means a Space differing from it by a given or assignable Difference; *i. e.* according to the Lemma premised to that Proposition, such a Difference, as, repeated a certain Number of Times, may exceed that Circle, as appears to any one that reads that Prop. and Lemma: which Lemma is the Foundation of the Method of Exhaustions, made use of by *Euclid* and *Archimedes* in these and many other Propositions.

9. Now if any one should object, that, notwithstanding of what *Archimedes* and *Euclid* have demonstrated in these Propositions, the Circle may be greater or less than the rectangular Triangle; and the Ratio of the Squares of the Diameters may be greater or less than the

¹ Lem. 1. of his Principles.

³ See Prop. 2. B. 12. of his Elements.

² See his *Κύκλου Μέτρησις*.

Ratio of the Circles, although not by any given or assignable Difference; yet by a Difference less than any given Difference : is not this the very same Objection as that raised against Sir *I. Newton's* Lemma? And therefore, if it be of no Weight against *Euclid* and *Archimedes*; no more is it against our Author. But the Truth of the Matter is, that a Difference less than any Thing assignable, is the same Thing as no Difference at all : for repeat it as often as you please, it can never be equal to any finite Quantity : and therefore can bear no Ratio to it, by Def. 4. B. 5. Elements, consequently it can be of no Importance to make the Thing greater or less ¹.

10. Def. 1. The ultimate Ratio of Quantities is the Limit to which the Ratio of variable Quantities continually approaches during any finite Time, and just attains to at the End of the Time.

11. Def. 2. The ultimate Ratio of evanescent Quantities is the Limit to which the Ratio of variable Quantities diminishing without Bound, continually approaches, to come nearer to it than by any given Difference; but which it never goes beyond : yet no sooner attains to, than the Quantities being diminished infinitely, vanish.

12. Def. 3. The prime Ratio of Quantities is the Limit from which the Ratio of variable Quantities continually recedes; during some finite Time, and with which it coincides at the very Beginning of the Time : and is only a different Way of conceiving the same Thing as is contain'd in Def. 1.

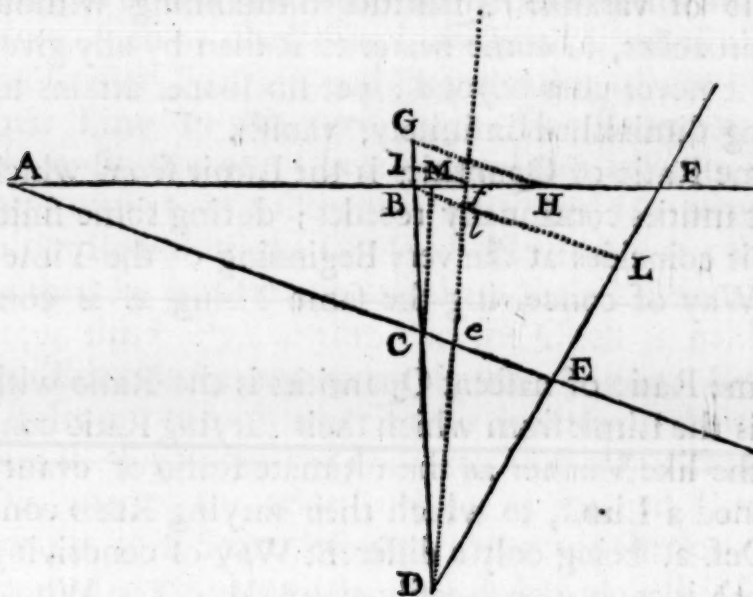
13. Def. 4. The prime Ratio of nascent Quantities is the Ratio with which they arise; or it is the Limit from which their varying Ratio continually recedes; after the like Manner as the ultimate Ratio of evanescent Quantities was defined a Limit, to which their varying Ratio continually approaches in Def. 2. being only a different Way of conceiving the same Thing; although it cannot be so easily express'd. The Whole is illustrated by what follows.

¹ *Archimedes* tells us in the Preface to his Treatise of the *Sphere* and *Cylinder*, that he assumes this for a Principle or Postulate.

"Ετι δὲ τῶν ἀνισῶν γραμμῶν, καὶ τῶν ἀνισῶν ἐπὶ φανείων, καὶ τῶν ἀνισῶν σφαιρῶν, τὸ μὲν αὐτὸ τῷ ἰσότητος ὑπερέχειν τοιούτῳ, ὃ συνισθήμενον ἑαυτοῦ ἑαυτῷ, δυνατόν ἐστιν ὑπερέχειν παντὸς τῷ προτιθέμενῳ τῶν πρὸς ἀλλήλα λεγομένων.

And he mentions the same in the Preface to his Treatise of *Spiral Lines*. Likewise in the Preface to his *Quad.* of the *Parab.* he tells us, that in order to demonstrate that a Segment of the Parabola is $\frac{4}{3}$ of a Triangle having the same Base and Altitude, he assumes this for a Foundation, that the Excess of unequal Spaces, whereby the greater exceeds the less, may be so often added to itself, as to exceed any propos'd finite Space. And he tells us, that Geometers before his Time had made use of the same Principle in demonstrating, that Circles are to one another in the duplicate Ratio of their Diameters; that Spheres are in the triplicate Ratio of their Diameters; that a Pyramid is a third Part of a Prism of the same Base and Altitude, &c.

14. Let the two right Lines AB, AC given in Position, and meeting in the Point A be intersected by a third right Line DCB, also given in Position, in the Points C and B: and let another right Line DEF revolve about the given Point D as a Pole, always cutting AC, AB in the Points E and F; and suppose it to approach DCB continually, until it at length fall in with it: then the Ratio of DC to CB is the ultimate Ratio of DE to EF. For the varying Ratio of DE to EF continually approaches to the given or invariable Ratio of DC to CB; so as that before DEF fall in with DCB, the Ratio of DE to EF comes nearer to the Ratio of DC to CB than by any given Difference; or, which is the same thing, than any other given or assignable Ratio does. For, produce DB unto G, so as that the Ratio of DC to CG be as near the Ratio of DC to CB, as that of DE to EF can ever arrive to, if possible: through G draw GH parallel to AC meeting AB in H: let DEF by revolving, come into the Position Def, so that *f* lye betwixt B and H: draw *f*I parallel to AC, meeting DB in I. Then, from parallel Lines, $De :: ef :: DC : CI$; but the Ratio of DC to CI is nearer to the Ratio of DC to CB, than that of DC to CG is; wherefore also the Ratio of *De* to *ef*



is nearer to the Ratio of DC to CB than that of DC to CG is, contrary to the Hyp. Whence it appears that the Ratio of DC to CB is the Limit to which the varying Ratio of DE to EF continually approaches, and comes nearer to than by any given Difference, and no sooner arrives at, than the Line DEF falls in with DCB; or, which is the same, ceases to move; therefore the Ratio of DC to CB, is * the ultimate Ratio of the variable Lines DE, EF.

* Art. 10.
of this Ex-
plication.

15. If the right Line DEF be suppos'd to move back again from DCB, where it is suppos'd to begin to move; then the Ratio of DC to CB is the * prime Ratio of DE to EF.

* Art. 12.
of this Ex-
plication.

16. Suppose the same Things as before: and let DEF, revolving upon D as a Pole, move towards DCB until it coincide with it, and the

the Points E and F coincide with the Points C and B, then I say the Ratio of the variable Lines CE, BF is continually changing: the compounded Ratio of DC to DB and AC to AB, is a Limit to which their Ratio continually approaches, and comes nearer to than by any given Difference; or than any other given Ratio doth; and that before the Points E and F fall in with C and B; but no sooner arrives at, than the variable Lines CE, BF being infinitely diminished, vanish; and the Points E and F fall in with C and B.

Through B draw BL parallel to AC meeting DF in L: then $CE : BF :: CE : BL + BL : BF$. Now $CE : BL :: DC : DB$ *, and therefore constant or given; but $BL : BF$ is continually varying: for when DLF comes into any other Position Dlf, if $BL : BF :: B/ : Bf$, LF would be parallel to lf*, which it never can be; and it's evident also that $B/ : Bf$ is greater than $BL : BF$; wherefore $CE : BF$ is continually varying and increasing until DEF fall in with DCB. 2°. If any given Ratio can be as near to the compounded Ratio of DC to DB and AC to AB as $CE : BF$ comes to: let it be $DC \times AC : DB \times AH$ a less Ratio than $DC \times AC : DB \times AB$, because it has been shewn that $CE : BF$ is always increasing; so that $AH > AB$. Assume the Point M any where in the Line AB, betwixt B and H, join CM: let DEF by revolving come into the Position Def parallel to CM. Then $Ce : Bf :: Ce : B/ + B/ : Bf :: DC : DB + AC : AM$ *; or $Ce : Bf :: DC \times AC : DB \times AM$; where for $Ce : Bf > DC \times AC : DB \times AH$, (because $DC \times AC : DB \times AM > DC \times AC : DB \times AH$.) contrary to the Hyp. 3°. $CE : BF$ never goes beyond $DC \times AC : DB \times AB$; for it has been demonstrated that $Ce : Bf :: DC \times AC : DB \times AM < DC \times AC : DB \times AB$: But it approaches nearer to it than by any given Difference; and that, while CE and BF are yet of a finite Magnitude: therefore $DC \times AC : DB \times AB$ is the Limit at which $CE : BF$ no sooner arrives, than CE and BF being infinitely diminished vanish; just while the Points E, F coincide with the Points C, B. So that $DC \times AC : DB \times AB$ is the ultimate Ratio of the continually diminishing and at last evanescent Lines CE, BF*.

17. If the right Line DEF be suppos'd to turn back again from DCB, where it's Motion begins, then the same Ratio, viz. $DC \times AC : DB \times AB$, is the prime Ratio of CE and BF considered as nascent*.

18. If any one should object that there can be no ultimate Ratio of continually diminishing and at last evanescent Quantities: because

* By this way of Notation I mean a Ratio compounded of the two Ratios of CE to BL and of BL to BF, the same as that of $CE \times BL$ to $BL \times BF$.

* By this Notation I mean the Ratio of BL to BF.

* Art. 11. of this Explication.

* Art 13. of this Explication.

before

before they vanish it is not the last; and after they vanish, they have no Ratio. The Answer is, that the ultimate Ratio is neither the Ratio of them before they vanish; nor after they vanish; but the Ratio wherewith they vanish, or the Limit to which their varying Ratio no sooner arrives, than they vanish. If there was any Thing in this Objection, it would infer that when a falling Body is stopt in it's Motion, it has no last or ultimate Velocity: for the Velocity before it was stopt is not the last, and the Velocity, after it is stopt, is none at all. But every one may see, that by the last Velocity is meant neither the one nor the other of these; but the Velocity it has at that very Instant it stops, which it does not arrive at before it is stopt; and no sooner arrives at but it is stopt. For since it moves with a continually accelerated Velocity, it has a different Velocity for every different Instant of Time, and therefore at that Instant it stops, it has acquired a Velocity different from the Velocity it had at any other Instant of Time. And the same is the Case of the ultimate Ratio of evanescent Quantities, whose Ratio is continually varying; it is that Ratio they have that very Instant they vanish: for, since they are supposed to have a different Ratio for every different Instant of Time, they must have a certain determinate Ratio, with which they vanish, otherwise they never can vanish, contrary to the Hypothesis: and that is the ultimate Ratio of the evanescent Quantities.

19. It signifies nothing to say ultimate Quantities cannot be assigned, in regard Quantity is divisible without End: for it is not the Quantities themselves that are hereby determined, but only their Ratio: which is capable of being determined, as has been already shewn*.

* Art. 16.
of this Ex-
plication.

20. As these are the only Objections that can be rais'd against this Doctrine of prime and ultimate Ratios, the noble Author was too clear sighted not to perceive them: and accordingly he has fully answered them elsewhere¹, to so great Satisfaction to every intelligent and unprejudiced Reader, that the great Dust which has been raised of late about the Whole of this Doctrine, must be owing to Weakness, or some worse Principle.

21. Having endeavoured to clear up the Meaning of prime and ultimate Ratios, and remove the Objections that are brought against that Doctrine: that I may clearly demonstrate the fundamental Proposition concerning the Proportions of Fluxions, laid down by our Author, and formerly mentioned*, I shall premise some Lemmas respecting Motion.

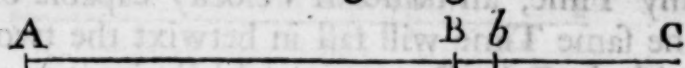
* Art. 3.
of the
Quadrature
of Curves.
Art. 5.
of this Ex-
plication.

¹ See Schol. to Sect. 1. of his Principles; and Lem. 2. B. 2.

L E M M A

L E M M A I.

22. If the right Line AC be described by the Motion of the Point B, with a Velocity continually accelerated, the Spaces described in equal Times will still increase, as the Time from the Beginning of it's Motion increases. For if the Motion was equable or uniform, the Spaces describ'd in equal Times would be equal, therefore when it is continually accelerated, the Spaces describ'd in equal Times must still increase, the further the Time is from the Beginning of the Motion.



L E M M A II.

23. If the Motion of the Point B be continually retarded, the Spaces described in equal Times will still diminish.

This is also evident by the same way of reasoning.

L E M M A III.

24. If the Point B move with a Motion continually accelerated or continually retarded, it's Velocity at any one Instant of Time is different from what it is at any other Instant of Time.

This is evident from the very Notion of a Motion continually accelerated, or continually retarded: for such Motion cannot continue the same for any the least Time.

L E M M A IV.

25. If the Motion of the Point B be continually accelerated or continually retarded, in such manner, as that the variable Velocity receive finite Additions or Diminutions, only in finite Times; and the Velocities at any two Instants of Time be taken, during which the Point B moves, it must pass through all possible intermediate Degrees of Velocity lying betwixt the two Velocities assumed, in the Space of Time lying betwixt the two Instants assumed. For let V denote any intermediate Velocity whatsoever, through which, if possible, the variable Velocity mentioned in this Lemma doth not pass; let M and N denote two Velocities, one on each Side of V, and the nearest to it which the variable Velocity mentioned in the Lemma passes through, then the Difference betwixt M and N is a finite Difference, (since there is a Velocity V assigned betwixt them) consequently the variable Velocity receives a finite Addition or Diminution in a Time less than any assignable, contrary to the Hyp. Therefore, &c.

G

L E M M A

L E M M A V.

26. If the Space or Augment Bb be described by the Point B , moving with a Velocity continually accelerated or continually retarded, in any Time, an uniform Velocity capable of describing the Space Bb in the same Time will fall in betwixt the two extreme Velocities, where-with the Point B moves at the Beginning and End of the Space or Augment Bb .

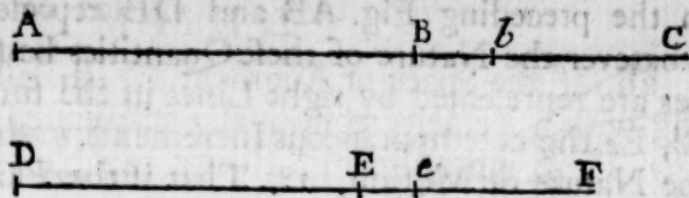
For when the Motion is accelerated, the Velocity with which the Point B moves at the Beginning of the Space Bb , continued the same, would not be capable of describing so great a Space as Bb , in the same Time it is described by the accelerated Motion: but the Velocity with which B moves at the End of the Space Bb , would produce a greater Space than Bb in that Time. Therefore an equable or uniform Velocity capable of describing Bb in the same Time it is described by an accelerated Velocity, must fall in betwixt the two, *i.e.* it must be greater than the first and less than the second, since, when uniform Velocities or equable Motions are compared, and the Time given, the Velocities are as the Spaces described directly. And it is evident, that the like way of reasoning may be applied to the Case of the continually retarded Motion: whence the Truth of the Lemma appears.

L E M M A VI.

27. If the Point B move with an uniform Motion, along the Line AC , and the Point E move along the Line DF with a Motion continually accelerated or continually retarded: and Bb and Ee be Spaces described or Augments generated by the Motion of these Points in the same or equal Times: the less the Spaces or Augments Bb , Ee be taken, or, which is the same thing, the less the Time be in which they are described, the nearer will the Ratio of Bb , Ee approach to the Ratio of the Velocities with which the Points B and E move at the Beginning of the Spaces or Augments: and by diminishing Bb and Ee more and more, their Ratio shall come nearer to the Ratio of these Velocities than by any given or assignable Difference: and therefore the ultimate Ratio of these Spaces or Augments Bb , Ee as they vanish (which is the same with their prime Ratio, considered as nascent) will be the Ratio of the Velocities mentioned.

The less the Space or Augment Ee is, and consequently, the less the Time is, in which it is described, the nearer will the Velocities, with which the Point E moves at the Beginning and End of that Space, approach

approach to one another: and by diminishing the Augment Ee , or the Time continually, these extreme Velocities may be made to approach nearer to one another than by any given Difference, as appears by Lem. 3; there-



fore much more, the uniform Velocity capable of producing Ee in the same Time, will still approach nearer and nearer to the Velocity with which the Point E moves at the Beginning of the Space Ee , the less that Space, or the Time wherein it is described, is: and by continually lessening that Space or Time, the uniform Velocity shall differ from the Velocity, with which the Point E moves at the Beginning of the Augment Ee , by less than any given Difference; since the uniform Velocity falls in betwixt the two extreme Velocities *. If then we put $\dot{B}$ for the Velocity with which the Point B moves, which is supposed to be constant or invariable; $\dot{E}$ for the Velocity with which Point E moves at the Beginning of the Space Ee ; and V for an uniform Velocity capable of describing Ee in the same Time it is described by the variable Velocity $\dot{E}$: then from the Nature of uniform Motion, $Bb : Ee :: B : V$; but the less that Ee is taken, the less is the Difference betwixt V and $\dot{E}$ *; so that when the Augment Ee is very small, it is nearly $\dot{B} : V :: \dot{B} : \dot{E}$, consequently, when the Spaces or Augments Bb , Ee are taken very small, or such as are describ'd in very small and equal Times, it will be nearly $Bb : Ee :: \dot{B} : \dot{E}$. Finally, since by continually diminishing Ee and consequently Bb , V approaches nearer to $\dot{E}$ than by any given Difference, it follows that $Bb : Ee$ approaches nearer to $\dot{B} : \dot{E}$ than by any given Difference: and therefore the ultimate Ratio of the evanescent Augments or Spaces Bb , Ee , or, which is the same, the prime Ratio of them considered as nascent, is the Ratio of $\dot{B}$ to $\dot{E}$. * Which Things were to be demonstrated.

* Art. 26. of this Explanation.

* Art. 26. of this Explanation.

* Art. 7. of this Explanation.

28. These Things being demonstrated, the Truth of the fundamental Proposition laid down by our Author *, will easily appear. For, Fluxions being defined *the Velocities of the Motions or Increments by which the flowing Quantities are generated*, suppose any one Quantity to flow uniformly, *i. e.* to have a constant or invariable Fluxion at all Times; and another Quantity of the same kind to flow according to any Law whatsoever, whether by an uniform Velocity, or a Velocity continually varying; and that at any rate, whether continually accelerated, or continually retarded: and let there be taken Increments of

* Art. 3. of the Quadrature of Curves.

these flowing Quantities, generated in the same or equal Terms: then in the preceding Fig. AB and DE represent the flowing Quantities, whatever the Nature of these Quantities be (just as all kind of Quantities are represented by right Lines in the fifth Book of *Euclid's Elem.*) Bb, Ee the cotemporaneous Increments. And therefore it appears from the Nature of Motion, 1°, That if the Fluents flow with constant or invariable Fluxions, the Fluxions are to one another as the Increments generated in any equal Times. And from what has been demonstrated, 2°, That if the one Fluent have a constant Fluxion, and the other a Fluxion continually varying, and their Augments be taken such as are produced in very small and equal Times, the Fluxions will be nearly as these Augments. 3°, In every Case accurately, the Fluxions are in the prime Ratio of the nascent Increments, or the ultimate Ratio of the evanescent Decrements, according as the flowing Quantities are supposed to increase or decrease. And further, because this is true when one of the Fluxions is given or constant, it follows that the same is true, when the Fluxions are all variable. For let P, Q denote any two homogeneous flowing Quantities, whose Fluxions are variable: A, any other Fluent of the same kind, which has an invariable Fluxion: let these Fluxions be denoted by the Symbols $\dot{P}$, $\dot{Q}$, $\dot{A}$ respectively; and let p , q , a denote the prime Ratios of their nascent Augments. Then, by what has been demonstrated $\dot{P} : \dot{A} :: p : a$ and $\dot{A} : \dot{Q} :: a : q$, therefore by Equality $\dot{P} : \dot{Q} :: p : q$, as was to be shewn.

Art. 4.
of the
Quadrature
of
Curves.

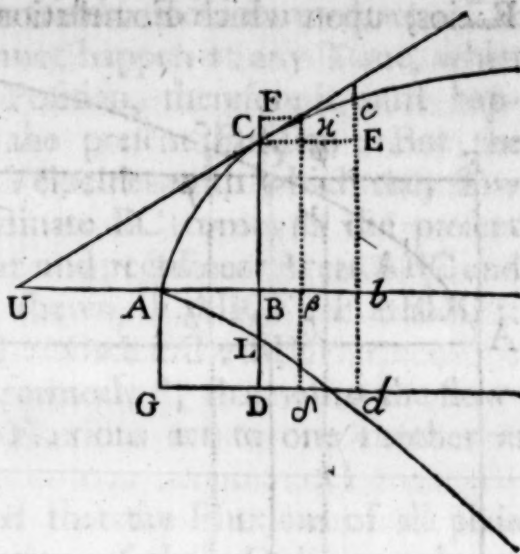
29. After laying down what we have been about demonstrating, touching the Proportion of Fluxions, our Author proceeds to shew the Application thereof, in determining the Fluxions of curvilinear Areas, in this Art.; and the Fluxions of the Abscisses, Ordinates and Curve Lines in the two succeeding Art. Now, what is laid down in Art. 4. may be thus demonstrated. Supposing the same Things as are there mentioned, I say $F.ABC^1 : F.ABDG :: BC : BD$, when ABDG is a Parallelogram. To demonstrate which, we need only shew that the prime Ratio of the nascent Increments, when the Areas are supposed to flow by increase; or the ultimate Ratio of the evanescent Decrements, when they are supposed to flow by decrease, is the Ratio of BC to BD. Let BC, BD, constituting one right Line, move forward, and come into the new Position bc , bd ; so that the Increments generated in the same Time be $BbcC$ and $BbdD$. Then it is evident, that, when cbd has got to any finite Distance from CBD, $BbcC : BbdD > BC : BD$; for by completing the Parallelogram $BCEb$, it is $BC : BD :: BCEb : BbdD$ * $< BbcC : BbdD$: but I say $BbcC : BbdD$ may be nearer to $BC :$

* Prop. 1.
B.6.Elem.

¹ F.ABC denotes the Fluxion of the Area ABC: and so of other like Notations.

BD,

BD, than by any given Difference; or than any given or assignable Ratio, by supposing bc to be still nearer and nearer to BC . For if possible let $BF : BD$ be as near to $BC : BD$ as $BbcC : BbdD$ can be: draw $F\kappa$ parallel to AB meeting the Curve in κ ; complete the Parallelograms $B\beta\kappa F$ and $B\beta\delta D$. Then $B\beta\kappa F : B\beta\delta D > B\beta\kappa C : B\beta\delta D$, but $B\beta\kappa F : B\beta\delta D :: BF : BD$, therefore $BF : BD > B\beta\kappa C : B\beta\delta D$, wherefore $B\beta\kappa C : B\beta\delta D$ is nearer to $BC : BD$ than $BF : BD$ is (since $B\beta\kappa C : B\beta\delta D$ can never be less than $BC : BD$, as was shewn) contrary to the Hyp. Therefore $BC : BD$ is the prime Ratio of the nascent



Augments $BbcC$ and $BbdD$ *. And it is the same thing to consider * Art. 13, of this Explanation.
 Bbc , $BbdD$ as Decrements, by supposing cbd to return back again into the Position CBD : for then it is demonstrated the same way that $BC : BD$ is the ultimate Ratio of $BbcC$ and $BbdD$ when vanishing. But the Fluxions of the Areas ABC and $ABDG$ are to one another in the prime Ratio of the nascent Increments *, or ultimate Ratio of the evanescent Decrements $BbcC$, $BbdD$ *. Therefore $F.ABC : F.ABDG :: BC : BD$, when $ABDG$ is a Parallelogram, as was to be shewn. * Art. 3. and 6. of the Quadra-

30. But further, although both Fluents were curvilinear Areas, I say their Fluxions shall be as the Ordinates, when the Ordinates do move with the same uniform Motion along the Absciss AB. For let ABL be another curvilinear Area described by the Ordinate BL, then, by what has been demonstrated, $F.ABC : F.ABDG :: BC : BD$ and $F.ABDG : F.ABL :: BD : BL$, therefore by Equality, $F.ABC : F.ABL :: BC : BL$.

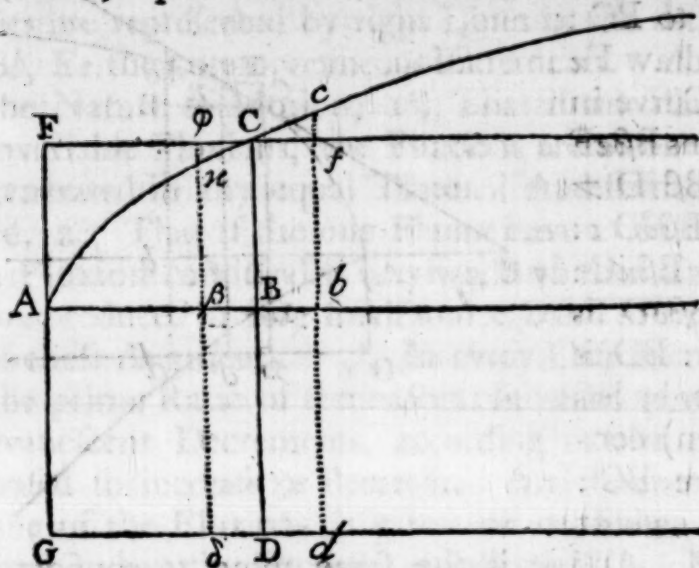
31. Although this Method of investigating and demonstrating the Proportions of the Fluxions of plain Figures, be laid down by Sir *J. Newton* in this Place; and, as I have shewn, is both certain and geometrical: yet such perhaps may be the Scruples of some; and such the Obstinacy of others, against the whole of this Doctrine of prime and ultimate Ratios of nascent and evanescent Quantities, that I hope it will not be unacceptable to the Reader, if I shall shew that the Proportions of the Fluxions of plain Figures (and consequently by Analogy of all other flowing Quantities) may be demonstrated in another manner, upon the most indisputable Principles of Geometry, without introducing either infinitely little Quantities, as has been done, without sufficient Caution,

* Art. 13,
of this Ex-
plication.

* Art. 3.
and 6.
of the
Quadra-
ture of
Curves.
Art. 28.
of this Ex-
plication.

by some; but purposely avoided by Sir *I. Newton* in this Treatise; or yet nascent and evanescent Quantities, with their prime and ultimate Ratios, upon which Foundation he has built this Doctrine.

* Art. 4.
of the
Quadrature
of Curves.
and Art.
29.
of this Ex-
plication.



Let ABC and ABDG be a curvilinear Area and Parallelogram described as before, the same things being suppos'd as formerly *: and further let ABCF be another Parallelogram describ'd at the same time by a given Ordinate equal to AF, so that all the three Areas ABC, ABDG and ABCF be described by the uniform Motion of their respective Ordi-

nates, lying always in the same right Line: then I say $F.ABC : F.ABDG :: BC : BD$.

For since the Ordinates of the Parallelograms ABDG, ABCF are given or invariable right Lines, and these Lines are supposed to move uniformly along the common Absciss AB, the Fluxions or Velocities of flowing of the Parallelograms must be constant and invariable, for the Spaces generated in equal Terms are evidently equal in the respective Parallelograms; and the Fluxions or Velocities of flowing in that

* Art. 28.
of this Ex-
plication.
* Prop. 1.
B. 6. Elem.

Case are as the Increments generated in any the same times *, or by Composition as the whole Areas generated, that is $F.ABCF : F.ABDG :: (ABCF : ABDG ::) BC : BD$ *, BC being considered as the invariable Ordinate of ABCF. But the Velocity with which the curvilinear Area ABC is described must be less than the Velocity with which the Parallelogram ABCF is described at any Time before the Ordinate BC arrive to it's present Position, as when it is in $\beta\kappa\phi$, because $\beta\kappa < \beta\phi$: again, the Velocity with which ABC flows, at any time after BC has past it's present Position, as when it is in bfc , is greater than the Velocity with which ABCF flows, since $bc > bf$: and these things are true, let $\beta\kappa\phi$ and bfc be as near to BC as you please, because the Curve AC and right Line FC intersect each other in one Point only: but the Velocity with which ABC flows, is continually or incessantly varying, since it's Ordinate BC never continues of the same Length, for any the least Time. Wherefore, betwixt the two Instants of Time that the Ordinate BC has the Positions $\beta\kappa$ and bc , the variable Velocity

The Quadrature of CURVES explained.

347

with which the Area ABC flows must pass through all possible intermediate Degrees *, therefore also through that Degree of Velocity with which ABCF flows, which is invariable and intermediate: But it has been demonstrated, that this cannot happen at any Time, when the Ordinate BC is out of the present Position, therefore it must happen, when the Ordinate BC obtains the present Position: But the Fluxions of flowing Quantities are the Velocities with which they flow or increase *: therefore, when the Ordinate BC comes to the present Position, the Fluxions of the curvilinear and rectilinear Areas ABC and ABCF are equal: but, as was already shewn, $F.ABCF : F.ABDG :: BC : BD$, therefore $F.ABC : F.ABDG :: \text{Ord. } BC : BD$.

* Art. 25.
of this Ex-
plication.

* Art. 2.
of the
Quadra-
ture of
Curves.

* Art. 30.
of this Ex-
plication.

32. From this, we may conclude as formerly *, that when the flowing Areas are both Curvilinear, their Fluxions are to one another as the Ordinates.

33. And thus we have demonstrated that the Fluxions of all plain Figures, described by the equable Motion of their Ordinates always lying in the same variable right Line, and moving along a common Absciss, are every where as their respective Ordinates; and that without introducing either infinitely small Quantities, or yet the Doctrine of prime and ultimate Ratios of nascent and evanescent Quantities. And therefore, the Foundation upon which the Doctrine of Fluxions is built, would still remain unshaken, although the Doctrine of prime and ultimate Ratios could be shewn to be uncertain and precarious; which yet all Sir *I. Newton's* Adversaries shall never be able to do.

From what has been said the following Corollaries may be deduced.

34. Cor. 1. The Fluxion of any plain Figure, generated by the uniform Motion of it's Ordinate along the Absciss, is every where proportional to the Ordinate.

35. Cor. 2. Although our Author has hitherto mentioned no Motion of the Ordinates but what is uniform and the same along the Abscisses, or rather common Absciss: yet the Motion of the Ordinates along the Abscisses of different Curves, *i. e.* the Fluxions of the Abscisses of different Curves, may be supposed different; yea and the Fluxion of the Absciss of the same Curve may be supposed variable, by being accelerated or retarded, and that according to any Law. But I say in every Case, the Fluxions of the Areas of plain Figures, described by the parallel Motion of their Ordinates along the Abscisses, are always to one another as the Rectangles contained under the Ordinates and the Fluxions of the respective Abscisses. For by what has been demonstrated, when the Fluxions of the Abscisses are uniform and the same in all, the Fluxions of the Areas are as the Ordinates directly: but if the Fluxion of the

the Absciss be increased or diminished in any Proportion, the Velocity with which the Area increases, *i. e.* it's Fluxion must be increased or diminished in the same Proportion; since the Space that would be described with that Velocity in any given Time, would be increased or diminished in that Proportion: and it is no matter, whether the Fluxion of the Absciss of the same Figure continue invariably the same, or vary according to any Law, by being accelerated or retarded; for it is only the present Fluxion of the Area: therefore it follows that the Fluxions of any plain Figures, are always as the Rectangles contained under the Ordinates of these Figures, and the Fluxions of their Abscisses, or right Lines proportional to and expounding the Fluxions of these Abscisses.

36. The same thing may be more shortly demonstrated by means of Symbols thus, as is done by our Author elsewhere¹, in the foregoing Figure, let the Ordinates BC, BD move at right Angles along the Absciss AB; let AG or BD be equal to Unity; the common Absciss AB = x , the curvilinear Area ABC applied to BC = z , or $ABC = z \times 1 = z$, then is $ABDG = (x \times BD = x \times 1 =) x$. Now putting $\dot{x}$ and $\dot{z}$ to stand for the Fluxions of x and z respectively, by what has been demonstrated, $\dot{z} : \dot{x} :: BC : BD = 1$: for it is all one whether the Fluxion of AB, *i. e.* $\dot{x}$ continue invariably the same, or vary according to any Law: therefore by reducing the Analogy to an Equation $\dot{z} = (\frac{\dot{x} \times BC}{BD} = \frac{\dot{x} \times BC}{1} =) \dot{x} \times BC$, *i. e.* $F.ABC = F.AB \times BC$, *viz.* the Fluxion of the Absciss multiplied into the Ordinate, *i. e.* in other Words, if the Ordinate of any plain Figure, continuing the same in Length, that it is in any given Place, were supposed to be carried perpendicularly along the Absciss with an uniform Velocity, equal to the Velocity with which the Absciss increases at that particular Place, the superficial Space thereby produced in any given Time, would be equal to the superficial Space produced or described in that same Time, by the Velocity with which the Area flows or increases at that Place, continuing uniformly the same.

37. But suppose the Ordinates not to move perpendicularly along their Abscisses; but in any given Angle, provided it be the same in all, the Fluxions of the flowing Areas will still be proportional to the Rectangles contained under the Ordinates and Fluxions of the Abscisses. For in the Demonstration of our Author's fourth Art. the Ordinates BC, BD may make any given Angle with the Absciss AB*: therefore it's easy to apply the Reasoning at Art. 35. to this Case.

* Art. 35.
of this Ex-
plication.

¹ See his Method of Fluxions and infinite Series, published by J. Colson, pag. 23.

38. Cor. 3. If two or more plain Figures begin to be described at the same Time, by their Ordinates moving along their Abscisses in a given Angle, and so be described in the same Time; or, which comes to the same purpose, if they be described in equal Times, and the Rectangles contain'd under the Ordinates and the Fluxions of the Abscisses or right Lines proportional to and expounding the Fluxions of the Abscisses, be equal in these Figures, at every the same or correspondent Instant of their flowing or increasing, throughout the whole Time of their Description, the Areas described shall be equal. For these Rectangles are proportional to the Fluxions of the Areas *, and therefore the Fluxions of the Areas are always equal to each other, *i. e.* the Velocities with which the Figures are described, are every where equal to each other: and therefore the Areas produced or described with these Velocities in the same or equal Times must be equal, from the Nature of Motion.

* Art. 35. and 37. of this Explication.

It would be the same thing, if the Ordinates of the Curves and the Fluxions of their Abscisses were reciprocally proportional: for then would the Rectangles mentioned be equal *.

* Prop. 14. B. 6. Elem.

39. Cor. 4. Conversely if the flowing Areas be every where equal to each other, the Rectangles contain'd under the Ordinates and the Fluxions of the Abscisses, shall be every where equal to each other: for, if not, the Fluxions of the plain Figures, *i. e.* the Velocities with which they flow or increase, would not be equal: and if the Velocities of Description were not every where equal, the Areas described could not be every where equal.

40. Schol. It may be here remarked, with respect to variable or flowing Quantities of any kind whatsoever, that, if they be at all times equal to each other, the Velocities of their Mutation, *i. e.* the Fluxions with which they flow at any the same Time, must also be equal: conversely, if the Velocities of Mutation or Fluxions of any variable Quantities be always equal, and moreover these Quantities arise together, when they flow by increase; or vanish together, when they flow by decrease, the variable Quantities shall always be equal at the same Time. For these variable or flowing Quantities may be represented and expounded by the flowing plain Figures; and their Velocities of Mutation, by the Fluxions of these Figures, mentioned in Cor. 3. and 4. Wherefore, since the Fluxions of these equal Figures are always equal; and conversely the Fluxions being equal, the flowing plain Figures are equal; hence the variable Quantities of any kind, and their Velocities of Mutation or Fluxions must be equal. Which is likewise evident in itself from the Nature of the Thing.

H

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the Curve draw the Ordinate βmn , meeting CE in e ; and VC in M. Then $Ce:ex > (Ce:em::) * VB:BC$. Again, $Ce:ex < (Ce:em::) * LB:BC$: therefore $Ce:ex$ falls in betwixt $VB:BC$ and $LB:BC$, and consequently $Ce:ex$ comes nearer to $VB:BC$ or $CE:ET$ than $LB:BC$ does, cont. Hyp. Therefore there is no assignable Ratio, which is so near the Ratio of VB to BC or CE to ET , as the Ratio of the Augments Bb , Ec may sometime or another be. Wherefore $CE:ET$ or $VB:BC$ is the prime Ratio of the nascent Augments Bb , Ec ; or, which is the same, it is the ultimate Ratio of these Parts, when vanishing.

Now with respect to the curvilinear Augment, Cc ; let the Ordinate bEc approach continually towards BC , and at length fall in with it, when the Points C , c and E , and likewise B and b coalesce; in the mean time let the Tangent CT and Chord Cc be supposed to be produced still to distant Parts H and K . Then 'tis evident the Angle HCK made by the Tangent and Subtense still diminishes, as the Arch Cc diminishes, or as bEc approaches to BC : because, as has been already demonstrated, the varying Ratio of Ec to ET is still increasing, and the Triangle CET is given in Species: but it has been shewn likewise, that the ultimate Ratio of CE to Ec is that of CE to ET ; and that therefore the ultimate Ratio of Ec and ET is a Ratio of Equality; and consequently the ultimate Ratio of the three Triangles CcE , $CxcE$ and CTE is a Ratio of Equality likewise; and as to their Form, they are at last similar. Therefore, since the Triangle CET is given in Species, and always similar to the Triangle VBC , it follows that the prime or ultimate Ratio of the nascent or evanescent Parts Bb , Ec and Cxc , whereby the Absciss, Ordinate and Curve-line are increased or diminished, is that of CE , ET and CT or VB , BC and VC . Consequently the Fluxions of the Absciss, Ordinate and Curve-line are proportional to these Lines *, and may be expounded by them: as our Author affirms.

* Art. 28. of this Explication.

42. Schol. From the Relations of the Fluxions of the Absciss, Ordinate and Curve-line just now demonstrated, are derived two principal Branches of the general and extensive Doctrine of Fluxions, viz. the Method of drawing Tangents to Curves: and the Rectification of Curve-lines, or determining their Lengths. The first belongs to the *direct*; the other to the *inverse* Method of Fluxions. The one depends upon this, that, if you call the Absciss $AB = x$, the rectangular Ordinate $BC = y$, and Curve-line $AC = z$; and denote their Fluxions by the Symbols $\dot{x}$, $\dot{y}$ and $\dot{z}$, then $\dot{y} : \dot{x} :: y : VB$, that is $VB = \frac{\dot{y}}{\dot{x}} \times y$: the other upon this that $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: both of which are

evident from what has been said: since the evanescent Triangle $CE\alpha C$ is ultimately similar to the right-angled Triangle VBC : and the ultimate Ratio of the evanescent Augments is the same with the Ratio of the Fluxions.

* Art. 7.
of the
Quadrature
of
Curves.
* Of this
Explication.

43. The Design of this Art. is to shew how the Fluxions of solid Figures; and also the Fluxions of their Curve-surfaces may be investigated and determined. In order to which, suppose, in the Fig. at Art. 29 *, a Circle described with BC , the Ordinate of the Curve ABC , for Radius; and that that Circle moves along the Absciss AB with a uniform Motion, keeping always perpendicular to it: the moving Circle will generate a solid Figure of such a Nature as corresponds to the plain Figure, of which the right Line BC , the Radius of the generating Circle, is the Ordinate perpendicularly applied to the Absciss AB : *i. e.* a Solid, which may be supposed to be otherwise generated by the Rotation of the plain Figure ABC about it's Axis AB . Now, to illustrate and demonstrate what our Author lays down in this Place, suppose another Circle, having the given or invariable Line AG or BD for it's Radius, to be drawn along the same Absciss AB , and by that means generate a Cylinder: let the generating Circles begin to move from A at the same Time, and be found ever after in the same Plane at the same Time; just as the right Lines BC and BD were formerly supposed to lye always in the same Direction. Then by what was formerly demonstrated *, the Fluxions of the Solid ABC and Cylinder $ABDG$ are in the prime Ratio of their nascent Increments. Let $BbcC$ and $BbdD$ be the solid Augments generated in the same Time: and then, as it was formerly demonstrated, that the prime Ratio of the Augments $BbcC$ and $BbdD$, *in statu nascenti*, when belonging to the plain Figures, was that of BC to BD ; so now, by the like way of reasoning, we may demonstrate that the prime Ratio of the Augments $BbcC$ and $BbdD$, *in statu nascenti*, when considered as belonging to the solid Figures, is that of the Circles having BC and BD for their Radius's; or which is the same, that of BC^2 to BD^2 (since Circles are as the Squares of their Radius's). For the varying Ratio of the solid Augments $BbcC$ and $BbdD$, by supposing the Distance Bb still less and less, comes nearer to the Ratio of BC^2 to BD^2 , than any other given or assignable Ratio; but can never go beyond it. For let BC be produced to F , so that, if possible, $BF^2 : BD^2$ may be as near to $BC^2 : BD^2$ as $BbcC : BbdD$ can in any case be. And, having made the same Construction as at Art. 29, suppose $BbEC$ and $B\beta\alpha F$ to represent two small Cylinders, having the right Lines BC or bE and BF or $\beta\alpha$ for the

* Art. 28.
of this
Explication.

the Radius's of their circular Bases. Now 'tis evident that when Bb is of any finite Length, $BbcC : BbdD > (BbEC : BbdD ::) * BC^2 : BD^2$; * Prop. 2. B. 12. Ele. therefore I have supposed BF greater than BC . But since $B\beta C < Cyl. B\beta F$, we have $B\beta C : B\beta dD < (B\beta F : B\beta dD ::) * BF^2 : BD^2$. Therefore $B\beta C : B\beta dD$ is less than $BF^2 : BD^2$; but it was shewn that it cannot be less than $BC^2 : BD^2$, wherefore it comes nearer to $BC^2 : BD^2$ than $BF^2 : BD^2$ is, contrary to the Hyp. Therefore the prime Ratio of the nascent Augments BbC and $BbdD$ is that of BC^2 to BD^2 *; or, which is the same, that of the Circles having BC and BD for their Radius's. Wherefore the Fluxions of the Solid ABC and Cylinder $ABDG$ are in the same Ratio *. Whence it's evident (by considering what was said in Art. 30.) that the Fluxions of all solid Figures, generated after this manner, the Fluxions of whose Abscisses are every where equal and uniform, are to one another in every Place, as the Circles, by whose Motion they are generated, are in that Place; or as the Squares of their Radius's: as our Author affirms.

* Art. 11. and 13. of this Explication. * Art. 28. of this Explication.

44. Cor. Hence it will follow, that, although the Motion of the generating Circles along the Abscisses be different and unequal, *i. e.* although the Fluxions of the Abscisses be different in different Solids; and even variable in the same Solid, yet universally, the Fluxions of Solids, generated by drawing the Circles along the Abscisses, are in a compounded Ratio of the generating Circles, and the Fluxions of the Abscisses: as may easily be demonstrated, by considering what was said at Art. 35. *

45. The other Thing to be demonstrated in this Art. is, that the Fluxion of the Curve-surface is as the Periphery of the generating Circle, having it's Radius BC , and the Fluxion of the Curve-line AC conjunctly; or in a compounded Ratio of these two. To illustrate and demonstrate which: suppose the Curve-line AC , and the Circumference of the Circle of which BC is Radius, to be both extended into right Lines; so as the latter, being made to move at right Angles along the former, may thereby generate or describe a plain Figure having the right Line equal to AC for it's Absciss; and the other, equal to the Perimeter of the Circle, whose Radius is BC , for it's Ordinate: and suppose the Velocity with which the rectilinear Absciss flows, or it's Fluxion, to be always the same with the Velocity or Fluxion of the Curve-line AC ; and it's Ordinate in every Place, equal to the Circumference of the Circle, having for it's Radius BC , in the corresponding Place; then it's evident that the Curve and plain Surfaces, generated in the same time, will be equal; and that therefore their Fluxions, at the

* Of this Explication.

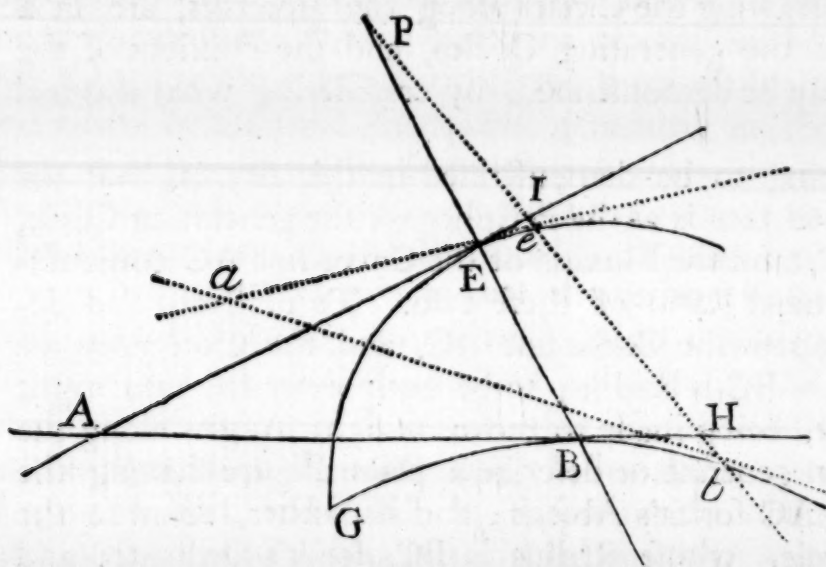
* Art. 40.
of this Ex-
plication.
* Art. 35.
of this Ex-
plication.

the same Time, will be equal *. But the Fluxion of the plain Surface is always as the Ordinate and Fluxion of the Absciss jointly *: therefore also the Fluxion of the Curve-surface is as the Perimeter of the Circle, having the Line BC for Radius, and the Fluxion of the Curve-line AC jointly. The plain meaning of which is this: that if there were any Number of Curve-surfaces, belonging to Solids, generated after the manner that has been explained, the Velocities with which the several Curve-surfaces flow, are such, that the superficial Spaces, which would be described by these Velocities in any given time, supposing these Velocities to continue invariable for that time, are as the Rectangles contain'd under the Perimeters of the several generating Circles, and right Lines proportional to the Fluxions of the several Curve-lines, corresponding to AC.

46. Schol. Upon these Properties of the Fluxions of Solids; and their Curve-surfaces, are founded two principal Branches of the Doctrine of Fluxions: *viz.* the Cubature of Solids; or the Method of finding their solid Content: and the Plaining of Curve-surfaces; or the Method of finding the superficial Content: both belonging to the inverse Method of Fluxions.

Art. 8,
9, 10.
of the
Quadra-
ture of
Curves.

47. These three Articles contain some more Examples, for the further Illustration of the Method of prime and ultimate Ratio's, both in



* Of this
Explica-
tion.

the Case of Right and Curve Lines, considered as flowing Quantities. The Example contained in Art. 8, is so plain as to need no further Illustration: that in Art. 9, is coincident with the one formerly adduced and explained Art. 16. *

From which Ex-

ample our Author's 10th Art. is deduced by way of Corollary: and is the only thing remaining in these Articles, that needs any further Explication. In order to which, let the right Line PB, revolving about the given Point P as a Pole, cut the Curves GB, GE given in position, in the Points B and E, and let the moveable right Lines AB and AE touch these Curves always in the moveable Points of Intersection B and E,

I

I say

I say the Fluxion of the Curve GB is to the Fluxion of the Curve GE in a Ratio compounded of AB : AE and PB : PE ; that is $F.GB : F.GE :: AB \times PB : AE \times PE$.

For let the revolving Line PB move forward from it's Place PB into the new Place Pb, while at the same time the Tangent-lines AB and AE, still touching the Curves come into the new Position Ab and Ae, so as b and e be the new Points of Contact and Intersection: and let the Line Pb cut AB and AE in the moveable Points H and I. Now suppose the Line Pb to return to it's former Place PB; by which means the Lines Ab and Ae will return to their former Places AB and AE, and the Points A and a, B, b and H; E, e and I at last coincide. In which Case, the ultimate Ratio of the evanescent Augments Bb and BH; Ee and EI is a Ratio of Equality, as may easily be collected from what was demonstrated the latter Part of Art. 41. *: for the ultimate Ratio of the Arch and Tangent, when vanishing, is a Ratio of Equality, whatever be the Angle, which the secant Line makes with the Curve or Tangent. Wherefore $Bb : Ee :: BH : EI$; when vanishing: but in that Case, $BH : EI :: AB \times PB : AE \times PE$ *. Therefore also the ultimate Ratio of the evanescent Augments Bb and Ee is $AB \times PB : AE \times PE$. Therefore $F.GB : F.GE :: AB \times PB : AE \times PE$ *. As was to be proved.

* Of this Explication.

* Art. 16. of this Explication.

* Art. 28. of this Explication.

48. Moreover if the Line PB, instead of revolving about the immovable Point P as a Pole, should perpetually touch a Curve given in Position in the movable Point P, the Ratio of the Fluxions of the Curves GB and GE, will be the same as before: since the ultimate Ratio of the evanescent Augments will be the same as before; as will easily appear by the same way of reasoning: so that there is not any necessity to insist particularly upon it.

49. This Article teaches us how to find the Fluxion of any Power of a flowing Quantity, *i. e.* to find the Relation betwixt the Fluxion of any Power of a flowing Quantity, and the Fluxion of the Root or Quantity itself *.

Art. 11. of the Quadrature of Curves.

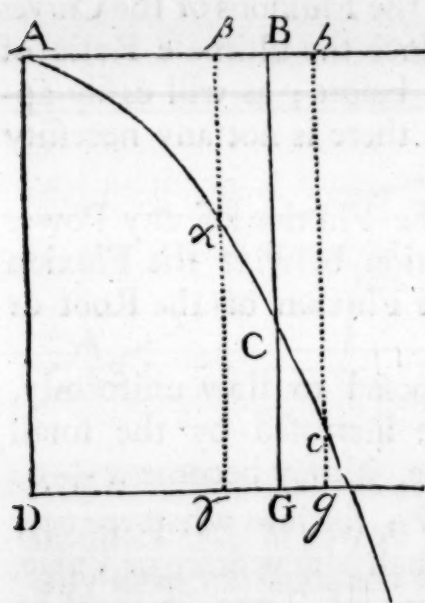
Let x represent any flowing Quantity, supposed to flow uniformly, and x^n any Power of that Quantity: let x be increased by the small Augment o , so that in any short Space of Time, it may become $x + o$: then in the same Time, x^n will become $\overline{x + o}^n$; for into whatever new State or Value x comes, x^n must come into such a new State or Value, as that still, in every Place, or at every Instant of Time, it must be the n th Power of the Value of x in that Place or at that Instant of Time. Now by Sir Isaac Newton's binomial Theorem, $\overline{x + o}^n = x^n +$

* Art. 6. of this Explication.

$\frac{n}{1} o$

$+ nx^{n-1}o + n \times \frac{n-1}{2} x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}o^3 + \&c.$ Wherefore, in the same Time that x becomes $x + o$, x^n becomes $x^n + nx^{n-1}o + n \times \frac{n-1}{2} x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}o^3 + \&c.$ From which two new Values subtracting the former Values x and x^n respectively, the Remainders, viz. o and $nx^{n-1}o + n \times \frac{n-1}{2} x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}o^3 + \&c.$ are the synchronal Increments: and if both be divided by o , we shall have 1 to $nx^{n-1} + n \times \frac{n-1}{2} x^{n-2}o + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}oo + \&c.$ for the Ratio of the Augments generated in any the same Time, and that whether the Increment o be big or small. Now the Fluxions of Quantities are to one another in the prime Ratio of the nascent or ultimate Ratio of the diminishing, and at length evanescent Augments *. But by diminishing o more and more continually, the prime Ratio of the nascent, or ultimate Ratio of the evanescent Augments will be that of 1 to nx^{n-1} , because the varying Ratio of 1 to $nx^{n-1} + n \times \frac{n-1}{2} x^{n-2}o + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}oo + \&c.$ (which is the Ratio of the Augments) by diminishing o continually; before o vanish, comes nearer to the Ratio of 1 to nx^{n-1} than by any given Difference. Therefore $F.x : F.x^n :: 1 : nx^{n-1}$.

* Art. 28.
of this Ex-
plication.



Line AD or BG = 1: then is AG = (1 x x =) x. Let BC be the Ordinate of the Figure ABC, whose Area is supposed equal to x^n . Now let the plain Figures ABC and AG be supposed first to flow by Increase,

50. The same thing might be demonstrated with greater Form, according to the same Principles: but waving that at this Time, I shall shew how it may be demonstrated by a geometrical Construction: in which the Quantities x and x^n are represented or expounded by plain Figures.

Suppose then the plain Figure ABC and Rectangle AG to be described in the same Time, by the uniform Motion of the right Line BCG along the common Absciss AB. Let AB = x , and BG (= AD) be the Ordinate of the flowing Rectangle AG, and let the given right

so that after a short Time, the right Line BCG may come into the new Place *bcg*: again let them be supposed to flow backwards, by decreasing, and let BCG come into the new Place $\beta\kappa\gamma$. Then let *bcg* and $\beta\kappa\gamma$ return again to BCG, and fall in with it at the same Time, so that B*b* and B β may be always equal. And, by what was formerly demonstrated *, the ultimate Ratio of the evanescent Augments B*b*cC and B*bg*G is BC: BG; and, for the same Reason, the ultimate Ratio of the evanescent Decrements B β κ C and B β γ G, is also BC: BG. Therefore, when these Increments and Decrements are just vanishing, we have B*b*cC: B*bg*G :: B β κ C: B β γ G: but because B β is always equal to B*b*, and therefore also when vanishing, hence B*bg*G = B β γ G, therefore also the ultimate Ratio of B*b*cC and B β κ C is a Ratio of Equality, Put then B*b* = 0 = B β , and consequently A*b* = $x + 0$ and A β = $x - 0$: and from the Property of the Figure ABC, we have A*b*c = $x + 0$ =

* Art. 29.
of this Ex-
plication.

(by Sir Isaac Newton's binomial Theorem) $x^n + nx^{n-1}0 + n \times \frac{n-1}{2} x^{n-2}0^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}0^3 + \mathcal{E}c$; and A β κ = $x - 0$ = $x^n - nx^{n-1}0 + n \times \frac{n-1}{2} x^{n-2}0^2 - n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}0^3 + \mathcal{E}c$. From the former subtract ABC = x^n ; and subtract the latter from ABC = x^n , and you'll have $nx^{n-1}0 + n \times \frac{n-1}{2} x^{n-2}0^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}0^3 + \mathcal{E}c$ = the Augment B*b*cC; and $nx^{n-1}0 - n \times \frac{n-1}{2} x^{n-2}0^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}0^3 - \mathcal{E}c$ = the Decrement B β κ C. Now F.ABC: F.AG :: B*b*cC: B*bg*G when vanishing *: and likewise F.ABC: F.AG :: B β κ C: B β γ G, when vanishing. Wherefore, I say, it follows that F.ABC: F.AG :: $nx^{n-1}0$: 0 :: nx^{n-1} : 1. For if F.ABC: F.AG < $nx^{n-1}0$: 0, then by diminishing of 0, $nx^{n-1}0 + n \times \frac{n-1}{2} x^{n-2}0^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}0^3 + \mathcal{E}c$: 0 might become at length less than $nx^{n-1}0$: 0, which can never be: again, if F.ABC: F.AG > $nx^{n-1}0$: 0, then $nx^{n-1}0 - n \times \frac{n-1}{2} x^{n-2}0^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}0^3 - \mathcal{E}c$: 0, by diminishing 0 continually, might at length become greater than $nx^{n-1}0$: 0, which can never be either, because in the Series $nx^{n-1}0 - n \times \frac{n-1}{2} x^{n-2}0^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}0^3 - \mathcal{E}c$. (where the Terms are the same with those of the 11th Power of the Residual $x - 0$, wanting the first Term, with the Signs changed) by diminishing 0 sufficiently, each Term will be greater than the succeeding one, and therefore the Terms being affected with the Signs + and - alternately, the whole Series, by diminishing 0 *in infinitum*, can never be

* Art. 28.
of this Ex-
plication.

come greater than $nx^{n-1}o$. Wherefore, since it is neither $F.ABC : F.AG < nx^{n-1}o : o$; nor yet $F.ABC : F.AG > nx^{n-1}o : o$, it follows that $F.ABC : F.AG :: nx^{n-1}o : o :: nx^{n-1} : 1$.

51. Schol. In this Demonstration, the Index n of x^n is positive; but supposing it to be negative, we can prove, by the same way of Reasoning, that the Fluxion of x is to the Fluxion of x^{-n} as 1 to $-nx^{-n-1}$. Which shews that when x flows by Increase, x^n flows by Increase also; but x^{-n} flows by Decrease, it's Velocity or Fluxion being negative. But farther, it appears, 1°. That, when x flows uniformly by Increase, having it's Fluxion constant and invariable, if n be positive and greater than Unity, the Velocity with which x^n increases, or it's Fluxion is continually accelerated: for such a Number being substituted for n , in the Expression nx^{n-1} , makes the Index $n-1$ positive: and therefore it is evident the greater x is, the greater will the Quantity nx^{n-1} be, which expresses the Fluxion of x^n , when 1 expresses the Fluxion of x . 2°. If n be positive, but less than Unity, the Index $n-1$, in the Expression of the Fluxion nx^{n-1} , becomes negative in the Numerator; or positive in the Denominator, therefore the greater x is, the less will the Value of the Expression nx^{n-1} be: wherefore the Velocity with which x^n increases, or it's Fluxion is continually lessened or retarded. 3°. If n be negative, whatever be it's Value, the Expression $-nx^{-n-1}$ or $-\frac{n}{x^n+1}$, having the Index of x negative in the Numerator; or positive in the Denominator, shews the greater x is, the less will the Value of the negative Expression $-nx^{-n-1}$ or $-\frac{n}{x^n+1}$ be. Therefore it appears from thence, that when x flows uniformly by Increase, the Quantity x^{-n} flows backward or by Decrease, with a Velocity continually lessened or retarded.

52. Cor. By what has been demonstrated $F.AG : F.ABC :: 1 : nx^{n-1}$; likewise $F.AG : F.ABC :: BG : BC$ *; hence $BG : BC :: 1 : nx^{n-1}$: but $BG = 1$, therefore $BC = nx^{n-1}$. Whence it follows, that if the Absciss, perpendicular Ordinate and Area of a Curve be called x , y , and z respectively; and $z = x^n$ then $y = nx^{n-1}$: and conversely if $y = nx^{n-1}$ expresses the Relation betwixt the Absciss and Ordinate, then $z = x^n$ expresses the Relation betwixt the Area applied to Unity and the Absciss: or, if the Value of x be taken from the Equation $y = nx^{n-1}$, and insert in place of it, in the Equation $z = x^n$, it becomes $z = \frac{y}{n} \frac{n}{n-1}$ for the Relation betwixt the Area applied to Unity and the Ordinate: or lastly by taking the Value of x^n , we will have $z = \frac{1}{n} yx$ for the Relation betwixt the Area of the Curve, and the Rectangle contained

* Art. 29. and 31. of this Explication.

ed under the Abscifs and Ordinate. And this is the Foundation of the Quadrature of simple Curves ¹.

From this it appears that if $nax^{n-1} = y$ express the Relation of the Abscifs and Ordinate of a Curve; where a denotes any given Quantity: then $ax^n = z$ shall exhibit the Relation betwixt the Abscifs and Area applied to Unity: and conversely, if $z = ax^n$, then $y = nax^{n-1}$.

For since nx^{n-1} and nax^{n-1} are in a given or constant Ratio, viz. that of 1 to a : it follows, by Composition, that the Areas of the Curves which have nx^{n-1} and nax^{n-1} for their Ordinates, and x for their common Abscifs, will be in the same given Ratio: but when $y = nx^{n-1}$, it is $z = x^n$ (as has been shewn) therefore when $y = nax^{n-1}$, it will be $z = ax^n$. Wherefore also conversely, if $z = ax^n$, it follows that $y = nax^{n-1}$. The same Thing may be easily demonstrated by the Help ^{* Of this Explication. Art. 12. of the Quadrature of Curves.} of Art. 38, 39, 40 *.

53. After our Author has observed in this Article, that by like ways of reasoning, in the Method of prime and ultimate Ratio's, the Fluxions of right and curve Lines in all Cases whatsoever; the Fluxions likewise of Surfaces, Angles and other Quantities may be investigated; he desires it may be remarked, that the manner of proceeding in the Doctrine of Fluxions, by means of prime and ultimate Ratio's, is perfectly agreeable to the Analysis of the Ancients: and that there is not any necessity of introducing into Geometry, by the Method of Fluxions, Figures infinitely small.

This is carefully to be observed: and the Truth of it will appear by considering, that the Method which he takes in the several Examples, contained in this Introduction to his Treatise of the Quadrature of Curves, is this: first he determines the Ratio's of the cotemporaneous or synchronal Increments, by such general Expressions, as are capable of being applied to these Increments whether taken great or small; or whether generated in a greater or less, only the same Time: then, from these general Expressions, he finds in particular, what must be the prime or ultimate Ratio's of these Increments, considered as just arising or vanishing. Which is certainly a just and geometrical way of proceeding; to give an universal Solution of a Problem, applicable to all possible Cases, and then apply it to one particular determinate Case. This Method of Solution is the most elegant; as well as most useful of any other, and what is commonly made use of both in the Analysis of the Ancients; and that of the Moderns. Thus in deter-

¹ See our Author's admirable Treatise, *Analysis by Equations of an infinite Number of Terms*, Rule first, and it's Demonstration towards the End of that Treatise: likewise his *Method of Fluxions and infinite Series*, Prob. 9.

mining the Fluxion of x^n , or rather the Ratio of the Fluxion of x^n to the Fluxion of x : he first shews that the Ratio of their synchronal Increments, whether these Increments be great or small, or, which is the same, whatever the Time be, wherein these cotemporaneous Increments are generated, is that of $nx^{n-1} + \frac{n^2-n}{2}x^{n-2}o + \frac{n^3-3n^2+2n}{6}x^{n-3}oo + \&c.$ to 1. Which expresses the Ratio of these Increments at all Times and in every Case: and therefore also, when these Increments are just arising; or just vanishing, as well as at any other Time: and then he shews that the foresaid Ratio, in that particular Case, when the Increments are just arising or vanishing, becomes the same with $nx^{n-1} : 1$. The same way of reasoning may be observed, in the other Examples proposed in this Introduction: thus in the Example at Art. 9, in which our Author proposes to determine the Ratio of the Fluxions of the flowing Lines AB and AE: he first demonstrates, that their synchronal Increments Bb and Ee, in all Cases, whether these Increments be great or small, are as $Ab \times PB$ to $Ae \times PE$: then he shews what that variable Ratio, still expressed by finite Lines, will be, in that particular Case, when the Augments are just arising; or just vanishing: and he demonstrates that it will, in that Case, become the same with the Ratio of $AB \times PB$ to $AE \times PE$, and therefore he infers, in consequence of what was formerly demonstrated, that the same is the Ratio of the Fluxions of the Lines AB and AE. And thus it is evident, by this way of proceeding, as our Author observes, that the Analysis is carried on, and the Fluxions of flowing or variable Quantities are expounded by finite Quantities: and although, as he observes, the Analysis may be performed by means of Figures either finite or infinitely small, which are supposed, similar to the evanescent Figures; or yet by Figures, which, by the Method of Indivisibles, are accounted as infinitely small; yet since the Method of Indivisibles is reckoned not so agreeable to the Principles of strict Geometry; and our Ideas of infinite-

¹ Hence it appears that there is no Fallacy or Paralogism in this Method of Reasoning, made use of by the ingenious Inventor, for finding the Fluxion of x^n , as is alledged by the Author of the *Analyst*. See Art. 13, 14, 15, 16 of that Piece. For it is most evident that

$nx^{n-1} + \frac{n^2-n}{2}x^{n-2}o + \&c : 1$, expresses the Ratio of the synchronal Increments of x^n and x in every Case: but when these Increments are supposed to be diminished continually until they vanish, there must be some last Ratio, with which they vanish, otherwise they never could vanish: and since it is demonstrated that the ultimate Ratio with which the Augments vanish can neither be less nor greater than the Ratio of nx^{n-1} to 1, it follows that the ultimate Ratio of the evanescent Increments of x^n and x , and consequently the Ratio of their Fluxions is that of nx^{n-1} to 1. For a further Answer, see a Letter to the Author of the *Analyst* by *Philalethes Cantabrigiensis*, pag. 54.

ly little Quantities are less clear and distinct, therefore I think these infinitely little Quantities; or such as are esteemed so, are not so proper to be introduced into Geometry, when we can obtain our End without them. And Sir Isaac seems to be of the same Mind, when he subjoins at the End of this Art. Provided one proceed cautiously.

54. The whole Doctrine of Fluxions, as was observed already, is reduced to two Problems: *viz. From the Fluents to find the Fluxions: and conversely, From the Fluxions to find the Fluents.* The last is by far the most difficult Problem: in the Solution of which there are different Steps and Cases, according to the different Suppositions made in the State of the Problem; and the Number of flowing Quantities, and Fluxions, which enter into it. The first Step of the Solution is equivalent to the Quadrature of Curves: and is in effect the same with this. *An Equation being given including one flowing Quantity, and the Fluxion of another uniformly flowing Quantity, to find this other in Terms made up of the first mentioned flowing Quantity and known Quantities.* Where such Value can be found in finite Terms, the Curve is geometrically quadrable, *i. e.* its Area may be compared with a right-lined Figure: such as the Parabola. But where such Value cannot be had in finite Terms, the Curve cannot be squared geometrically: such are the Circle and Hyperbola. What our incomparable Author hath done in this Affair will appear from what follows.

Notes on the Quadrature of Curves.

S E C T. II.

Containing Notes on Articles 13 and 14.

55. **O**UR Author in these two Articles gives us a short Account of the Origin and Notation of Fluxions; and of their Gradation or different Orders. The first two Things stand in no need of any further Illustration: but in regard the Doctrine of second, third and superior Orders of Fluxions is represented as altogether chimerical by a late Author¹ already mentioned, without any sufficient Foundation, I judge it will be necessary, especially for the sake of such as have been less conversant about these abstract Ideas, for whom these Notes are principally designed, to explain and illustrate at some Length the different Orders of Fluxions, mentioned but slightly by our Author in this Place.

¹ See the Analyst, Art. 36-48.

56. If a variable or flowing Quantity flow uniformly, so as to acquire equal Increments in equal Times, then it is evident it's Fluxion or Celerity of flowing being constant and invariable, that Fluxion must be the same in every Place, and at every Time: wherefore it can have no Fluxion or Mutation: Fluxion being only applicable to that which is variable, or flows from one Value to another. So that in this Case there is no Fluxion of the Fluxion, or no second Fluxion of the flowing Quantity.

57. But if the flowing Quantity don't flow equably or uniformly; but be either continually accelerated or retarded, then it is evident that it's Fluxion, or Celerity of flowing, being different at different Times, is itself a variable, indeterminate or flowing Quantity; and therefore admits of a Fluxion: and this is called the Fluxion of the Fluxion, or the second Fluxion of the first flowing Quantity.

58. Again if this last or second Fluxion be constant and invariable, there is no third Fluxion: but if it be variable and inconstant, then that variable Fluxion or Celerity, may be itself considered as a flowing Quantity: and therefore does admit of a Fluxion, as well as any other variable Quantity, which is also called the second Fluxion of the first Fluxion; and the third Fluxion of the first flowing Quantity. And so we may proceed to superior Orders of Fluxions, and that without end. For the Fluxion of any flowing Quantity, being nothing else but it's Celerity of flowing: and Celerity being itself a Quantity, there is no Reason why, when it is variable, it should not be considered in the same Light with any other flowing Quantity, in this Respect: *i. e.* why it may not be considered as having a Fluxion, which expresses the swifter or slower Mutation, with which that Celerity flows or changes: and that notwithstanding of what has been said, by the Author of the Analyst, of the Absurdity of supposing a Celerity of a Celerity. But because it is so strenuously insisted upon, that the Conception of any Fluxion, particularly of a second, third or any subsequent Fluxion, is altogether impossible, I shall endeavour fully to explain this Affair, in what follows.

59. The Fluxion of any variable or flowing Quantity, being the Velocity or Swiftneſs with which it flows or changes, by Increase or Decrease, according as it is supposed either to be continually increasing or diminishing, the most natural and easy way of representing and explaining a Fluxion, will be by such Quantities as Space may be properly applied to, or predicated of: such are geometrical Quantities, otherwise called *Magnitudes*, *viz.* Lines, Surfaces and Solids, to which one or more Dimensions are applicable.

60. Now

60. Now as the Velocity with which any geometrical Quantity flows, by augmenting or diminishing, includes Time and Space in the Notion of it, whereby it may be measured and determined; it will be proper to observe, that, when the variable or flowing Magnitude is a Line, it is supposed to be generated or produced by the continual Motion or Flowing of a Point: and then, the Fluxion of the flowing Line, at any Time, is most naturally measured by the Length which the moving Point would run over in a certain given Time, supposing the Velocity, with which the Point moves, continued invariably the same, during the given Time, that it was at that Instant, or at that particular Place, when and where it's Fluxion is spoke of and enquired after. If the flowing Quantity be a Superficies, it is supposed to be generated or produced by the continual Motion of a Line: and the Fluxion of the flowing superficial Space at any Place, or Instant of Time, during the flowing, is naturally measured by the Quantity of superficial Space, that would be described in a certain given Time, by supposing the generating Line, continuing invariably the same, to be carried along another Line, with an uniform Motion, and thereby describe a Space, which increases equably, and just as fast through the whole given Time, as the flowing superficial Space increased at that Place or Instant of Time. Lastly, if the flowing Quantity be a Solid, it is supposed to be generated by the Motion of a plain Figure: and the Fluxion of the Solid at any Time or Place, is naturally measured by the Quantity of solid Space that would be described by the generating plain Figure continuing invariably the same, and drawn along a right Line, with an uniform Motion, so as to make the solid Space increase, through the whole given Time, just as fast as the flowing Quantity increases at that particular Place or Instant of Time: when it's Fluxion is sought.

61. This is the most easy and natural Way to conceive the Measure of the Fluxion of flowing Magnitudes: but notwithstanding of this, the Fluxion of any Magnitude may be represented or expounded, by a Magnitude of any kind, provided it be always proportional to that Fluxion: thus the Fluxions of Superficies's and Solids; as well as of Lines, may be expounded by Lines, which are the most simple kind of Magnitudes, provided these Lines, by which they are expounded and represented, be always proportional to these Fluxions: just as *Euclid* has represented all Magnitudes by right Lines, in the 5th Book of the Elements. Thus, as has been shewn already, when a plain Figure is generated by the uniform Motion of the Ordinate along the Absciss, the Fluxion of the plain Figure is always as the Ordinate: and therefore may be expounded thereby.

62. When

62. When Fluxion or Velocity of flowing is applied to other Quantities, it must be understood in an analogical Sense, it being applied to them, because of the Similitude there is betwixt the increasing and diminishing of any Quantity whatsoever, which is supposed to be continually passing from one State or Value to another; and the increasing and diminishing of flowing or variable Magnitudes.

Now the whole Business of the different Orders of Fluxions may be explained in the following manner.

63. Let ABC be a plain Figure generated or described, by the uniform or equable Motion of it's Ordinate BC, along the Base or Absciss AB, always at right Angles to each other: and putting the Absciss $AB = x$, let the Nature of the Figure be such, that it's Area $ABC = ax^n$; where a is any given or invariable Quantity; and n any indefinite Index to x : and let $AG = 1$. Now suppose the Fluxion of the Absciss AB, *i. e.* the Velocity with which the Point B moves, which is equable and uniform, to be such that in a given Time, suppose a Second of Time, it moves through a Space equal to $AG = 1$. Draw AH perpendicular to AG and equal to it: and let the Rectangle HB be described, at the same Time with ABC, by it's invariable Ordinate equal to AH or Unity. Then 'tis evident, that in the same Time, the Point B moves through a linear Space equal to AG or 1, the Ordinate of the Rectangle BH will move through or describe a superficial Space equal to $AG \times AH$ or the Square of 1: that is, it will do it in a Second of Time. But, if the Velocity with which the Area ABC increases, were continued invariably the same, for a second of Time, or which is the same *, if the Ordinate BC continued invariably the same, moving along AB, the superficial Space described by BC, in a Second of Time, would be $BC \times AG = BC \times 1$. Therefore $AG \times AH$ or a superficial Unite, being the Measure of the Fluxion of the Rectangle $HB = AB \times 1$; $BC \times AG = BC \times 1$ will be the Measure of the Fluxion of the flowing Area ABC. But since $ABC = ax^n$, it appears from what was formerly demonstrated *, that the Ordinate $BC = nax^{n-1}$, therefore the superficial Space, which is the Measure of the Fluxion of ABC, may likewise be expressed by $nax^{n-1} \times 1$ or nax^{n-1} .

* Art. 31.
of this Ex-
plication.

* Art. 47.
of this Ex-
plication.

* Art. 46.
of this Ex-
plication.

64. Now the Expression nax^{n-1} , when $AB = x$ flows, denotes a variable or flowing Quantity, unless when $n = 1$: and a Quantity flowing by Increase, when $n - 1$ is positive *, and x increases: therefore the Fluxion of ABC is itself a variable or flowing Quantity, *i. e.* the Celerity with which ABC flows, is continually flowing or varying as to it's Value, which, in this Case, grows still greater and greater. Suppose

denotes the Celerity of the Mutation of the second Fluxion. This will happen, when ABC is a right angled Triangle, having it's Base equal to $2ax$: and then ABD is a Rectangle, having the Absciss AB for it's variable Altitude, and the invariable Line denoted by $2a$ for it's Base. But if n be not equal to 2, the Expression $n^2 - n \times ax^{n-2}$ is variable: and if $n > 2$, it is always increasing, as x increases*. In which Case, suppose ABE another plain Figure, having it's Area equal to $BD \times AG$ or $BD \times 1 = n^2 - n \times ax^{n-2} \times 1$ or $n^2 - n \times ax^{n-2}$: and let it be described by the Motion of it's Ordinate BE along the common Absciss AB, at the same Time ABC and ABD are described by their Ordinates BC and BD. Then it appears, by the same way of reasoning as before, that the Area ABE is always proportional to, or the Exponent of the second Fluxion of ABC: and further, that the Velocity with which the flowing Area ABE increases is measured by $BE \times AG = BE \times 1$: Therefore the Velocity with which ABE increases or it's Fluxion will be always proportional to, or the Exponent of the second Fluxion of ABD; and of the third Fluxion of ABC; or, which is the same thing $BE \times AG = BE \times 1$ is always proportional to and consequently the Exponent of the third Fluxion of ABC. Moreover since $ABE = n^2 - n \times ax^{n-2}$, the Ordinate $BE = \frac{n^2 - n \times ax^{n-2}}{ax^{n-3}} = n^2 - 3n^2 + 2n \times ax^{n-3}$, therefore $n^2 - 3n^2 + 2n \times ax^{n-3}$ is likewise the Exponent of the first Fluxion of ABE; of the second Fluxion of ABD; and of the third Fluxion of the first flowing Area ABC. Now if, in the Expression ax^n , which is the Value of the Area ABC, we suppose $a = \frac{8}{105}$ and $n = \frac{7}{2}$, we shall have $BE = (n^2 - 3n^2 + 2n \times ax^{n-3}) = x^{\frac{1}{2}}$: which shews that ABE is the Apollonian, or common Parabola, having $AG = 1$ for the Parameter of the Axis.

* Art. 51.
of this Ex-
plication.

* Art. 52.
of this Ex-
plication.

Again if ABF α be a plain Figure, described by the uniform Motion of it's Ordinate BF along AB, together with the other plain Figures; such, that it's Area ABF $\alpha = BE \times AG = BE \times 1 = n^2 - 3n^2 + 2n \times ax^{n-3} \times 1$, then the Area ABF α , will also be proportional to, or the Exponent of the first Fluxion of ABE; of the second Fluxion of ABD; and of the third Fluxion of the first flowing Area ABC.

66. And thus we might proceed to subsequent Orders of Fluxions; which will run on infinitely, if the indefinite Index of x , viz. n being positive, be not an Integer; or whatever it be, if negative: because in every such Case, the Exponent of x in the Expression of any of the flowing Areas, in the Series of plain Figures, can never become nothing: and therefore the Exponents of their Fluxions will always be variable: and consequently no Fluxion of any Order, in that Case, can ever

ever be a constant Quantity. Only it may be observed, that when the Exponent of the Power of x , in the Expression of the Area of any Curve, in the Progression of Curvilinear Areas, becomes negative, the Fluxion of that Area must likewise be negative*: for that Area diminishes as x increases; and therefore is posited below the Ordinate; consequently it is so far from having any Velocity of Increase, that upon the contrary, it decreases or diminishes, and so the Velocity with which it diminishes, is call'd a negative Fluxion. Thus, if we suppose $a = \frac{8}{10}$, and $n = \frac{7}{2}$, as formerly, and consequently ABE a Parabola, then the Curve αF will be of the hyperbolical Kind, approaching the Asymptotes AB and AH continually: and it's Area $ABF\alpha$ being equal to $BE \times 1 = x^{\frac{1}{2}} \times 1 = x^{\frac{1}{2}}$, it's Ordinate $BF = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{x}}$ *, and therefore it's Fluxion will be expounded by a Curvilinear Area lying below the Ordinate, for according to the Order of the Progression, that Area must be $\frac{1}{2\sqrt{x}} \times 1$ or $\frac{1}{2}x^{-\frac{1}{2}}$, the Fluxion of which must be negative.

* Art. 51.
of this Ex-
plication.

* Art. 52.
of this Ex-
plication.

67. The Progression of Curvilinear Areas, as Exponents of the various Orders of Fluxions and Fluents, may be supposed to be carried on likewise the contrary way. Thus if $AB\Delta$ be a Curvilinear Figure, having it's Ordinate $B\Delta$ equal to the Area ABC applied to AG or Unity: and if there were another Curve, whose Ordinate is equal to $AB\Delta$ applied to AG, and so on; that would constitute a Series of Curves running the contrary way, such, that every Curvilinear Area in this Series will be a Fluent, of which the immediately preceding is the Exponent of it's Fluxion; that two Steps before it, is the Exponent of it's second Fluxion, &c. And thus the Progression of Curves, reckoning from ABC, may be supposed to be carried on both ways *in inf.*

68. And thus, I have endeavoured to explain and illustrate the whole Affair of the different Orders of Fluxions, so as to shew and demonstrate that they have a real Foundation in Nature; and may be distinctly conceived.

69. Since the Curvilinear Areas ABD, ABE, $ABF\alpha$, &c. are equal to the Ordinates BC, BD, BE, &c. multiplied by AG or 1 respectively: therefore the Ordinates BC, BD, BE, &c. will also be the Exponents of the first, second, third, &c. Fluxions of the flowing Area ABC.

70. But it may be proper here to caution the Beginner, that he don't understand me, as if I said, that the Areas ABD, ABE, $ABF\alpha$, &c. or the Ordinates BC, BD, BE, &c. were really the first, second, third, &c. Fluxions of the flowing Area ABC. They are only Exponents of these Fluxions, as being proportional to them: for a Fluxion,

which denotes the Celerity of Flowing or Mutation, is one thing; and a Surface, or a Line is another: although a Surface or a Line may be proportional to, and therefore the Exponent of a Celerity. So that the true Meaning of the thing is this: Suppose BCFDE, and *bfcde*, to be two different Positions of the Ordinates of the Curves already mentioned, generated and related to one another as above, the first Fluxion of ABC is to the first Fluxion of *Abc* as ABD to *Abd*; or as BC to *bc*: the second, third, &c. Fluxions of ABC are to the second, third, &c. Fluxions of *Abc* respectively, as ABE, ABFa, &c. to *Abe*, *Abfa*, &c. or as BD, BE, &c. to *bd*, *be*, &c. respectively.

71. From what has been said, the Truth of what our Author advances in Art. 14. will easily appear, viz. If $\dot{z}$, $\dot{z}$, $\dot{z}$, $\dot{z}$, $\dot{z}$, $\dot{z}$, &c. $\sqrt{az - zz}$, $\sqrt{az - zz}$, $\sqrt{az - zz}$, $\sqrt{az - zz}$, $\sqrt{az - zz}$, &c. $\frac{az + zz}{a - z}$, $\frac{az + zz}{a - z}$, $\frac{az + zz}{a - z}$, $\frac{az + zz}{a - z}$, $\frac{az + zz}{a - z}$, &c. be Series's of Quantities, so related to each other, that every subsequent Quantity in any of these Series's, is the Fluxion of the preceding Quantity; and conversely every preceding Quantity being considered as Fluent have the succeeding one it's Fluxion: I say if this be the Relation of the Quantities to each other, then every preceding Quantity is as the Area of a Curvilinear Figure, whereof the immediately following is the rectangular Ordinate and z the Absciss.

* Art. 29,
31, 63.
of this Ex-
plication.

For it has been demonstrated * that the Fluxion of any plain Figure, as ABC (see the preceding Fig.) is such, that when it is compared with the Fluxion of AB the Absciss; or rather with the Fluxion of the Rectangle $AB \times 1$, there arises this Analogy $F.AB \times 1 : F.ABC :: 1 : BC$. Wherefore if the Fluxion of $AB \times 1$ or AB be denoted by 1 (which is supposed) then the Fluxion of ABC will be denoted by BC, i. e. the rectangular Ordinate. Consequently if any Quantity such as $\sqrt{az - zz}$ express the Area of a Curve, whose Absciss is z : and $\dot{z} = 1$, as is supposed, then $F.\sqrt{az - zz}$ i. e. $\sqrt{az - zz}$ = the rectangular Ordinate, as our Author affirms: and so of others.

S E C T. III.

Containing Notes on Art. 15 — 24.

Explication of Prop. I.

72. **A**NY Difficulty there is in our Author's Demonstration of this Proposition, arises from assuming ox , oy and oz for the synchro-
nical Increments of the flowing Quantities x , y and z : and neglect-
ing these Terms, which involve 0 at the Result of the Operation.

73. If the synchronal Increments of x , y and z , be supposed to have already acquired any finite Magnitude, it must be acknowledged that ox , oy and oz cannot express such Increments: because ox , oy and oz are proportional to $\dot{x}$, $\dot{y}$ and $\dot{z}$, viz. to the Fluxions of x , y and z ; or the Velocities with which they flow or increase, just as the Increments are beginning to arise: but finite Increments are not always proportional to the Fluxions of the flowing Quantities, since the Velocities, with which these finite Increments are generated, may vary during the Time they are generated: therefore ox , oy , oz cannot properly express finite Increments. But it was shewn formerly *, that the less the synchronal Increments of flowing Quantities are; or the less the Time be, in which they are produced, the nearer will their Ratio approach to the Ratio of the Fluxions: so as that, by diminishing these Increments; or the Time in which they are generated, continually more and more, the Ratio of the Increments, shall approach nearer to the Ratio of the Fluxions, than by any given Difference; and consequently their ultimate Ratio be the same with the Ratio of the Fluxions. Therefore by ox , oy , oz we must understand synchronal Increments in this Sense, viz. such as are indefinitely small, called by our Author *Moments* or *synchronal momentaneous Increments*. And surely, if there can be no Increments how small soever assigned, but there are Increments yet less and less, assign as small ones as you please; he may be allowed to assume a Notation denoting nascent or evanescent Increments, which are indefinitely small, although we have no positive adequate Idea of such Increments, or Moments. For a negative Idea is sufficient, in many Cases, for determining the Properties and Proportions of Quantities. What other than a negative Idea have I of the infinite Decimal .6666, &c. Yet I can demonstrate that it is exactly equal to $\frac{2}{3}$. What other than a negative and very imperfect Idea has any one of $\sqrt{2}$? Yet it can be shewn that it is a mean Proportional betwixt 1 and 2. And although all Surds are negative Ideas; and involve some kind of Infinity in them, yet they admit of arithmetical Operations: and can be compared with one another. Thus though none can tell how much $\sqrt{2}$ and $\sqrt{8}$ are separately: yet we know that their Product is 4; and that the latter is double the former. After the same manner, though the momentaneous Increments of flowing Quantities cannot be distinctly and adequately conceived by the Mind, yet we may make use of some kind of Notation, by way of *Symbol* or *Representation* of them: and their Relations may be determined and express'd by finite Quantities, which are distinctly conceived: and this is all this Proposition requires or proposes.

* Art. 27.
of this Ex-
plication.

74. The

74. The other thing I mentioned, as having any Difficulty, in the Demonstration given by Sir Isaac of this 1st Prop. is the neglecting the Terms of the Equation, at the Result of the Operation, wherein the Quantity 0 is found. The Reason of it is assigned briefly by himself: viz. because the Quantity 0 is infinitely diminished and vanishes. For the Equation, expressing the Relation of the synchronal Augments, will only then express the Relation of the Fluxions, when the Quantity 0 is just vanishing: but when that Quantity vanishes, all the Terms wherein it is found must vanish likewise; which is the Reason of their being neglected and thrown out.

75. But in regard this Proposition is a fundamental one; and contains the Substance of the direct Method of Fluxions, I shall shew how it may be demonstrated at fuller Length. In order to which, it will be necessary to shew how the Fluxion of any simple Quantity, which contains the Product of any Number of flowing Quantities may be investigated.

76. Case 1. When the flowing Quantity is the Rectangle or Product of two flowing Quantities, as xy ; suppose $\dot{x}$ and $\dot{y}$ to denote the Fluxions of the Factors x and y , I say $F.xy = \dot{x}y + y\dot{x}$.

Put $z = xy$; and let x, y, z denote any synchronal Increments, great or small of the flowing Quantities x, y, z : so that in the same Time that x becomes $x + \dot{x}$, y and z may become $y + \dot{y}$ and $z + \dot{z}$ respectively. Then because the Equation $xy = z$ exhibits the Relation of the flowing Quantities x, y and z indifferently, or at all times, into whatever new Values they pass, therefore $z + \dot{z} = x + \dot{x} \times y + \dot{y} = xy + \dot{x}y + xy + \dot{y}x$: take away z from the one Side of the Equation and xy equal to it, from the other; and we have $\dot{z} = \dot{x}y + y\dot{x}$ an Equation defining the Relation of the Increments generated in the same Time, great or small: and by dividing by $\dot{x}$, it is $\frac{\dot{z}}{\dot{x}} = y + \frac{\dot{y}}{\dot{x}}x$ + y . Now suppose the Increments x, y and z to be diminished infinitely, so as to be just vanishing, and in their ultimate Ratio: in that Case the Ratio of the evanescent Increments, is the same with the

* Art. 28. Ratio of the Fluxions *, i. e. $\frac{\dot{y}}{\dot{x}} = \frac{y}{x}$ and $\frac{\dot{z}}{\dot{x}} = \frac{z}{x}$: which Values being inserted the Equation becomes $\frac{\dot{z}}{\dot{x}} = y + \frac{y}{x}x + y$; where the Term y , being infinitely diminished, vanishes and goes out: consequently we have $\frac{\dot{z}}{\dot{x}} = y + \frac{y}{x}x$, for the Relation of the Fluxions, i. e. $\dot{z} = \dot{x}y + xy$.

* Art. 40. of this Explication. But since $z = xy$, $\dot{z} = F.xy$ *, therefore $F.xy = \dot{x}y + xy$.

77. Cor. 1. $F.ax = ax$. For suppose $ax = z$: then by what has been said $a \times \dot{x} + x$ or $a\dot{x} + ax = z + z$: subtract $ax = z$, there remains $a\dot{x} = z$ or $a = \frac{z}{\dot{x}}$: therefore the Increments of z and x are in a given Ratio: consequently their Fluxions are in the same given Ratio *, * Art. 28. of this Explication.
i.e. $a = \frac{z}{\dot{x}}$, whence $a\dot{x} = z$: therefore $F.ax = az$. The same thing easily appears from Art. 40 *.

78. Cor. 2. $F.xx = 2xx$: for it is $xx + x\dot{x} = 2xx$.

79. Case 2. When the Fluent is the Product of three flowing Quantities as xyz , I say $F.xyz = x\dot{y}z + xy\dot{z} + x\dot{y}z$. For put $xy = u$, then $xyz = uz$, therefore $F.xyz = F.uz = \dot{u}z + u\dot{z}$ *: but since $xy = u$, $\dot{x}y + x\dot{y} = \dot{u}$. Wherefore if we insert $\dot{x}y + x\dot{y}$ for $\dot{u}$ and restore xy for u , in the Equation $F.xyz = \dot{u}z + u\dot{z}$, we have $F.xyz = x\dot{y}z + xy\dot{z} + x\dot{y}z$. * Art. 40. of this Explication. * Art. 76. of this Explication.

80. Cor. $F.axz = a\dot{x}z + ax\dot{z}$. $F.ax^2 = 2ax\dot{x}$. For the flowing Quantities xz , axz ; and x^2 , ax^2 , are in a given or constant Ratio, viz. that of 1 to a : therefore their Fluxions are in the same given Ratio *: but $F.xz = \dot{x}z + x\dot{z}$, and $F.x^2 = 2xx$, therefore $F.axz = a\dot{x}z + ax\dot{z}$ and $F.ax^2 = 2ax\dot{x}$. Likewise $F.x^3 = 3xx\dot{x}$: for it is $xxx + x\dot{x}x + x\dot{x}x = 3xx\dot{x}$. * Art. 40. of this Explication.

81. Case 3. When the Fluent is the Product of any Number of flowing Quantities, the general Rule for finding the Fluxion in every Case is this. Multiply the Fluent by the Index of the Power of every flowing Quantity it involves; and in every Multiplication, change the Root of the Power into it's Fluxion: and the Aggregate of all these Products is the Fluxion required.

For when the Fluent consists of four Dimensions, and as many different Letters, as $xyzv$, it's Fluxion is $x\dot{y}zv + xy\dot{z}v + xyz\dot{v} + xyz\dot{v}$: which is demonstrated by reducing it to three, viz. by putting $t = xy$, i.e. making t a 4th Proportional to 1, x and y : and then $xyzv = tvz$, therefore $F.xyzv = F.tvz = \dot{t}zv + t \times \dot{v}z + tv\dot{z}$ *. But since $t = xy$, $\dot{t} = \dot{x}y + x\dot{y}$ *, therefore by Restitution, $F.xyzv = \dot{x}y + x\dot{y} \times zv + xy \times \dot{v}z + tv\dot{z}$. * Art. 79. of this Explication.

$\times zv + zv = x\dot{y}zv + xy\dot{z}v + xyz\dot{v} + xyz\dot{v}$. And thus by means of a Quantity of four Dimensions, you may discover the Fluxion of a Fluent of five Dimensions, and so on: in such manner, that he who rightly considers the Process, will see this Truth demonstrated, viz. That the Fluxion of any Fluent, which is itself a simple Quantity, i.e. a Quantity of one Term, be it's Dimensions what they will, is found by setting down the Quantity as often as there are flowing prime Divisors in

in it, only changing each prime Divisor into it's Fluxion, in order, one after another, and taking the Sum of all, after the Manner of the preceding Examples. Which is coincident with the general Rule prescribed under this 3^d Case: or at least manifestly included in it.

However that the general Rule may be more clearly demonstrated, take the Quantity $x^ny^mz^p$ with indefinite Exponents: and I say $F.x^ny^mz^p = nx^{n-1}y^mz^p + mx^n y^{m-1}z^p + px^n y^m z^{p-1}$. For let it be $1:x^n:r :: 1:y^m:s :: 1:z^p:t ::$, or $x^n=r$, $y^m=s$ and $z^p=t$: then $F.x^ny^mz^p = F.rst = \dot{r}st + r\dot{s}t + rst\dot{}$: but because $x^n=r$; $y^m=s$ and $z^p=t$, therefore $F.x^n = nx^{n-1}\dot{x} = \dot{r}$: $F.y^m = my^{m-1}\dot{y} = \dot{s}$: and $F.z^p = pz^{p-1}\dot{z} = \dot{t}$. Substitute for $\dot{r}$, $\dot{s}$ and $\dot{t}$ these their Values, and replace x^n , y^m and z^p , for r , s and t in the Equation $F.x^ny^mz^p = \dot{r}st + r\dot{s}t + rst\dot{}$, and we have $F.x^ny^mz^p = nx^{n-1}\dot{x}y^mz^p + mx^n y^{m-1}\dot{y}z^p + px^n y^m z^{p-1}\dot{z}$, according to the general Rule.

* Art. 40.
of this Ex-
plication.

If the Fluent have any constant or invariable Quantity among it's Factors, as $a^2xy^3z^4$ the general Rule still holds. For $F.a^2xy^3z^4 = a^2\dot{x}y^3z^4 + 3a^2xy^2\dot{y}z^4 + 4a^2xy^3\dot{z}z^3$, as appears from what has been said already, since xy^3z^4 and $a^2xy^3z^4$ are in a given Ratio, and therefore their Fluxions in that same given Ratio*: But that likewise is the Fluxion of $a^2xy^3z^4$ according to the general Rule above, since $\dot{a} = 0$, and therefore the Term which involves $\dot{a}$ equal to nothing likewise, being $2\dot{a}axy^3z^4$.

82. And now these things being understood, the Solution of our Author's first Prop. will be very plain and easy, viz. thus: An Equation being given involving any number of flowing Quantities, transpose all the negative Terms to the opposite Side, that all may become positive, then find the Fluxions of both Sides, by taking the Aggregate of the Fluxions of all the Terms, found by the general Rule above: put the Fluxions of both Sides equal to each other: and so you shall have an Equation exhibiting the Relation of the Fluxions. Thus if the first Equation mentioned in this Prop. were given, viz. $x^3 - xy^2 + a^2z - b^3 = 0$, make it by Transposition stand thus $x^3 + a^2z = xy^2 + b^3$. Then take the Fluxions of both Sides, which, being the Sum of the Fluxions of all the Terms, I do, by taking the Fluxion of each Term, and adding them together: whereby we will have $3\dot{x}x^2 + a^2\dot{z} = \dot{x}y^2 + 2xy\dot{y}$; or by transposing all to one Side, $3\dot{x}x^2 - \dot{x}y^2 - 2xy\dot{y} + a^2\dot{z} = 0$, for the Fluxion of b^3 is nothing. Which is the same with that discovered by our Author: and defines the Relation of the Fluxions, in as far as that Relation can be determined from one Equation only including the flowing Quantities.

The

The Demonstration of this Solution is founded upon Art. 40 *. Where it was shewn that the Fluxions of equal Quantities are equal. * Of this Explication.

83. It is evident that the Solution will be the same, if all the Quantities being brought to one Side, and made equal to nothing, the Fluxion of the whole be taken and put equal to nothing, it being understood that every Part of the Fluxion of a negative Term, must have an opposite Sign to what it would have, if the Term were positive. And so the Solution will be the same with our Author's.

84. But because fractional and surd Quantities may often occur in the fluential Equation proposed, as in our Author's 2^d Example *, it will be proper to shew more particularly how the Fluxions of such Quantities are to be found. And first with Respect to fractional Quantities. Let $\frac{x}{y}$ represent any fractional Quantity, having both Num^r. and Den^r. flowing Quantities. Put $y:x::1:v$; or $\frac{x}{y}=v$ or yet $x=vy$: then $\dot{x}=\dot{v}y+v\dot{y}$ *: and by Transposition and Division $\frac{\dot{x}-v\dot{y}}{y}=\dot{v}$. Restore $\frac{x}{y}$ for v , and it is $\frac{\dot{x}-\frac{x}{y}\dot{y}}{y}=\frac{\dot{x}y-x\dot{y}}{yy}=\dot{v}$, i.e. (be- * Art. 18. of the Quadrature of Curves.

cause $\dot{v}=\frac{\dot{x}}{y}-\frac{x\dot{y}}{y^2}$) $\frac{\dot{x}}{y}-\frac{x\dot{y}}{y^2}=\frac{\dot{x}y-x\dot{y}}{y^2}$. If either Numerator or Denominator was a constant Quantity, it is only putting $\dot{x}$ or $\dot{y}$ equal to nothing, and the Fluxion will be investigated the same way. By which way of reasoning, we deduce this general Rule for finding the Fluxion of a Fraction, viz. Multiply the Fluxion of the Numerator into the Denominator; subtract from that Product the Fluxion of the Denominator multiplied into the Numerator: and divide the Difference by the Square of the Denominator. Thus $\frac{a^3}{x^3}=\frac{0-3a^2\dot{x}x^2}{x^6}=-\frac{3a^2\dot{x}x^2}{x^6}$ or $-\frac{3a^2\dot{x}}{x^4}$ or $-3a^2\dot{x}x^{-4}$. Which Expressions of the Fluxion are all the same in effect: the first being the Notation arising from the Rule for finding the Fluxion of a Fraction; the last Notation arises immediately from the Application of the general Rule above *: for $\frac{a^2}{x^3}=a^2x^{-3}$. Likewise * Art. 40. of this Explication.

$\frac{ax+x^2}{b-x}=\frac{ax+2xx\dot{x}\times b-x-\dot{a}x+x^2\times-\dot{x}}{b^2-2bx+xx}=\frac{abx+2b\dot{x}x-x\dot{x}^2}{b^2-2bx+x^2}$. By the same way of reasoning $\frac{1}{x^n}=-\frac{nx^{n-1}\dot{x}}{x^{2n}}=-\frac{n}{x^{n+1}}\dot{x}$ which is a further Demonstration of Art. 51 *. * Art. 81. of this Explication.

By a fluential Equation is meant an Equation containing flowing Quantities, whereby their Relation is determined: the Equation thence deduced according to this first Prop.; or even any other proposed, which contains Fluxions, and thereby determines the Relation of the Fluxions, is called a fluxional Equation.

L 85. Now

* Of this Explication.

85. Now with respect to the Fluxions of Surds: let $\sqrt[n]{ax - x^2}$ or $(ax - x^2)^{\frac{1}{n}}$ be a Surd with an indefinite Index. In order to find it's Fluxion, suppose $ax - x^2 = v^n$ or $ax - x^2 = v^n$. Take the Fluxions of both, and we have $a\dot{x} - 2x\dot{x} = nv^{n-1}\dot{v}$: or, by dividing both by nv^{n-1} , $\frac{a\dot{x} - 2x\dot{x}}{nv^{n-1}} = \dot{v}$. Whence, by inserting the Value of v in place of it, we have $\dot{v} = \frac{a\dot{x} - 2x\dot{x}}{n \times (ax - x^2)^{1 - \frac{1}{n}}}$ or $\frac{1}{n} \times (ax - x^2)^{\frac{1}{n} - 1} \times a\dot{x} - 2x\dot{x}$. Which is in effect the same with what it ought to be, according to the Rule for finding the Fluxion of any Power of a flowing Quantity, Art. 49 and 50*: or by the general Rule at Art. 81*. Likewise F. $x^{\frac{n}{m}} = \frac{n}{m} x^{\frac{n}{m} - 1} \dot{x}$. For $x^{\frac{n}{m}} = (x^{\frac{1}{m}})^n$, whose Fluxion, by what has been now said, is $\frac{1}{m} x^{\frac{1}{m} - 1} \dot{x} \times nx^{\frac{n}{m} - 1} \dot{x} = \frac{n}{m} x^{\frac{n}{m} - 1} \dot{x}$. And F. $x^{-\frac{n}{m}} = -\frac{n}{m} x^{-\frac{n}{m} - 1} \dot{x}$. For $x^{-\frac{n}{m}} = \frac{1}{x^{\frac{n}{m}}}$, whose Fluxion, by the last Art. and by

* Of this
Explica-
tion.
* Of this
Explica-
cation.

what has been just now shewn, is $\frac{-\frac{n}{m} x^{\frac{n}{m} - 1} \dot{x}}{x^{\frac{n}{m}}} = -\frac{n}{m} x^{-\frac{n}{m} - 1} \dot{x}$.

Wherefore universally the Rule for finding the Fluxion of any Power of a flowing Quantity, be it's Exponent positive or negative; integral or fractional, and whether the Quantity affected by the Exponent be simple or compound, is this, multiply the flowing Quantity by the Exponent of the Power, diminish the Exponent by a Unity, and multiply further by the Fluxion of the Quantity or Root: agreeably to Art. 50 and 51*.

* Of this
Explica-
tion.

86. Suppose now the Equation $x^3 - xy^2 + aa\sqrt{ax - y^2} - b^3 = 0$, our Author's 2^d Ex. was proposed, to find the Fluxions: By taking the Fluxions of all the Terms, according to the preceding Rules, it gives $3x\dot{x}^2 - \dot{x}y^2 - 2xy\dot{y} + \frac{1}{2}aa \times (ax - y^2)^{-\frac{1}{2}} \times a\dot{x} - 2y\dot{y} = 0$, or, which is the same thing, $3x\dot{x}^2 - \dot{x}y^2 - 2xy\dot{y} + \frac{a^2\dot{x} - 2a^2y\dot{y}}{2\sqrt{ax - y^2}} = 0$. Which exhibits the Relation of $\dot{x}$ and $\dot{y}$. And so of others.

87. Schol. When Fractions and Radicals, involving the flowing Quantities, enter the fluential Equation, you may investigate the Equation exhibiting the Relation of the Fluxions, by a Substitution; thus, let the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay + x^2} = 0$ express the Relation

tion of x and y : put $\frac{by^3}{a+y} = z$ and $xx\sqrt{ay+x^2} = v$: and so you will have three Equations, viz. $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ax^4y + x^5 - v^2 = 0$: the first of which gives $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$, the second gives $a\dot{z} + \dot{z}y + zy - 3b\dot{y}y^2 = 0$, and the third, $4a\dot{x}x^3y + ax^4\dot{y} + 5x^4 - 2v\dot{v} = 0$, for the Relations of $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$. Now by Reduction of Equations from the two last, you may find $\dot{z} = \frac{3b\dot{y}y^2 - zy}{a+y}$ and $\dot{v} = \frac{4a\dot{x}x^3y + ax^4\dot{y} + 5x^4}{2v}$: which Values of $\dot{z}$ and $\dot{v}$ being substituted in place of them, in the first of the three fluxionary Equations, there arises $3\dot{x}x^2 - 2a\dot{y}y + \frac{3b\dot{y}y^2 - zy}{a+y} - \frac{4a\dot{x}x^3y + ax^4\dot{y} + 5x^4}{2v} = 0$, i.e. by restoring the Values of z and v , $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^3 + 2b\dot{y}y^2}{a^2 + 2ay + yy} - \frac{4a\dot{x}xy + ax^2\dot{y} + 6\dot{x}x^3}{2\sqrt{ay + xx}}$: which expresses the Relation of $\dot{x}$ and $\dot{y}$. And after the same manner you may proceed, when there are other more complicate Terms, in the fluential Equation proposed.

88. The Fluxions we have been speaking of hitherto, under this first Prop. are first Fluxions. But our Author proceeds at Art. 19. * to * Of the shew how second, third and following Fluxions may be investigated. Quadra- And he tells us, it is only by repeating the same Operation, by which Curves. the first Fluxions are found.

Thus if we take the fluential Equation $zy^3 - z^4 + a^4 = 0$ of Art. 19. *: * Of the from it we deduce this fluxional Equation $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$, Quadra- for the Relation of the first Fluxions $\dot{z}$ and $\dot{y}$. Which Equation may Curves. be reduced to an Analogy $\frac{\dot{z}}{\dot{y}} = \frac{3zy^2}{4z^3 - y^3}$ or $\dot{z} : \dot{y} :: 3zy^2 : 4z^3 - y^3$. Again from the Equation $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$, by considering $\dot{z}$ and $\dot{y}$ as flowing Quantities as well as z and y , we deduce this other Equation $\ddot{z}y^3 + 3\dot{z}\dot{y}y^2 + 3\dot{z}y\dot{y}^2 + 3z\ddot{y}y^2 + 6z\dot{y}\dot{y}^2 - 4\ddot{z}z^3 - 12\dot{z}\dot{z}z^2 = 0$, or $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3\dot{z}y\dot{y}^2 + 6z\dot{y}\dot{y}^2 - 4\ddot{z}z^3 - 12\dot{z}\dot{z}z^2 = 0$, containing the second Fluxions of z and y . After the same manner, by considering $\ddot{z}$ and $\ddot{y}$, as well as $\dot{z}$, $\dot{y}$, z and y , as flowing Quantities; and applying the general Rule contained in this Prop. we shall have $\ddot{z}y^3 + 3\dot{z}\dot{y}y^2 + 6\dot{z}\dot{y}y^2 + 6\dot{z}y\dot{y}^2 + 12\dot{z}\dot{y}\dot{y}^2 + 3\dot{z}\dot{y}y^2 + 3z\dot{y}\dot{y}^2 + 6z\dot{y}\dot{y}^2 + 6\dot{z}\dot{y}^2y + 12z\dot{y}\dot{y}^2 + 6z\dot{y}^2\dot{y} - 4\ddot{z}z^3 - 12\dot{z}\dot{z}z^2 - 24\dot{z}\dot{z}z^2 - 24\dot{z}^2z = 0$, or, by adding similar Terms $\ddot{z}y^3 + 9\dot{z}\dot{y}y^2 + 9\dot{z}y\dot{y}^2 + 18\dot{z}\dot{y}^2y + 3z\dot{y}\dot{y}^2 + 18z\dot{y}\dot{y}^2 + 6z\dot{y}^3 - 4\ddot{z}z^3 - 36\dot{z}\dot{z}z^2 - 24\dot{z}^2z = 0$, an Equation containing the third Fluxions of the flowing Quantities x and y . And so we might proceed to Equations containing the subsequent Orders of Fluxions. The Reason of which Operations is plainly contained in

what has been said already, with respect to the finding an Equation containing the Relation of the first Fluxions from a fluent Equation proposed. For although the fluent Equation contain three or more flowing Quantities, a fluxional Equation, may notwithstanding, be deduced from it, exhibiting the Relation of the Fluxions: by proceeding according to the general Rule.

89. If we resume the fluent Equation at the Beginning of the last Art. viz. $zy^3 - z^4 + a^4 = 0$, and the Relation of z and y thence derived, viz. $\frac{z}{y} = \frac{3zy^2}{4z^3 - y^3}$, the first expresses the Relation of the flowing Quantities x and y at all Times, or in every State; and the other, the Relation of their Fluxions, at all Times, in Terms made up of z and y . Wherefore if we assume a particular determinate Value for z or y , the corresponding Value of the other may be found, from the first Equation and thereby the Ratio of $\dot{z}$ and $\dot{y}$ or $\frac{\dot{z}}{\dot{y}}$ will be wholly known. Thus suppose that at any Time $z = 2a$: then by substituting $2a$ for z , in the Equation $zy^3 - z^4 + a^4 = 0$, we have $2ay^3 - 15a^4 = 0$, whence $y^3 = \frac{15}{2}a^3$ or $y = a\sqrt[3]{\frac{15}{2}}$. Substitute these Values of z and y in the Expression of the Ratio of $\dot{z}$ and $\dot{y}$, and it becomes $\frac{\dot{z}}{\dot{y}} =$

$$\frac{6a^3\sqrt[3]{\frac{225}{2}}}{32a^3 - \frac{15a^3}{2}} = \frac{12\sqrt[3]{\frac{225}{2}}}{49} = \frac{\sqrt[3]{97200}}{\sqrt[3]{117649}} \text{ wholly a known Quantity. Which shews}$$

that, at what Time or Place, the variable Quantity z becomes equal to the known determinate Quantity $2a$, the Velocity with which z flows, at that Time, is to the Velocity with which y flows at the same Time, in the subtriplicate Ratio of 97200 to 117649; or as the Cube Roots of these Numbers. And if z be taken of any other Value, another known Relation of $\dot{z}$ and $\dot{y}$ will arise.

90. Hence it appears, that when a fluent Equation is proposed, containing two variable or flowing Quantities only, the Relation of the first Fluxions of these two flowing Quantities, may always be had in Terms containing the two variable Quantities and known Quantities only: and therefore by assuming one of the flowing Quantities at pleasure, the fluent Equation, by the Rules of Algebra, will give a corresponding known Value of the other flowing Quantity: wherefore the Relation which the Fluxions of these two Quantities, bear to one another at that Time, will be fully known, and determined, by substituting the particular determined Values of the flowing Quantities, in place of them, in the fluxionary Equation.

91. Take

91. Take another Example. Let the fluential Equation $px - y^2 = 0$ be proposed: which belongs to the common Parabola: x being the Absciss; y , the Ordinate; and p , the Parameter of the Diameter. The fluxionary Equation thence arising, is $p\dot{x} - 2y\dot{y} = 0$, whence $\dot{x} : \dot{y} :: 2y : p$. *i. e.* the variable Ratio of $\dot{x}$ and $\dot{y}$, will be at all Times, and in every Place as twice the Ordinate in that Place to the Parameter. Now suppose I would know, what that Ratio would be when $x = p$, I insert p for x , in the Equation $px - y^2 = 0$, and it becomes $p^2 - y^2 = 0$ or $p = y$: so that when the Absciss is equal to the Parameter; so is the Ordinate also. Wherefore insert p for y , in the Value of the Ratio of $\dot{x}$ to $\dot{y}$, and it is $\dot{x} : \dot{y} :: 2p : p :: 2 : 1$, so that, at that Time, the Velocity, with which the Absciss increases, is double the Velocity with which the Ordinate increases. And so by assuming other Values of x ; or of y , other known Relations of $\dot{x}$ and $\dot{y}$ will arise.

92. But if there be three variable or flowing Quantities in a fluential Equation proposed, another Equation including at least two of them, ought also to be given, that the Relation of their Fluxions may be fully determined; and also their Relation among themselves. Thus let the fluential Equation $ax + by^2 - cxz = 0$ including three flowing Quantities x , y and z be given. The fluxional Equation thence deduced is $a\dot{x} + 2b\dot{y}y - c\dot{x}z - cx\dot{z} = 0$, which gives the Relation of the Fluxions $\dot{x}$, $\dot{y}$ and $\dot{z}$, as far as that Relation can be determined from one fluential Equation only. But the Relation of the Fluents in the fluential Equation, and of their Fluxions in the fluxionary Equation, will not be fully determined, unless another fluential Equation be given. As if it were supposed that $x - ay + z = 0$. From whence we deduce $\dot{x} - a\dot{y} + \dot{z} = 0$, for another Relation of the Fluxions, besides the former. Therefore by comparing the two fluential Equations together; and the two fluxional Equations, thence deduced, together, you may exterminate any one of the flowing Quantities; and also any one of the Fluxions, and thereby you may obtain an Equation, which will entirely determine the Relation of the other two, whether flowing Quantities or Fluxions. Thus if I wanted to have the Relation of $\dot{x}$ and $\dot{y}$, fully determined: The first fluxional Equation gives $\dot{z} = \frac{a\dot{x} + 2b\dot{y}y - cx\dot{z}}{cx}$: the other, which is deduced from the 2^d fluential Equation, &c. either given; or assumed, if not given, will give $\dot{z} = a\dot{y} - \dot{x}$: therefore, by equating these Values of $\dot{z}$, we have $\frac{a\dot{x} + 2b\dot{y}y - cx\dot{z}}{cx} = a\dot{y} - \dot{x}$. Again by means of one of the fluential Equations, as of $x - ay + z = 0$ take a Value of z , *viz.* $z = ay - x$, and put that in place of z in the Equation $\frac{a\dot{x} + 2b\dot{y}y - cx\dot{z}}{cx}$

$= ay - \dot{x}$, and it becomes $ax + 2byy - acxy - acxy + 2c\dot{x}x = 0$, or $\dot{x} : y :: acx - 2by : a - acy + 2cx$. Where if you take x at pleasure, the fluential Equations will give y ; and so the Relation of $\dot{x}$ and y will be entirely determined.

More shortly thus. From the fluential Equation $ax + by^2 - cxz = 0$, proposed; and the other $x - ay + z = 0$, either given or assumed, find another Equation free of z ; which will be $ax + by^2 = acxy - cx^2$: and from that you'll have $ax + 2byy - acxy - acxy - 2c\dot{x}x$, as formerly.

93. It may be observed here, that, when there are three flowing Quantities, and but one Equation to determine their Relation, the Fluxions, as well as the Quantities themselves, admit of an infinite Variety of different Relations: and therefore when in that Case, I assume another Equation, as above, I only thereby determine one of these Relations: which, from the Nature of the Thing, are indefinitely many.

94. If there are four, five or any other Number of flowing Quantities, there ought to be given as many Equations save one, as there are Quantities, in order to the full Determination of the Relations of the first Fluxions of these flowing Quantities: as plainly appears from what has been said.

* Art. 6.
of this Ex-
plication.

95. It was observed formerly *, that the Fluxions of homogeneous Quantities are still to be considered in relation to one another: for without such a Consideration, we can make nothing of the Doctrine of Fluxions. And therefore, since every fluxionary Equation contains the Fluxions of two flowing Quantities at least, either express'd or understood; and thereby determines only the Relation of these Fluxions, it is left at liberty, to suppose one of these flowing Quantities to flow or change at any rate, either equably and uniformly; or according to any Law of Acceleration or Retardation, we think fit to frame or suppose: because such a Supposition never alters the Relation of the Fluxions or Velocities of flowing. Hence it appears, that our Conception of the Relation of the Fluxions, will then be most clear and distinct, when we suppose one of the flowing Quantities, to flow with an uniform and invariable Velocity, and call it Unity. For thereby that uniform Fluxion is made a common Standard, by which to measure the rest. Thus in the Equation $px - y^2 = 0$, mentioned above, to a Parabola, from which the fluxionary Equation $p\dot{x} - 2y\dot{y} = 0$ is deduced, I suppose the Absciss x to flow uniformly, and put it's Fluxion $\dot{x} = 1$. By which means the fluxional Equation becomes $p - 2y\dot{y} = 0$, *i. e.* $\dot{y} = \frac{p}{2y}$: so that the Fluxion of the Absciss being always 1; the Fluxion

of

of the Ordinate, will always be express'd by the Quantity $\frac{p}{2y}$. I might have likewise supposed the Parabola to be so described, by the Motion of the Ordinate along the Axis, that the Velocity, with which the Ordinate increases, is always invariably the same, and call it Unity: then $px - 2yy = 0$ becomes $px - 2y = 0$, or $\dot{x} = \frac{2y}{p}$: i. e. in such a Case, the Velocity, with which the Absciss flows will be always express'd by $\frac{2y}{p}$. And so of others.

96. This Supposition is of further Use, when we have occasion to proceed to 2^d , 3^d and other superior Orders of Fluxions. For in all the Equations, for determining the Relations of these superior Orders of Fluxions, the Symbol, for expressing the first Fluxion of the uniformly flowing Quantity, may be dashed out, wherever it is found, being Unity; and all these Terms thrown out, in which a second, third or any subsequent Fluxion of the uniformly flowing Quantity, would be involved: for since it's first Fluxion is Unity, a constant and invariable Quantity; every one of the subsequent Fluxions is nothing. Which our Author observes in Art. 20. * and by this means the Equations, for determining the Relations of the Fluxions, are considerably abridg'd, as is there to be seen.

* Of the Quadrature of Curves.

97. But hence it must happen, that a fluxionary Equation, will sometimes appear not to have the Symbols of the Fluxions of the same Dimensions, through all the Terms: which yet it ought to have, if we consider the Manner of forming any fluxional Equation from it's fluent. Fluxions of the same Order, or whose Symbols are of the same Dimensions, are such as these, $\dot{x}$, $\dot{y}$, $\dot{z}$ of one Dimension; $\ddot{x}$, $\ddot{x}^2$, $\dot{x}\dot{y}$; of two Dimensions; $\ddot{x}$, $\ddot{x}\dot{y}$, $\dot{x}^3$, $\dot{x}^2\dot{y}$, $\dot{x}\dot{y}\dot{z}$ of three Dimensions, &c. That the Symbols of the Fluxions, in each Term of any fluxionary Equation, as it is formed from the corresponding Fluent, whether it be a fluxionary Equation of the 1^{st} , 2^d , 3^d , or any other Order, must always be of the same Dimensions, will easily appear to any one that considers, how the several Orders of fluxional Equations are formed above *. And it is necessary it should be so, since Fluxions are Quantities of a different kind from the Quantities, whereof they are the Fluxions. Thus taking the preceding Example $px - yy = 0$: hence arises $px - 2yy = 0$, where the Symbols of the Fluxions, in both Times, are of the same Dimension. But if we put $\dot{x} = 1$, it becomes $p - 2yy = 0$, where one Term contains no Symbol of a Fluxion at all. Thus also, when we have the Equation $xy^3 - x^4 + a^4 = 0$ mentioned Art. 88 *, the third fluxional Equation, thence deduced, was shewn to be $\ddot{y}^3 + 9\ddot{y}\dot{y}^2 + 9\dot{y}\ddot{y}^2$

* Art. 88. of this Explication, and Art. 18. and 19. of the Quadrature of Curves.

* Of this Explication.

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$+ 18\dot{z}\dot{y}^2y + 3z\dot{y}y^2 + 18z\ddot{y}y + 6z\dot{y}^3 - 4\dot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$: where all the Terms contain Symbols of Fluxions of three Dimensions. But if we suppose $\dot{z} = 1$, it appears in another Form, thus $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\ddot{y}y^2 + 18z\ddot{y}y + 6z\dot{y}^3 - 24z = 0$.

98. Therefore, whenever a fluxionary Equation is proposed, where the Symbols of the Fluxions, in all the Terms, are not of the same Dimensions, these Dimensions must be completed, or supposed to be completed, by multiplying the lower Terms, by the Fluxion of some flowing Quantity, whose Fluxion is understood to be Unity, until the Symbols of the Fluxions arise to the same Number of Dimensions, in all the Terms. Thus the fluxionary Equation, just now mentioned, is to be completed, by multiplying the two first Terms $9\ddot{y}y^2$ and $18\dot{y}^2y$ by $\dot{z}$; and the last Term $-24z$, by $\dot{z}^3$: $\dot{z}$ being supposed equal to Unity. And so the Equation will appear thus $9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\ddot{y}y^2 + 18z\ddot{y}y + 6z\dot{y}^3 - 24\dot{z}^3z = 0$. Where the Dimensions are completed, by multiplying the lower Terms as often as is necessary by $\dot{z}$, which is the Fluxion of a flowing Quantity z , viz. such a one as is found in the given fluxional Equation, while in the mean Time, it's Fluxion is not found.

99. But if there be no variable or flowing Quantity in the given fluxionary Equation, but what has it's Fluxion in it likewise, then the Order of the Fluxions in the lower Terms, must be completed, by supposing some other variable or flowing Quantity, which flows uniformly, whose Fluxion must be understood to be Unity, and multiplying the lower Terms by that Fluxion, so often, until the Dimensions of the fluxionary Symbols be the same through all the Terms. Thus if the Equation $\dot{x} + \dot{x}y - ax^2 = 0$ were proposed, we are to suppose some third Quantity as z , which flows uniformly, having it's Fluxion $\dot{z} = 1$: and then multiply the first Term by $\dot{z}$, and the last Term by $\dot{z}z$. And so we must conceive of the given fluxional Equation $\dot{x} + \dot{x}y - ax = 0$, as if it had been derived from this one $\dot{x}z + \dot{x}y - a\dot{z}zx^2 = 0$, by putting $\dot{z} = 1$. Likewise if $aav - av + vv = 0$ were given, I complete the Dimensions, by supposing another flowing Quantity z , whose Fluxion $\dot{z} = 1$. As if the Equation had been originally $aav - av\dot{z} - v^2\dot{z} = 0$. And so of others.

100. This first Prop. of our Author's, which I have been explaining, lays the Foundation of the whole Doctrine of Fluxions: particularly it contains the Solution of the first of the two general Problems mentioned at the Beginning*, viz. *the Length of the Space described, being continually or at all Times given, to determine the Velocity of the Motion at any Time proposed*: which in effect amounts to the same thing, with the

* Art. 2.
of this Ex-
plication.

the Proposition as here express'd: and is called the *direct* Method of Fluxions. The Application of these two general Problems is very extensive: particularly by them we are enabled to solve the following useful Problems, *viz.*

- 1°. To determine the *Maxima* and *Minima* of Quantities.
- 2°. To draw Tangents to all Sorts of Curves, whether geometrical or mechanical.
- 3°. To determine the Points of contrary Flexure, and Retrogression in Curves.
- 4°. To determine the Quantity of Curvature at a given Point of a given Curve: *i.e.* to find the Length of the Radius of an equicurve Circle.

Under this may be comprehended the finding the Points, where a Curve has any given Degree of Curvature: where it has the greatest or least Curvature: and the Determination of the Locus of the Center of Curvature.

5°. To determine the Quality of Curvature at a given Point of a given Curve: *i.e.* it's Form, as it is more or less inequable; or as it varies more or less in it's Progress through different Parts of the Curve.

6°. To find as many Curves as we please, which may be squared: *i.e.* to find the Nature and Property of all such Curves, whose Areas are exhibited by finite Equations*.

* Prop. 2.

7°. To find as many Curves as we please, the Relation of whose Areas to the Area of any given Curve is assignable by finite Equations.

8°. To find as many Curves as we please, whose Lengths may be express'd by finite Equations. To this Problem and the 4th belongs the determining the Evolutes of Curves.

9°. To find as many Curves as you please, whose Lengths may be compared with the Length of any Curve proposed; or with it's Area applied to a given Line.

10°. To find the Caustics to all Sorts of Curves, whether by Reflection or Refraction.

11°. The *Quadrature* of Curves; or finding their Areas.

12°. The *Rectification* of Curve-lines; or finding their Lengths.

13°. The *Plaining* of Curve-surfaces; or reducing them to plain Surfaces.

14°. The *Cubature* of Solids; or finding their solid Contents.

15°. The finding the Centers of *Gravity*: and of *Oscillation* or *Percussion* of all Sorts of *Lines*, *Superficies's* and *Bodies*.

101. These Problems for most part belong immediately to the Geometry of Curves; but by their means, serve to other excellent Purposes

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The Quadrature of CURVES explained.

in Mathematics and natural Philosophy: witness that inimitable Work, the mathematical Principles of natural Philosophy: a great Part of which is founded upon the Geometry of Curves: and Principles of the same or like Nature with those of Fluxions.

102. To go through all these Problems would take up a great deal of Time; and be foreign to our Design: however some of them that come in most naturally, shall be explain'd and resolv'd in the Progress of this Work: particularly the 1st, 2^d, 3^d, 6th, 7th, 8th, 12th, 13th, 14th. The 11th is that whose Explication and Solution takes up the great Part of this Work.

103. Among geometrical Problems, one of the most noble and useful is the *Quadrature* of Curves: and accordingly the greatest Geometricians, in all Ages, have laboured this Point with all the Force of Invention; and exerted themselves to the utmost upon this Head. Yet there was very little done in that way, till after the Middle of the last Century, that the Doctrine of infinite Series was introduced; founded upon the Arithmetic of Infinites, made use of by Dr. Wallis. Which Method of infinite Series was improved and perfected by the Discovery of the Method of Fluxions by our celebrated Author about the Year 1666¹: whereby the Quadrature of Curves is rendered universal.

104. This is a new kind of Analysis: by means of which the Areas of all Sorts of Curves are investigated and determined, either to a perfect Exactness; or, when the Nature of the Curve will not admit of that, to any Degree of Exactness, one can possibly require. And such is the Subject of this Treatise.

105. But if any one should think that the Quadrature of Curves, being accounted but one Branch of the Application of the Doctrine of Fluxions, scarcely deserves to be insisted upon at so great length: it must be known, that, the Quadrature of Curves, is not only among the most sublime and useful Discoveries in Geometry, considered in itself: but in effect includes under it, the finding the Length of Curvelines; the Solidity of Bodies, the Contents of their Surfaces, and Innumerable other Problems. Add to this that the original Principles, and

¹ See Dr. Wallis's *Treatise of Algebra*, Chap. 82, 85, 91, and the *Commercium Epistolicum de varia re Mathematica*, particularly a Letter from Sir I. Newton to Mr. Oldenburg, dated October 24, 1676. Some time after Sir I. Newton had discovered his *Doctrine of infinite Series and Fluxions*, Mr. James Gregory Professor of Mathematics at St. Andrews, about the End of the Year 1670, fell upon a like general Method, serving for the same Purposes; having discovered many particular Series's long before that Time: as appears from the *Commercium Epist.* See Mr. Gregory's Letters to Mr. Collins, dated Sept. 5. Nov. 23. Dec. 24. Anno 1670. Feb. 15. Anno 1671, likewise Mr. Collins's Letter to Dr. David Gregory: and Sir I. Newton's second Letter to Mr. Oldenburg already mentioned.

justest Conceptions of the Method of Fluxions, are deduced from the Consideration of the Description of curvilinear Areas, after the Manner already explained. Upon which account, the Quadrature of Curves, here laid down by our Author, ought to be fully understood by all such as would have a just and comprehensive Notion of Fluxions.

Explication of Prop. 2.

106. By this Proposition, the Relation betwixt the Area of any Curve and it's Absciss or rather the Rectangle contain'd under the Absciss and an invariable right Line, supposed to be Unity, being exhibited, you can always determine the Nature of the Curve, *i.e.* the Relation betwixt the Absciss and Ordinate: or you may assume or feign any Relation you please, betwixt the curvilinear and rectilinear Areas ABC and ABED¹, and thence determine the Relation of the corresponding Ordinates AB and BC.

107. Although the two following Prop. be Examples of this, yet, for the sake of a further *Illustration* of this Prop. I shall shew it's Application, in some Examples, which are more simple and easy: by which, at the same Time, the Demonstration of it will be made more evident.

Ex. 1. Calling the Absciss $AB = z$; the Area $ABC = t$; the Ordinate $BC = y$, and the Ordinate $BE = (AD =) 1$: and consequently the Rectangle $ABED = z \times 1 = z$: Let the Relation of the two Areas ABC and ABED, *i.e.* of t and z , be express'd by this Equation $z^2 = t$: then taking the Fluxions by Prop. 1, we have $2\dot{z}z = \dot{t}$: *i.e.* * Art. 46. by reducing to an Analogy, $1 : 2z :: \dot{z} : \dot{t}$: but $\dot{z} : \dot{t} :: EB : BC$ * of this Explication. therefore by Equality, $1 : 2z :: (EB : BC ::) 1 : y$: whence $2z = y$: * Art. 29. Which defines the Relation of the Absciss and Ordinate of the Figure of this Explication. ABC: and shews it to be a right-angled Triangle, having the Base BE double the Altitude AB.

Ex. 2. Suppose we assume $\frac{2}{3}z\sqrt{az} = t$, or $\frac{2}{3}a^{\frac{1}{2}}z^{\frac{3}{2}} = t$, for the Relation of the Areas ABED and ABC: then, by taking the Fluxions, we have $a^{\frac{1}{2}}\dot{z}z^{\frac{3}{2}} = \dot{t}$, *i.e.* $1 : a^{\frac{1}{2}}z^{\frac{3}{2}} :: (\dot{z} : \dot{t} ::) EB = 1 : BC = y$: Whence $\sqrt{az} = y$ or $az = y^2$: which shews the Curve to be the common Parabola.

Ex. 3. Let $az^3 = 3b^2t + 3bcz^3t$ exhibit the Relation of the Areas: thence, by taking the Fluxions, we have $3a\dot{z}z^2 = 3b\dot{b}t + 9bc\dot{z}z^2t + 3bcz^3\dot{t}$; or, by dividing by 3 and transposing $a\dot{z}z^2 - 3bcz^3\dot{t} \times \dot{z} = b\dot{b} + bc\dot{z}^3 \times \dot{t}$, *i.e.* $b^2 + bc\dot{z}^3 : a\dot{z}z^2 - 3bcz^3\dot{t} :: (\dot{z} : \dot{t} ::) 1 : y$:

¹ See the Fig. belonging to this Prop.

whence, by multiplying the Extremes and Means, we have $b^2y + bcx^3y = ax^2 - 3bcx^2t$ for the Relation of z , y and t . Now that t may be exterminated: from the Equation expressing the Relation of the Areas, viz. $ax^3 = 3b^2t + 3bcx^3t$, find the Value of t , which is $t = \frac{ax^3}{3b^2 + 3bcx^3}$; substitute this Value in the Equation just now found, viz. $b^2y + bcx^3y = ax^2 - 3bcx^2 \times \frac{ax^3}{3b^2 + 3bcx^3}$; i. e. by Reduction, $y = \frac{ax^2}{b^2 + 2bcx^3 + c^2x^6}$; which shews the Relation of z and y or AB and BC, and so determines the Nature of the Curve ABC sought.

Ex. 4. Let the Equation $2a^2t - 4xt - a^4 = tt$ be assumed for the Relation of z and t : by taking the Fluxions, we have $2a^2\dot{t} - 4x\dot{t} - 4t\dot{x} = 2t\dot{t}$; or $a^2\dot{t} - 2x\dot{t} - t\dot{x} = t\dot{t}$, i. e. $a^2 - 2x - t : 2t :: (\dot{z} : \dot{t} ::) 1 : y$, whence $y = \frac{2t}{a^2 - 2x - t}$; and in order to exterminate t , let its Value be found from the Equation $2a^2t - 4xt - a^4 = tt$, expressing the Relation of the Areas: which will be $t = a^2 - 2x \pm 2\sqrt{x^2 - a^2x}$: insert this Value of t in place of it, in the Equation $y = \frac{2t}{a^2 - 2x - t}$, and we shall have $y = \left(\frac{2a^2 - 4x \pm 4\sqrt{x^2 - a^2x}}{a^2 - 2x - a^2 + 2x \pm 2\sqrt{x^2 - a^2x}} = \frac{a^2 - 2x \pm 2\sqrt{x^2 - a^2x}}{\pm \sqrt{x^2 - a^2x}} \right) \pm \frac{2x - a^2}{\sqrt{x^2 - a^2x}} - 2$ for defining the Nature of the Curve.

Ex. 5. Let the Equation $\frac{-2ab + acx^2}{3cc} \sqrt{b + cx^2} = t$ be given or assumed for expressing the Relation of z and t : then by taking the Fluxions, we have $\frac{2axx}{3c} \sqrt{b + cx^2} + \frac{cx}{\sqrt{b + cx^2}} \times \frac{-2ab + acx^2}{3cc} = \dot{t}$; or by due Reduction $\frac{axx^3}{\sqrt{b + cx^2}} = \dot{t}$: whence it is $1 : \frac{ax^3}{\sqrt{b + cx^2}} :: (\dot{z} : \dot{t} ::) 1 : y$; or $y = \frac{ax^3}{\sqrt{b + cx^2}}$.

* Art 29, 31. of this Explication.

* Art. 95. of this Explication.

108. Schol. Since we have always $\dot{t} : \dot{z} :: y : 1$ *, thence $\dot{z}y = \dot{t}$, is an Equation expressing the Relation of $\dot{z}$ and $\dot{t}$ in every Case: in which Equation, if we suppose $\dot{z} = 1$, as we may do*, it becomes $y = \dot{t}$. Whence the Solution of this Problem may be a little more expeditious, thus: From the Equation propos'd or assum'd, expressing the Relation of t and z , find the Equation expressing the Relation of the Fluxions $\dot{t}$ and $\dot{z}$: in which Equation, put 1 for $\dot{z}$; or dash out $\dot{z}$ every where, and then the resulting Value of $\dot{t}$, will be the Value also of y : therefore substitute y for $\dot{t}$: and so you shall have an Equation expressing the Nature of the Curve. Thus in the 2^d Ex. $\frac{2}{3}a^2z^{\frac{2}{3}} = t$, the Relation of the

the Fluxions is $a^{\frac{1}{2}}\dot{z}z^{\frac{1}{2}} = \dot{t}$, i. e. by substituting 1 for $\dot{z}$ and y for $\dot{t}$, $\sqrt{az} = y$. Likewise in the 4th Ex. viz. $2a^2t - 4zt - a^4 = tt$, the Relation of the Fluxions is $a^2\dot{t} - 2\dot{z}t - 2z\dot{t} = t\dot{t}$, i. e. $a^2y - 2t - 2zy = yt$, by means of which Equation, and the proposed Equation, t being exterminated, we obtain, as formerly $y = \frac{1}{\sqrt{z^2 - aaz}} - 2$ for defining the Nature of the Curve. And so in all other Cases.

Our Author's 3^d Prop. is an Example of the same thing: and is so plainly demonstrated by himself, that, if what hath been said in the foregoing Schol. be considered, there cannot remain any Difficulty in it: and therefore I shall pass on to the next Proposition.

Demonstration of Prop. 4.

109. The Demonstration of this Prop. is omitted by our Author, because it may be easily supplied: if the Demonstration of Prop. 3^d be duly considered: and it is thus. Let us suppose the Area $z^{\theta}R^{\lambda}S^{\mu} = v$: then by taking the Fluxions according to Prop. 1, we have $\theta\dot{z}z^{\theta-1}R^{\lambda}S^{\mu} + \lambda z^{\theta}R^{\lambda-1}\dot{R}S^{\mu} + \mu z^{\theta}R^{\lambda}\dot{S}S^{\mu-1} = \dot{v}$.

Instead of $R^{\lambda}S^{\mu}$, in the first Term of the Equation, substitute $RR^{\lambda-1}SS^{\mu-1}$; instead of $z^{\theta}S^{\mu}$, in the second Term, $zz^{\theta-1}SS^{\mu-1}$: and instead of $z^{\theta}R^{\lambda}$, in the third Term, substitute $zz^{\theta-1}RR^{\lambda-1}$: which Expressions are evidently equal to the others: and it becomes

$$\theta\dot{z}z^{\theta-1}RR^{\lambda-1}SS^{\mu-1} + \lambda z z^{\theta-1}\dot{R}R^{\lambda-1}SS^{\mu-1} + \mu z z^{\theta-1}RR^{\lambda-1}\dot{S}S^{\mu-1} = \dot{v}$$

$$\text{Or } \theta\dot{z}RS + \lambda z\dot{R}S + \mu zR\dot{S} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} = \dot{v}.$$

But, by Hyp. $R = e + fz^{\eta} + gz^{2\eta} + \mathcal{E}c$. and $S = k + lz^{\eta} + mz^{2\eta} + \mathcal{E}c$.

From which two last Equations, find the Relations of the Fluxions by Prop. 1, and you'll have $\dot{R} = \eta f\dot{z}z^{\eta-1} + 2\eta g\dot{z}z^{2\eta-1} + \mathcal{E}c$. And $\dot{S} = \eta l\dot{z}z^{\eta-1} + 2\eta m\dot{z}z^{2\eta-1} + \mathcal{E}c$.

Substitute these Values of $\dot{R}$ and $\dot{S}$, in place of them in the preceding fluxional Equation; and likewise the Values of R and S in place of them: and put $\dot{z} = 1$. And so you'll have $\theta \times e + fz^{\eta} + gz^{2\eta} + \mathcal{E}c$.

$$\times \frac{k + lz^{\eta} + mz^{2\eta} + \mathcal{E}c}{k + lz^{\eta} + mz^{2\eta} + \mathcal{E}c} + \frac{\lambda z \times \eta f z^{\eta-1} + 2\eta g z^{2\eta-1} + \mathcal{E}c}{k + lz^{\eta} + mz^{2\eta} + \mathcal{E}c} \times \frac{\mu z \times e + fz^{\eta} + gz^{2\eta} + \mathcal{E}c}{k + lz^{\eta} + mz^{2\eta} + \mathcal{E}c} \times \frac{\eta l z^{\eta-1} + 2\eta m z^{2\eta-1} + \mathcal{E}c}{k + lz^{\eta} + mz^{2\eta} + \mathcal{E}c} \text{ into } z^{\theta-1}R^{\lambda-1}S^{\mu-1} = \dot{v} = {}^1(y=) BC *$$

That is, if you actually multiply the several Factors; and observe to rank the Terms according to the Dimensions of the variable Quantity z ; *Art. 108. of this Explication.

* y is supposed to stand for the Ordinate of the Curve.

Areas.	Ordinates.
114. $x^{\theta \pm \sigma} R^{\lambda} \dots \frac{\theta \pm \sigma}{\lambda \eta} x^e \dots \frac{\theta \pm \sigma}{\lambda \eta} f x^{\eta} \dots \frac{\theta \pm \sigma}{2 \lambda \eta} g x^{2 \eta} \dots \frac{\theta \pm \sigma}{3 \lambda \eta} h x^{3 \eta} \dots + \&c. x^{\theta \pm \sigma - 1} R^{\lambda - 1}$	
Or $\frac{\theta \pm \sigma}{\lambda \eta} x^e \dots \frac{\theta \pm \sigma}{\lambda \eta} f x^{\eta} \pm \sigma \eta \dots \frac{\theta \pm \sigma}{2 \lambda \eta} g x^{2 \eta} \pm \sigma \eta \dots \frac{\theta \pm \sigma}{3 \lambda \eta} h x^{3 \eta} \pm \sigma \eta \dots + \&c. x^{\theta - 1} R^{\lambda - 1}$	
115. $x^{\theta \pm \sigma} R^{\lambda} S^{\mu} \dots \frac{\theta \pm \sigma}{\lambda \eta} x^e \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x f k x^{\eta} \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x g k x^{2 \eta} \dots$	
	$\frac{\theta \pm \sigma}{\lambda \eta} x f l x^{2 \eta} \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x g l x^{3 \eta} \dots$
	$\frac{\theta \pm \sigma}{\mu \eta} x e l x^{\eta} \dots \frac{\theta \pm \sigma}{\lambda \eta} x f m x^{3 \eta} \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x g m x^{4 \eta} \dots$
Or $\frac{\theta \pm \sigma}{\lambda \eta} x^e \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x f k x^{\eta} \pm \sigma \eta \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x g k x^{2 \eta} \pm \sigma \eta \dots$	
	$\frac{\theta \pm \sigma}{\mu \eta} x e l x^{\eta} \pm \sigma \eta \dots \frac{\theta \pm \sigma}{\lambda \eta} x f l x^{2 \eta} \pm \sigma \eta \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x g l x^{3 \eta} \pm \sigma \eta \dots$
	$\frac{\theta \pm \sigma}{2 \mu \eta} x e m x^{2 \eta} \pm \sigma \eta \dots \frac{\theta \pm \sigma}{\lambda \eta} x f m x^{3 \eta} \pm \sigma \eta \dots \frac{\theta \pm \sigma}{2 \lambda \eta} x g m x^{4 \eta} \pm \sigma \eta \dots$

And it is further evident that, if any of the Signs of the Letters entering the Values of R or S, in the Expression of the Areas, be negative, there must be a corresponding Change made upon the Signs in the Expression of the Ordinates: which is easily known from the common Rules of Algebra. Which will be clear by applying the general Expressions to a particular Example. Wherefore, let $\frac{3a^2x - 3x^3}{\sqrt{b^3 + ccx - dx^2}}$ be the

Area of a Curve whose Absciss is x: that it may be compared with the general Form of Areas in Prop. 4, I bring it to stand thus $x^2 \times \frac{3a^2x - 3x^3}{\sqrt{b^3 + c^2x - dx^2}} \times 3a^2 * - 3x^2$: by comparing which with the general Expression $x^{\theta} R^{\lambda} S^{\mu}$, we have $x = x$. $R = b^3 + c^2x - dx^2$, $S = 3a^2 - 3x^2$. $\theta = 1$. $\lambda = -\frac{1}{2}$, $\mu = 1$. $e = b^3$, $f = c^2$, $g = -d$. $k = 3a^2$, $l = 0$, $m = -3$. Insert these particular Values in the general Form of the Ordinate in Prop. 4, and you'll have $\frac{3a^2b^3 + 2a^2c^2x - a^2dx^2 - 8c^2x^3 + 7dx^4 - 9b^3}{\sqrt{b^3 + ccx - dx^2}}$

$\times \frac{3a^2x - 3x^3}{\sqrt{b^3 + ccx - dx^2}}$ for the Ordinate of the Curve whose Area is $\frac{3a^2x - 3x^3}{\sqrt{b^3 + ccx - dx^2}}$.

The last Factor, corresponding to S, is represented as wanting the 2^d Term, viz. thus $3a^2 * - 3x^2$, because, since the Powers of x belonging

longing to R and S, in the general Form, are suppos'd to have their Indexes going on by the same Differences, being η , 2η , 3η , &c. and since the Value of η , in this particular Case, must be Unity: therefore it is evident why the second Term of the Value of S must be nothing, and therefore $l=0$.

And moreover by the same way of reasoning it appears that, if it be $R=e+fx^{-\eta}+gz^{-2\eta}+hz^{-3\eta}+\&c.$ in Prop. 3: and $R=e+fx^{-\eta}+gz^{-2\eta}+\&c.$ and likewise $S=k+lx^{-\eta}+mx^{-2\eta}+nx^{-3\eta}+\&c.$ in Prop. 4: where the Indexes of the Powers of z under the Vinculums are all negative: then the Sign of η must be chang'd every where into it's Opposite in the Expressions of the Ordinates of the Curves, belonging to these Prop. respectively.

115. It may likewise be observed that, if the general Forms of the Areas, viz. $z^{\theta}R^{\lambda}$ and $z^{\theta}R^{\lambda}S^{\mu}$ be multiplied by any constant or given Quantity, as A, and so become $Az^{\theta}R^{\lambda}$ and $Az^{\theta}R^{\lambda}S^{\mu}$, the corresponding Ordinates of the Curves must be multiplied by the same given Quantity A. For the Fluents being increased or diminished in a given Ratio, the Fluxions are increased or diminished in the same given Ratio*: and the Areas being the Fluents, the Ordinates are as the Fluxions in this Case*, because the Fluxions of the Abscisses are the same.

* Art. 40.
of this Ex-
plication.
* Art. 32.
of this Ex-
plication.

116. By the same way of reasoning as that made use of in these two Prop. it appears that, if the Area of a Curve be $z^{\theta}R^{\lambda}S^{\mu}T^{\nu}$, where ν is any given Number, and $T=p+qx^{\eta}+rx^{2\eta}+\&c.$ and the other Symbols stand for the same things as before, the Ordinate of such Curve may be found: and so for other Areas more complex.

117. It follows from Prop. 3, and what has been said that if

$$\frac{\theta+\sigma\eta}{\lambda\eta}Az^{\theta+\sigma\eta} \times \frac{\theta+\sigma\eta}{\lambda\eta}fAz^{\theta+\sigma\eta} \times \frac{\theta+\sigma\eta}{2\lambda\eta}gAz^{2\eta+\sigma\eta} \times \frac{\theta+\sigma\eta}{3\lambda\eta}hAz^{3\eta+\sigma\eta} + \&c. \times Az^{\theta-1}R^{\lambda-1}$$

be the Ordinate of a Curve, the Area of the Curve is $Az^{\theta+\sigma\eta}R^{\lambda}$: for if not, the Area $Az^{\theta+\sigma\eta}R^{\lambda}$ would belong to a Curve having a different Ordinate, contrary to what has been demonstrated. And the like is to be understood with respect to the Area and Ordinate contain'd in the general Expressions in Prop. 4.

S E C T. IV.

Containing Notes on Art. 27—31.

Some Lemmas serving to demonstrate several Things contain'd in the Author's Remarks upon Prop. 5.

L E M M A. I.

119. **A**N infinite Series of Powers of a proper Fraction, whose Indexes constitute an arithmetical Progression ascending, having finite Coefficients¹, will converge: i. e. a finite Number of Terms of the Series, will differ from the Sum of the whole Series by less than any given Difference.

Let $\frac{N}{D}$ represent any proper Fraction, $\frac{N}{D}^n \cdot \frac{N}{D}^{n+p} \cdot \frac{N}{D}^{n+2p} \cdot \frac{N}{D}^{n+3p}$ &c. in inf. an infinite Series of Powers of that Fraction, whose Exponents are $n \cdot n+p \cdot n+2p \cdot n+3p$, &c. then it is evident, the Series is an infinite geometrical Progression descending, having $\frac{N}{D}^p$ for the common Multiplier: wherefore, according to the Nature of such a

Progression, the Sum of the whole is $\frac{\frac{N}{D}^n}{1 - \frac{N}{D}^p}$. Let q be any given

Quantity how small soever; and T a Term of the Series less than $q \times 1 - \frac{N}{D}^p$ (for, it is evident, such a Term may be found) so that

$T < q \times 1 - \frac{N}{D}^p$. Now, the Sum of all the Terms after T inclusive, from the Property of a geometrical Progression, is $\frac{T}{1 - \frac{N}{D}^p}$: therefore

that Sum is less than $\frac{q \times 1 - \frac{N}{D}^p}{1 - \frac{N}{D}^p} = q$. Now, supposing the several Terms

to have any finite Coefficients prefix'd, so that none of them be greater than A : then you need only assume any Term of the Series, which

¹ By finite Coefficients in this Lem. I mean such as the greatest of them is not greater than any assignable Quantity.

N

without

without it's Coefficient, is less than $\frac{q}{A} \times 1 - \frac{N^p}{D}$: and then all the subsequent Terms from that Term inclusive, without the Coefficients,

will be less than $\frac{\frac{q}{A} \times 1 - \frac{N^p}{D}}{1 - \frac{N^p}{D}} = \frac{q}{A}$, as has been shewn: wherefore, since

no Coefficient is greater than A, the Sum of all these Terms with their Coefficients prefixt, cannot be so great as $\frac{q}{A} \times A = q$, i. e. the Sum of them all is less than any given Quantity.

120. Schol. Hence it appears that an infinite Series of Powers of any Quantity greater than Unity, whose Indexes decrease in an arithmetical Progression, without Coefficients, or with finite Coefficients, will converge. For if $A \times N^n \cdot B \times N^{n-p} \cdot C \times N^{n-2p}$ &c. in *inf.* represent such a Series: it is equivalent to this other $AN^n \times \frac{1}{N|^0} \cdot BN^n \times \frac{1}{N|^p} \cdot CN^n \times \frac{1}{N|^{2p}}$ &c. which converges by the foregoing Lemma.

LEMMA. II.

121. An infinite Series, such as these just now mentioned, which converges when all the Terms are affected with the same Sign, will also converge when the Terms are affected with different Signs.

For since a Term may be found, when all are positive, or all negative, such, as that the Sum of it with all that follow shall be less than any given Quantity; much more will the Amount of that Term with all that follow be less than any given Quantity, when the Terms are affected with different Signs: and therefore the Series converges: since the Amount of the whole Series is supposed not to be less than a finite Quantity. Q. E. D.

LEMMA. III.

122. If any rational integral Fluent, including only one indeterminate or flowing Quantity x : and it's Fluxion, have any common prime Divisor, which includes x : I say the Fluent shall contain such common prime Divisor oftener by once and no more, than it's Fluxion doth.

Let $a + bx^n + cx^{2n} + dx^{3n} + \&c.$ be a Fluent, where the Quantities $a, b, c, d, \&c.$ are constant or invariable; and any of them may be equal to nothing. It's Fluxion is $nbxx^{n-1} + 2ncxx^{2n-1} + 3ndxx^{3n-1}$

+

The Quadrature of CURVES explained.

95

+ &c*. Now it is demonstrated by Writers upon Algebra¹, that, * Art. 82. of this Explication.
if the several Terms of the Quantity $a + bx^n + cx^{2n} + dx^{3n} + \&c.$ be multiplied by the Exponents of the Powers of x in these Terms respectively, and the Products thence arising be divided by x : and if the Quantity which results, viz. $nbx^{n-1} + 2ncx^{2n-1} + 3ndx^{3n-1} + \&c.$ have any common prime Divisor with the Quantity proposed $a + bx^n + cx^{2n} + dx^{3n} + \&c.$ that prime Divisor shall be contain'd just once oftener in the Quantity $a + bx^n + cx^{2n} + dx^{3n} + \&c.$ than in the Quantity $nbx^{n-1} + 2ncx^{2n-1} + 3ndx^{3n-1} + \&c.$: therefore if the flowing Quantity proposed $a + bx^n + cx^{2n} + dx^{3n} + \&c.$ and it's Fluxion $nbxx^{n-1} + 2ncxx^{2n-1} + 3ndxx^{3n-1} + \&c.$ have any common prime Divisor, including the variable Quantity x , such Divisor must be contain'd just once oftener in the Fluent than in the Fluxion. Q.E.D.

Explication of Prop. 5, and of the Author's Remarks upon it.

Although our Author's Demonstration of this Proposition may be evident enough to any one that carefully considers it, and understands what hath been already taught: yet for the sake of a farther Illustration; and some Observations to be made upon it, I shall exhibit the Demonstration at some further Length.

123. Wherefore, supposing the same things as are supposed by Sir Isaac; and moreover that the Factor $a + bx^n + cx^{2n} + dx^{3n} + \&c.$ continued one Term further, is $a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \&c.$ so that the Ordinate of the Curve may be $x^{\lambda-1}R^{\lambda-1} \times a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \&c.$; likewise that $R = e + fx^n + gx^{2n} + bx^{3n} + iz^{4n} + \&c.$; as also that $v + \lambda = u$: then by what has been demonstrated*, the following Ordinates and Areas set opposite to them, will belong to the same Curves.

* Art. 114, 116. of this Explication.

	Ordinates.	Areas.
1 ^o .	$\frac{0}{\lambda} \times A + \frac{0}{2\lambda} \times Bx^n + \frac{0}{3\lambda} \times Cx^{2n} + \frac{0}{4\lambda} \times Dx^{3n} + \&c.$	$Ax^{\lambda} R^{\lambda}$
2 ^o .	$\frac{0}{\lambda} \times Bx^n + \frac{0}{2\lambda} \times Cx^{2n} + \frac{0}{3\lambda} \times Dx^{3n} + \frac{0}{4\lambda} \times Ex^{4n} + \&c.$	$Bx^{\lambda+n} R^{\lambda}$
3 ^o .	$\frac{0}{\lambda} \times Cx^{2n} + \frac{0}{2\lambda} \times Dx^{3n} + \frac{0}{3\lambda} \times Ex^{4n} + \&c.$	$Cx^{\lambda+2n} R^{\lambda}$
4 ^o .	$\frac{0}{\lambda} \times Dx^{3n} + \frac{0}{2\lambda} \times Ex^{4n} + \&c.$	$Dx^{\lambda+3n} R^{\lambda}$
5 ^o .	$\frac{0}{\lambda} \times Ex^{4n} + \&c.$	$Ex^{\lambda+4n} R^{\lambda}$

¹ See Reynau's *Analyse démontrée*, B. 4. Sect. 4.

Therefore if the Sum of these Ordinates be suppos'd equal to the given Ordinate $a + bz^n + cz^{2n} + dz^{3n} + ez^{4n} + \&c. \times z^{\theta-1} R^{\lambda-1}$, the Sum of the corresponding Areas, viz. $A + Bz^n + Cz^{2n} + Dz^{3n} + Ez^{4n} + \&c. \times z^{\theta} R^{\lambda}$ shall be equal to the Area of the Curve, whereof the Ordinate propos'd, is the Ordinate.

Wherefore that the Values of A, B, C, D, E, &c. which are determinate Quantities, may be discovered, let the corresponding Terms of the two Ordinates be compared, by putting them equal: and thence arise the following Equations.

$$1^{\circ}. \theta A = a$$

$$2^{\circ}. \theta + \lambda n \times f A + \theta + n \times e B = b$$

$$3^{\circ}. \theta + 2\lambda n \times g A + \theta + n + \lambda n \times f B + \theta + 2n \times e C = c$$

$$4^{\circ}. \theta + 3\lambda n \times h A + \theta + n + 2\lambda n \times g B + \theta + 2n + \lambda n \times f C + \theta + 3n \times e D = d$$

$$5^{\circ}. \theta + 4\lambda n \times i A + \theta + n + 3\lambda n \times h B + \theta + 2n + 2\lambda n \times g C + \theta + 3n + \lambda n \times f D + \theta + 4n \times e E = e$$

From which, by Reduction of Equations, we obtain the following Equations, expressing the Values of A, B, C, &c. viz.

$$1^{\circ}. A = \frac{a}{\theta}$$

$$2^{\circ}. B = \frac{b - \frac{\theta + \lambda n \times f A}{\theta + n \times e}}{\theta + n \times e}$$

$$3^{\circ}. C = \frac{c - \frac{\theta + 2\lambda n \times g A - \frac{\theta + n + \lambda n \times f B}{\theta + 2n \times e}}{\theta + 2n \times e}}{\theta + 2n \times e}$$

$$4^{\circ}. D = \frac{d - \frac{\theta + 3\lambda n \times h A - \frac{\theta + n + 2\lambda n \times g B - \frac{\theta + 2n + \lambda n \times f C}{\theta + 3n \times e}}{\theta + 3n \times e}}{\theta + 3n \times e}}{\theta + 3n \times e}$$

$$5^{\circ}. E = \frac{e - \frac{\theta + 4\lambda n \times i A - \frac{\theta + n + 3\lambda n \times h B - \frac{\theta + 2n + 2\lambda n \times g C - \frac{\theta + 3n + \lambda n \times f D}{\theta + 4n \times e}}{\theta + 4n \times e}}{\theta + 4n \times e}}{\theta + 4n \times e}}{\theta + 4n \times e}$$

&c. in inf.

Hence it appears, that, if the given Ordinate $a + bz^n + cz^{2n} + dz^{3n} + ez^{4n} + \&c. \times z^{\theta-1} R^{\lambda-1}$ be suppos'd equal to the Sum of the Ordinates above; and consequently the Area of the Curve, whose Ordinate is given, equal to the Sum of the corresponding Areas, the Values of A, B, C, &c. must be those express'd by the last Equations just now mentioned. But the Sum of the Areas is $A + Bz^n + Cz^{2n} + Dz^{3n} + Ez^{4n} + \&c. \times z^{\theta} R^{\lambda}$. Wherefore, by inserting the Values of A, B, C, &c. found above, we shall have

$z^{\theta} R^{\lambda}$

$$\begin{aligned}
 z^{\theta} R^{\lambda} \times & + \frac{a}{\theta} \\
 & + \frac{b - \theta + \lambda \eta \times fA}{\theta + \eta \times e} z^{\eta} \\
 & + \frac{c - \theta + 2\lambda \eta \times gA - \theta + \eta + \lambda \eta \times fB}{\theta + 2\eta \times e} z^{2\eta} \\
 & + \frac{d - \theta + 3\lambda \eta \times bA - \theta + \eta + 2\lambda \eta \times gB - \theta + 2\eta + \lambda \eta \times fC}{\theta + 3\eta \times e} z^{3\eta} \\
 & + \frac{e - \theta + 4\lambda \eta \times iA - \theta + \eta + 3\lambda \eta \times bB - \theta + 2\eta + 2\lambda \eta \times gC - \theta + 3\eta + \lambda \eta \times fD}{\theta + 4\eta \times e} z^{4\eta} \\
 & + \text{Ec.}
 \end{aligned}$$

for the Value of the Area of the Curve, having the propos'd Ordinate

$$z^{\theta-1} R^{\lambda-1} \times a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + ez^{4\eta} + \text{Ec.}$$

124. Which Area, by putting $\frac{\theta}{\eta} = r$ or $\theta = r\eta$, $r + \lambda = s$, $s + \lambda = t$, $t + \lambda = v$, $v + \lambda = u$; and inverting the Order of all the Terms save the first, in the several Numerators, stands thus

$$\begin{aligned}
 z^{\theta} R^{\lambda} \times & + \frac{1}{r} \frac{a}{e} \\
 & + \frac{1}{r+1} \frac{b - fA}{e} z^{\eta} \\
 & + \frac{1}{r+2} \frac{c - s+1 \times fB - t gA}{e} z^{2\eta} \\
 & + \frac{1}{r+3} \frac{d - t+2 \times fC - v+1 \times gB - u bA}{e} z^{3\eta} \\
 & + \frac{1}{r+4} \frac{e - u+3 \times fD - v+2 \times gC - w+1 \times bB - u iA}{e} z^{4\eta} \\
 & + \text{Ec.}
 \end{aligned}$$

The same with the Expression of the Area delivered by our Author; save only that he hath not set down the first and last Terms of the last Numerator, viz. $\frac{1}{r}$ and $-uiA$; which I have inserted.

And the Law of Progression of the Series is manifest.

125. But that the Properties of it; and the Laws or Conditions, according to which it either terminates; or is continued infinitely, may the

the more evidently appear: and that the Expression of the Area may be wholly in known Terms, let us insert instead of the Capitals A, B, C, D, &c. their Values at length, which are easily deduced from the foregoing Equations*. Which done, the Series expanded at full Length, is

* Art. 123.
of this Ex-
plication.

$$z^0 R^1 \times + \frac{1}{r} z^0 = A.$$

$$+ \frac{1}{r+1} z^1 - \frac{f \times \frac{1}{r}}{r \times r + 1 \times e^2} z^1 = B z^1$$

$$+ \frac{1}{r+2} z^2 - \frac{s+1 \times f \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times e^2} z^2 + \frac{s+1 \times f^2 \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times e^3} z^2 - \frac{t g \times \frac{1}{r}}{r \times r + 2 \times e^2} z^2 = C z^2$$

$$+ \frac{1}{r+3} z^3 - \frac{s+2 \times f \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times r + 3 \times e^2} z^3 + \frac{s+2 \times s+1 \times f^2 \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times r + 3 \times e^3} z^3 - \frac{s+2 \times t g \times \frac{1}{r}}{r \times r + 2 \times r + 3 \times e^2} z^3 + \frac{t+1 \times g \times \frac{1}{r}}{r \times r + 1 \times r + 3 \times e^2} z^3 - \frac{t+1 \times f g \times \frac{1}{r}}{r \times r + 1 \times r + 3 \times e^3} z^3 - \frac{u h \times \frac{1}{r}}{r \times r + 3 \times e^2} z^3 = D z^3$$

$$+ \frac{1}{r+4} z^4 - \frac{s+3 \times f \times \frac{1}{r}}{r \times r + 3 \times r + 4 \times e^2} z^4 + \frac{s+3 \times s+2 \times f^2 \times \frac{1}{r}}{r \times r + 2 \times r + 3 \times r + 4 \times e^3} z^4 - \frac{s+3 \times s+2 \times s+1 \times f^3 \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times r + 3 \times r + 4 \times e^4} z^4 + \frac{s+3 \times s+2 \times s+1 \times f^4 \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times r + 3 \times r + 4 \times e^5} z^4 - \frac{s+3 \times s+2 \times t f^2 g \times \frac{1}{r}}{r \times r + 2 \times r + 3 \times r + 4 \times e^4} z^4 + \frac{s+3 \times t+1 \times f g \times \frac{1}{r}}{r \times r + 1 \times r + 3 \times r + 4 \times e^3} z^4 - \frac{s+3 \times t+1 \times f^2 g \times \frac{1}{r}}{r \times r + 1 \times r + 3 \times r + 4 \times e^4} z^4 + \frac{s+3 \times u f h \times \frac{1}{r}}{r \times r + 3 \times r + 4 \times e^3} z^4 - \frac{t+2 \times g \times \frac{1}{r}}{r \times r + 2 \times r + 4 \times e^2} z^4 + \frac{t+2 \times s+1 \times f g \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times r + 4 \times e^3} z^4 - \frac{t+2 \times s+1 \times f^2 g \times \frac{1}{r}}{r \times r + 1 \times r + 2 \times r + 4 \times e^4} z^4 + \frac{t+2 \times t g^2 \times \frac{1}{r}}{r \times r + 2 \times r + 4 \times e^3} z^4 - \frac{u+1 \times b \times \frac{1}{r}}{r \times r + 1 \times r + 4 \times e^2} z^4 + \frac{u+1 \times s f b \times \frac{1}{r}}{r \times r + 1 \times r + 4 \times e^3} z^4 - \frac{u i \times \frac{1}{r}}{r \times r + 4 \times e^2} z^4 = E z^4$$

3

+ &c.

The

The same with the former Series, by inserting the Values of the Capitals A, B, C, &c. And that it may be reduced to regular Series's,

begin at the first Term $\frac{1}{r} z^0$, and running through all the rest in order, set down in one Series, all these Parts of each complex Term, which will constitute a Series of geometrical Proportionals going on by the common Multiplier $\frac{z^n}{e}$, with their proper Coefficients: then begin-

ing with the first Part of the second complex Term, viz. $\frac{1}{r+1} \times \frac{z^n}{e}$, and running through all the subsequent Terms in order, setting down in another Series, all those Parts of each complex Term that will constitute a second geometrical Progression, going on by the same common Multiplier as before $\frac{z^n}{e}$, with their proper Coefficients prefix'd: then beginning

with the first Part of the third complex Term, viz. $\frac{1}{r+2} \times \frac{z^{2n}}{e}$ take out of the subsequent Terms, as before, such Parts as constitute a third geometrical Progression proceeding by the same common Multiplier still, viz. $\frac{z^n}{e}$ with their proper Coefficients: and so do continually. And thence the Area of the Curve, having the propos'd Ordinate $z^{b-1} R^{a-1} \times a + bz^n + cz^{2n} + dz^{3n} + ez^{4n} + \&c.$ shall be express'd as follows,

$$\begin{aligned}
 & z^b R^a \times + \frac{1}{r} \times \frac{z^n}{e} - \frac{f \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+1} \times \frac{z^n}{e^2} + \frac{\overline{s+1} \times f^2 \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+1 \times r+2} \times \frac{z^{2n}}{e^3} - \frac{\overline{s+2} \times \overline{s+1} \times f^3 \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+1 \times r+2 \times r+3} \times \frac{z^{3n}}{e^4} \\
 & + \frac{\overline{s+3} \times \overline{s+2} \times \overline{s+1} \times f^4 \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+1 \times r+2 \times r+3 \times r+4} \times \frac{z^{4n}}{e^5} - \&c. \\
 & + \frac{1}{r+1} \times \frac{z^n}{e} - \frac{\overline{s+1} \times f \times \frac{1}{r+1} \times \frac{z^n}{e}}{r+1 \times r+2} \times \frac{z^{2n}}{e^2} + \frac{\overline{s+2} \times \overline{s+1} \times f^2 \times \frac{1}{r+1} \times \frac{z^n}{e}}{r+1 \times r+2 \times r+3} \times \frac{z^{3n}}{e^3} - \frac{\overline{s+3} \times \overline{s+2} \times \overline{s+1} \times f^3 \times \frac{1}{r+1} \times \frac{z^n}{e}}{r+1 \times r+2 \times r+3 \times r+4} \times \frac{z^{4n}}{e^4} + \&c. \\
 & - \frac{f g \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+2} + \frac{\overline{s+2} \times f g \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+2 \times r+3} - \frac{\overline{s+3} \times \overline{s+2} \times f^2 g \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+2 \times r+3 \times r+4} \\
 & + \frac{\overline{s+1} \times f g \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+1 \times r+3} - \frac{\overline{s+3} \times \overline{s+1} \times f^2 g \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+1 \times r+3 \times r+4} \\
 & - \frac{\overline{s+2} \times \overline{s+1} \times f^2 g \times \frac{1}{r} \times \frac{z^n}{e}}{r \times r+1 \times r+2 \times r+4} +
 \end{aligned}$$

$$+ \frac{\frac{1}{n}c}{r+2} \times \frac{z^{2n}}{e} - \frac{\frac{1}{n}f \times \frac{1}{n}c}{r+2 \times r+3} \frac{z^{3n}}{e^2} + \frac{\frac{1}{n}g \times \frac{1}{n}f \times \frac{1}{n}c}{r+2 \times r+3 \times r+4} \frac{z^{4n}}{e^3} - \&c.$$

$$\begin{aligned} & - \frac{\frac{1}{n}g \times \frac{1}{n}b}{r+1 \times r+3} + \frac{\frac{1}{n}f \times \frac{1}{n}g \times \frac{1}{n}b}{r+1 \times r+3 \times r+4} \\ & - \frac{\frac{1}{n}b \times \frac{1}{n}a}{r \times r+3} + \frac{\frac{1}{n}f \times \frac{1}{n}b \times \frac{1}{n}a}{r \times r+3 \times r+4} \\ & + \frac{\frac{1}{n}g \times \frac{1}{n}f \times \frac{1}{n}b}{r+1 \times r+2 \times r+4} \\ & + \frac{\frac{1}{n}g \times \frac{1}{n}a}{r \times r+2 \times r+4} \\ & + \frac{\frac{1}{n}f \times \frac{1}{n}b \times \frac{1}{n}a}{r \times r+1 \times r+4} \end{aligned}$$

$$+ \frac{\frac{1}{n}d}{r+3} \times \frac{z^{3n}}{e} - \frac{\frac{1}{n}f \times \frac{1}{n}d}{r+3 \times r+4} \frac{z^{4n}}{e^2} + \&c.$$

$$\begin{aligned} & - \frac{\frac{1}{n}g \times \frac{1}{n}c}{r+2 \times r+4} \\ & - \frac{\frac{1}{n}b \times \frac{1}{n}c}{r+1 \times r+4} \\ & - \frac{\frac{1}{n}a}{r \times r+4} \end{aligned}$$

$$+ \frac{\frac{1}{n}i}{r+4} \times \frac{z^{4n}}{e} - \&c.$$

&c. in inf.

Which Series exhibits the Value of the Area; and is the same with the immediately preceding one; only that it is reduced into Order; and divided into so many distinct Series's: the Progression of which, when duly considered, will be manifest. For the Coefficients of the first Terms of each distinct Series, constituting the first perpendicular Row,

are $\frac{1}{n}a$, $\frac{1}{n}b$, $\frac{1}{n}c$, $\frac{1}{n}d$ &c. then the Coefficients of the subsequent perpendicular Rows, are all formed from the preceding in order: the second

second perpendicular Row from the first; the third from the second; the fourth from the third, &c. after such an easy and natural way, as will easily appear to any one that considers it. Moreover the Progression of the uppermost horizontal Rows, in each particular Series is evident.

126. And now by considering these two Forms of the Expression of the Area, *viz.* that exhibited by our Author; and the immediately preceding one, we shall be enabled to discover several Properties, and Laws or Conditions, according to which, the Series will terminate or stop, after a certain Number of Terms from the Beginning; and when it will run out infinitely: and in what Cases it may or may not converge. Of all which by and by.

127. Our Author observes * that every Ordinate of a Curve is capable of being resolv'd into a Series two different Ways: *viz.* by making the Index η either positive, or negative. Thus let the Ordinate

* Art. 28.
of the
Quadrature of
Curves.

$\frac{3k - lz^2}{z^2 \sqrt{kz - lz^3 + mz^4}}$ be proposed: it is reduced to the general Form men-

tioned in this Prop. 1^o. by bringing the Quantity $z^2 \sqrt{kz - lz^3 + mz^4}$ or $z^2 \times \sqrt{kz - lz^3 + mz^4}^{\frac{1}{2}}$ out of the Denominator into the Numerator: which is done by changing the Signs of the Indexes of the Factors z^2 and $\sqrt{kz - lz^3 + mz^4}^{\frac{1}{2}}$ into their Opposites: by which the Ordinate is express'd thus $z^{-2} \times \sqrt{kz - lz^3 + mz^4}^{-\frac{1}{2}} \times 3k - lz^2$.

2^o. By purging the radical Expression of $z^{-\frac{1}{2}}$: which is done by dividing it by $z^{-\frac{1}{2}}$, and multiplying the other Factor z^{-2} by $z^{-\frac{1}{2}}$, which two Operations compensate each other: for the radical Expression must be made as simple as may be: and so the whole Expression of the Ordinate propos'd, is reduc'd to this $z^{-\frac{5}{2}} \times \sqrt{k - lz^2 + mz^3}^{-\frac{1}{2}} \times 3k - lz^2$;

or rather $z^{-\frac{5}{2}-1} \times \sqrt{k - lz^2 + mz^3}^{\frac{1}{2}-1} \times 3k - lz^2$, capable of being compared with the general Form $z^{\theta-1} R^{\lambda-1} \times$

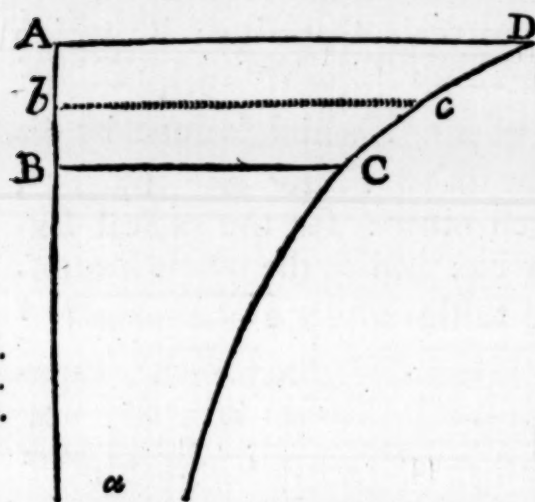
$a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.$ i. e. $z^{\theta-1} \times e + fz^{\eta} + gz^{2\eta} + bz^{3\eta} + \&c.$ $^{\lambda-1}$
 $\times a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.$ This is one Form of the Ordinate, where the Indexes of z , in the two Factors $\sqrt{k - lz^2 + mz^3}^{-\frac{1}{2}}$ and $3k - lz^2$ are positive. But it may be brought to another Form, in which these Indexes are negative, *viz.* thus: divide the radical or surd Factor $\sqrt{k - lz^2 + mz^3}^{-\frac{1}{2}}$ by $z^{\frac{1}{2}}^{-\frac{1}{2}}$, and multiply the other Factor $z^{-\frac{5}{2}}$ by the same $z^{\frac{1}{2}}^{-\frac{1}{2}}$ or $z^{-\frac{1}{2}}$, and then the Ordinate stands thus $z^{-4} \times \sqrt{m - lz^{-1} + kz^{-3}}^{-\frac{1}{2}} \times 3k - lz^2$: further, divide the Factor $3k - lz^2$ by z^2 , and multiply z^{-4} by the same z^2 , and so you'll

O

have

have $z^{-2} \times m - lz^{-1} + kz^{-3} \sqrt{-1 - 3kz^{-2}}$, or, which is the same, $z^{-1-1} \times m - lz^{-1} + kz^{-3} \sqrt{-1 - 3kz^{-2}}$, where the Indexes of the Powers of z in the two last Factors, which were formerly positive, are made negative. And after the like manner, every Ordinate of a Curve may be express'd after these two different ways. Accordingly, when any Ordinate is proposed, it is to be reduced to these two different Forms: and both Cases are to be tried. Which is done by applying the general Form of the Area, exhibited in this Prop. to the particular Case proposed, in such manner as our Author shews, viz. by substituting the particular Values instead of the corresponding general Expressions. By doing of which, the Series expressing the Area, puts on two different Forms: one or both of which will terminate or become finite, by the Terms vanishing at length; or else both of them will run out into infinite Series's. If one or both of the Expressions of the Area terminate, you have the Area exactly, and so the Curve is quadrable: but if both the Expressions of the Area run out infinitely, one of them will converge, and give the Area by Approximation, when the Curve is not quadrable.

128. Let DCa be a Curve, describ'd by the Ordinate BC moving perpendicularly along the Absciss AB .



*Art. 127.
of this Ex.
plication.

Let $AB = z$, $BC = y$, and a , a given Line: and let the Equation to the Curve be $y = \frac{a}{1+z}$. This Ordinate brought to due Form is $z^0 \times 1 + z^{-1} \times a$ or $z^{1-1} \times 1 + z^{-1-1} \times a$, or yet, by the Method shewn above*, $z^{-2} \times 1 + z^{-1} \times a$, i.e. $z^{-1-1} \times 1 + z^{-1-1} \times a$. By comparing the first Form of the Ordinate, viz. $z^{1-1} \times 1 + z^{-1-1} \times a$

with the general Form $z^{\theta-1} R^{\lambda-1} \times a + bz^{\eta} + \mathcal{E}c$. we find $\theta = 1$, $\eta = 1$. $\lambda = -1$. $r = (\frac{\theta}{\eta}) = 1$. $s = (r + \lambda) = 0$. $e = 1$. $f = 1$. $g = 0 = b$, $\mathcal{E}c$. $a = a$. $b = 0 = c$, $\mathcal{E}c$. Whence, by Substitution,

the general Expression for the Area, $z^{\theta} R^{\lambda} \times \frac{1-a}{re} + \frac{1-b-fA}{r+1 \times e} z^{\eta} + \mathcal{E}c$.

becomes $z \times 1 + z^{-1} \times a$ or $\frac{az}{1+z}$: All the Terms except the first vanishing.

nothing. But the other Form of the Ordinate proposed is $z^{-1-1} \times \frac{1}{1+z^{-1}}^{-1-1} \times a$: which compar'd with the general Form, gives $\theta = -1 \cdot \eta = -1 \cdot \lambda = -1 \cdot r = (\frac{\theta}{\eta} =) 1 \cdot s = (r + \lambda =) 0$, the rest as formerly. Whence the Area, corresponding to this Form of the Ordinate, will be found, by a due Substitution of Values, in the general Expression for the Area, $z^{-1} \times \frac{1}{1+z^{-1}}^{-1-1} \times -a$, or $\frac{-a}{1+z}$: all the Terms save the first vanishing in this Case also.

129. Hence it appears that we obtain two different Areas, both belonging to one and the same Curve DCa . But the first $\frac{az}{1+z}$ is the Area $ABCD$ lying above the Ordinate BC , and comprehended betwixt it and the same Line AD drawn parallel to BC through the Beginning of the Absciss A : whereas the other Area, viz. $\frac{-a}{1+z}$ is represented by the Space aBC , lying below the Ordinate BC , and adjacent to the Absciss infinitely produced towards a . Now that there is nothing strange in this; but on the contrary, that it is what ought naturally to happen, will appear, if we consider that the two Areas $ABCD$ and aBC have one and the same common Fluxion: because the Velocity with which the one flows by Increase, is the same with that whereby the other flows by Decrease: or the positive Fluxion of the one is the negative Fluxion of the other; accordingly while the one increases, the other decreases, and contrarily. And since a positive Fluxion produces a positive Fluent; and a negative Fluxion, a negative Fluent, hence $AB (=z)$ and consequently the Area $ABCD$, being both supposed to flow by Increase, that Area is represented positive, being $\frac{az}{1+z}$; but the other aBC is represented negative, being $\frac{-a}{1+z}$. And so it always happens, that two Areas, terminating at the same Ordinate of a Curve, and situate upon different Sides of that Ordinate; but upon the same Side of the Absciss or Base, are affected with opposite Signs. And therefore since the Area above the Ordinate, because of it's positive Fluxion is positive; the Area below the Ordinate, if it lie upon the same Side of the Absciss produc'd, must be negative.

130. There are several other Things respecting the Position and true Values of curvilinear Areas, of which more afterwards. Let it suffice just now to shew how one may determine the initial Limit of any Area, or the Term whence it commences. To do this, all that is necessary, is to suppose the Expression of the Area equal to nothing, and the Value of the Absciss z , agreeing to that Supposition, will shew the initial Limit.

Limit. Thus in the Example above, the first of the two Areas was $\frac{az}{1+z}$: wherefore put $\frac{az}{1+z} = 0$: hence $az = 0$, *i. e.* $z = 0$, since a is a given Quantity: therefore, when the Area $\frac{az}{1+z}$ is nothing, the Absciss z is nothing; and in no other Circumstances can the Expression $\frac{az}{1+z}$ be nothing, but when the Absciss z vanishes. And this shews that the initial Limit of the Area is at the Beginning of the Absciss, or is to be computed from the Line AD, drawn through A, parallel to BC: whence it follows that it is the Area ABCD. Again put the other Expression of the Area, *viz.* $\frac{-a}{a+z} = 0$: and in that Case, z must be infinitely great, since otherwise the Expression $\frac{-a}{a+z}$ could not be equal to nothing: which shews that the Area designed thereby, lies along the Absciss infinitely produced below the Ordinate BC: and therefore denotes the Area aBC , as was said.

131. Now, although the initial Limit, from whence any curvilinear Area commences, may be different from either of these; and fall in any other Place; yet wherever it falls, it may be determin'd the same way. Thus taking the Area $-2\sqrt{\frac{k-lz^2+mx^3}{z^3}}$, which is shewn by our

Author to belong to the Curve, whose Ordinate is $\frac{3k-lz^2}{z^2\sqrt{kz-lz^3+mx^4}}$: put

$-2\sqrt{\frac{k-lz^2+mx^3}{z^3}} = 0$, and thence you'll have this Equation $k-lz^2+mx^3=0$, by the Construction of which cubical Equation the Length of z the Absciss is determined, through the Extremity of which an Ordinate being drawn gives the initial Limit of the Area. And thus far with respect to the Effect of the two-fold Expression or Form of the Ordinate, in determining the initial Limit and Position of the Area of the Curve, in order to explain and illustrate what is said by our Author in Art. 28 *.

* Of the
Quadrature
of
Curves.

* Art. 128.
of this Ex-
plication.
* Art. 28.
of the
Quadrature
of
Curves.

132. Therefore when an Ordinate of any Curve is proposed, both Expressions or Forms of the Ordinate must be tried: and sometimes, both the corresponding Areas will be finite, and exactly determined, as in the Example above *: sometimes, one of them only can be exactly found, as in the Example adduced by Sir Isaac *. One or other of these will happen, if the Curve be quadrable, *i. e.* admit of a geometrical Comparison with a rectilinear Figure. But if the Curve be not quadrable, both Expressions of the Area will run out into an infinite Series: yet one or other of them will converge, and thereby approximate to the Value

Value of the Area. To which notwithstanding there are some Exceptions. 1°. When the Value of r in both Series's expressing the Area is 0 or a negative Integer; although it be not 0 in both; nor a negative Integer in both at the same Time. For since $r, r+1, r+2, r+3, \&c.$ are Factors in the Denominators of the Terms of the Series expressing the Area of the Curve, according to the Author's Form of it*, * Art. 26. of the Quadrature of Curves. hence it follows that if $r=0$ or a negative Integer, some Term must become infinitely great, the Denominator vanishing: and therefore if this be the Case with respect to both Forms of the Series, both that in which the Powers of z have positive Exponents; and that also in which they have negative Exponents, the Value of the Area in both Cases is infinitely great, *i.e.* the Area lying on both Sides of the Ordinate. Whereas if r be 0 or a negative Integer in one of the Cases only, you conclude from this that the Area upon the one Side of the Ordinate is infinite; but finite upon the other: which must be observed, lest the Ambiguity in the Manner it is express'd by Sir Isaac, should lead one into a Mistake. Thus taking the Equation to the equilateral Hyperbola $\frac{1}{1+z} = y$: which reduced to Form is either $z^{1-1} \times 1+z^{0-1} = y$, or $z^{0-1} \times 1+z^{-1} = y$; by comparing the last of which with the general Form of Ordinates $z^{\theta-1} \times e + fz^{\eta} = y$, you have $\theta=0=\lambda$, and therefore $r = (\frac{\theta}{\eta}) = 0$: whence you conclude that the Area lying along the Absciss infinitely produced beyond the Ordinate, is infinitely great. Whereas, since in the former Case it is $\theta=1=\eta$, hence $r = (\frac{\theta}{\eta}) = 1$, and therefore the Area adjacent to the Absciss and terminated at the Ordinate is finite, and so, according to this Prop. is thus express'd $z \times 1 - \frac{1}{2}z + \frac{1}{3}z^2 - \frac{1}{4}z^3 + \&c.$ in *inf.* or $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 \&c.$ which converges, and approximates to the Value of the Area, if z do not exceed 1; otherwise not. But it must be observed that this *Exception* must be taken with this Limitation, *viz.* That the Term whose Denominator vanishes, have not it's Numerator equal to 0 at the same Time: for if that should happen, the whole Term vanishes, and the Area is finite. There is another Exception mentioned by our Author against the general Rule about the Convergency of the Series by which the Area is express'd: which is this, if $\frac{z^{\eta}}{e} = 1$ the Series does not converge. In this Case the Powers of $\frac{z^{\eta}}{e}$ are all the same, *viz.* 1, and therefore by observing the Form and Progression of the Series in Art. 125*, * Of this Explication. where $\frac{z^{\eta}}{e}$ is the common Multiplier, in producing the last Factor of each

each Term from that of the preceding, it appears that the Series cannot converge, unless it be by means of the Coefficients, which seldom happens: but that this sometimes happens in the present Case, we shall demonstrate just now: and therefore this Exception must be taken with some Limitation: as likewise must that which follows in this Art. *viz.* That if $\frac{z^n}{e}$ be less than 1, the Series will converge in which the Index n is positive; but if $\frac{z^n}{e}$ be greater than 1, the other Series will converge. For, that this will not always hold, will appear by the following Example. Let there be given this Equation $y = \frac{1}{e + fz} = z^{1-1} \times e + fz^{0-1}$: by comparing which with the general Expression $z^{\theta-1} \times e + fz^{\lambda-1}$, you have $\theta = 1$, $\eta = 1$, $\lambda = 0$, $r = (\frac{\theta}{\eta}) = 1$, $s = 1$: whence the Series for the Area is $\frac{z}{e} \times 1 - \frac{1}{2} \times \frac{fz}{e} + \frac{1}{3} \times \frac{f^2 z^2}{e^2} - \frac{1}{4} \times \frac{f^3 z^3}{e^3} + \frac{1}{5} \times \frac{f^4 z^4}{e^4} - \&c. in inf.$ where, if we suppose $\frac{fz}{e}$ to be any thing greater than Unity, the Series will not converge, whatever the Value of $\frac{z}{e}$ be. Thus suppose $f = 3$, $e = 2$, $z = 1$, and therefore $\frac{fz}{e} = \frac{3}{2}$: then the Series for the Area

will be $\frac{1}{2} \times 1 - \frac{1}{2} \times \frac{3}{2} + \frac{1}{3} \times \frac{9}{4} - \frac{1}{4} \times \frac{27}{8} + \frac{1}{5} \times \frac{81}{16} - \&c. in inf.$ where, it's evident, the Value of the Terms after the third, continues to increase still more and more, and therefore the Series doth not converge although $\frac{z^n}{e} = \frac{z}{e} = \frac{1}{2}$ be less than 1. 2°. If it be $\frac{z}{e} = 1$, yet if f be any thing less than 1, the Series will converge by Art. 119,

* Of this
Explica-
tion.

121 *. for in that Case $\frac{fz}{e} < 1$. And so having shewn by this Example that the Exception, and Rules mentioned by our Author, with respect to the Value of $\frac{z^n}{e}$ in determining the Convergency or Non-convergency of the Series, must be taken with some Limitation, I shall here shew how you may determine when the Series for the Area of a binomial Curve will not converge; and when it will converge.

133. In the Series for the Area of a binomial Curve, if $\frac{fz^n}{e}$ be greater than 1, the Series in which the Exponents of the Powers of z are positive cannot converge, whatever be the Value of $\frac{z}{e}$. For the Series for binomial Curves, putting A, B, C, D, &c. for the first, second, third, fourth, &c. Terms (and not for their Coefficients) with their proper Signs, stands thus $z^{\theta} R^{\lambda} \times$

$$\frac{1}{r} - \frac{s}{r+1} \times \frac{fz^\eta}{e} A - \frac{s+1}{r+2} \times \frac{fz^\eta}{e} B - \frac{s+2}{r+3} \times \frac{fz^\eta}{e} C - \frac{s+3}{r+4} \times \frac{fz^\eta}{e} D - \&c. in inf.$$

Now, if you suppose s to be less than $r + 1$, then it's evident, that, since the Coefficients $\frac{s}{r+1}, \frac{s+1}{r+2}, \frac{s+2}{r+3}, \&c.$ go on continually by the Addition of Units to the Numerator and Denominator, some Coefficient must at length become a proper positive Fraction, and always continue after that to increase, since s is less than $r + 1$ (where a negative Quantity is considered as less than 0, or any positive Quantity, or yet a less negative) because the Difference continuing the same, the Ratio of the less to the greater will increase: therefore it's evident that if $\frac{fz^\eta}{e} > 1$, the Coefficient multiplied into $\frac{fz^\eta}{e}$, will at length exceed 1; and so the Terms, after that, must increase: but if s be greater than $r + 1$, then it's evident that whether $\frac{fz^\eta}{e}$ be greater than 1; or equal to 1, the Value of the Terms must increase as the Series proceeds; and therefore whatever be the Values of r and s , the Series in which the Exponents of the Powers of z are positive cannot converge, if it be $\frac{fz^\eta}{e} > 1$.

134. In any binomial Curve thus generally express'd $ax^{\theta-1} \times [e + fz^\eta]^{\lambda-1} = y$, if the Quantity $\frac{fz^\eta}{e}$ be less than 1, the Series for the Area in which the Exponent η is affirmative, will converge: but if $\frac{fz^\eta}{e}$ be greater than 1, the other Series, in which η is negative. For the Series for the Areas of Curves, by applying it to the Case of bino-

mial Curves, may stand thus $\frac{ax^\theta}{\eta e} R^\lambda \times 1 - \frac{s}{r+1} \times \frac{fz^\eta}{e} + \frac{s \times s+1}{r+1 \times r+2} \times \frac{fz^\eta}{e}^2$
 $- \frac{s \times s+1 \times s+2}{r+1 \times r+2 \times r+3} \times \frac{fz^\eta}{e}^3 + \frac{s \times s+1 \times s+2 \times s+3}{r+1 \times r+2 \times r+3 \times r+4} \times \frac{fz^\eta}{e}^4 - \&c.$ where, if it be $s < r + 1$, in the Sense already mentioned, the Coefficients, after a certain Number of Terms from the Beginning, must continually diminish (as is evident) and therefore the Series, in this Case, will converge by Lemmas 2. and 4. preceding. Again, if it be $s > r + 1$, and you take n an integral Number not less than $\frac{sfz^\eta - r+1 \times e}{e - fz^\eta}$, the Terms of the Series after the $n + 2$ Place shall continually diminish faster and faster, or in a greater Proportion continually; and therefore the Series shall converge. For calling P the Term which immediately

diately precedes the $n + 2$ Term, and Q the next: from the Nature of the Series, you'll have $Q = \frac{s+n}{r+n+1} \times \frac{fz^n}{e} P$: where, by inserting for n , the Quantity $\frac{fz^n - r + 1 \times e}{e - fz^n}$, and reducing, you'll have $Q = \left(\frac{s-r-1 \times e f z^n}{s-r-1 \times e f z^n} P \right) = P$; consequently if n be taken greater, Q will be less than P : so that hence it appears that the Value of the Terms after this constantly diminishes: and it is moreover evident that they must diminish still in a greater Ratio, since $\frac{s+n}{r+n+1} > \frac{s+n+1}{r+n+2}$: wherefore, in this Case also, the Series will converge. But finally if it be $s = r + 1$, then the Curve is quadrable: for $\lambda = 1$, since $s = r + \lambda$.

Again, if it be $\frac{fz^n}{e} > 1$, I say the Series will converge in which the Powers of z are negative: for since $\frac{fz^n}{e} > 1$, it must be $\frac{ez^{-n}}{f} < 1$, and therefore, since the Powers of $\frac{ez^{-n}}{f}$ which is less than 1, run through the Terms of the Series, it appears by Lemma 2, and 4. above, and what has been just now demonstrated, that the Series must converge. Therefore in binomial Curves, which are not quadrable, whether $\frac{fz^n}{e}$ be less than nothing or greater than nothing, one or other of the Series's for the Area will converge: and if it be $\frac{fz^n}{e} < 1$, the Convergence will be the quicker (*cæteris paribus*) the less $\frac{fz^n}{e}$ is; but if $\frac{fz^n}{e} > 1$, the quicker, the greater $\frac{fz^n}{e}$ is.

There is a third Case, *viz.* when $\frac{fz^n}{e} = 1$. In which Case the Convergency depends entirely upon the Coefficients: therefore if $s > r + 1$, or, which is the same, if $\lambda > 1$, neither of the Series's for the Area will converge, because the Quantity $\frac{s+n}{r+n+1}$, by which the preceding Term of the Series is multiplied, becomes greater than Unity. But if $s < r + 1$, or which is the same, $\lambda < 1$, both the Series's for the Area will converge; for the Quantity $\frac{s \times s+1 \times s+2 \times s+3, \&c.}{r+1 \times r+2 \times r+3 \times r+4, \&c.}$ must continually decrease within a finite Number of Terms, and become less than any given Quantity, and that in both Forms of the Series:

Series: and the Terms being affected with the Signs $+$ and $-$ alternately, each Term will be greater than all the subsequent Part of the Series. For let $a - b + c - d + e - f$, &c. represent a Series of an infinite Number of Terms, which continually diminish, and are alternate, the Amount of the Series is $a + c + e$, &c. $- b - d - f$, &c. now $a + c + e$, &c. $> b + d + f$, &c. but $b + d + f$, &c. $> c + e$, &c. therefore $a > a - b + c - d + e - f$, &c. in inf. Lastly, if $\lambda = 1$, the Curve is quadrable.

135. In Art. 29. our Author desires to remark, that, if the Ordinate of a Curve be contain'd under a rational Factor, which he denotes by Q , and another irreducible surd Factor, denoted by R^π ; and the Quantity under the radical Sign, viz. R , do not divide the rational Factor Q , then the Expression of the Ordinate, remaining as it stands; $\lambda - 1$, the Index of R in the general Form of Ordinates, viz. $z^{\lambda-1} R^{\lambda-1}$

Art. 29—
31. of the
Quadrature of
Curves.

$\times a + bz^n + cz^{2n} + \&c.$) must be put equal to π : so that $R^{\lambda-1} = R^\pi$. For in this Case, the Expression of the given Ordinate cannot be rendered more simple. And so in the Application of the general Theorem, we'll have $\lambda = \pi + 1$. But if Q can be divided by R , let it be divided, and then put $\lambda - 1 = \pi + 1$, or $R^{\lambda-1} = R^{\pi+1}$, so that $\lambda = \pi + 2$. Again, $Q \times R^\pi$ expressing any Ordinate of a Curve, if R divide Q twice, thrice, &c. after Q is divided twice, thrice, &c. respectively by R , you must put $\lambda - 1 = \pi + 2$, $\lambda - 1 = \pi + 3$, &c. respectively; or $R^{\lambda-1} = R^{\pi+2}$; $R^{\lambda-1} = R^{\pi+3}$, &c. This Operation reduces the Expression of the Ordinate to due Form; but it is evident it alters not it's Value: for $Q \times R^\pi = \frac{Q}{R} \times R^{\pi+1} = \frac{Q}{R^2} \times R^{\pi+2} = \frac{Q}{R^3} \times R^{\pi+3}$, &c.

Thus, y and z standing for the Ordinate and Absciss, let it be $y = \sqrt{a^2 - z^2} \times \sqrt{a + z}^{-\frac{1}{2}}$; here $Q = a^2 - z^2$, $R = a + z$, $\pi = -\frac{1}{2}$ and $R^\pi = \sqrt{a + z}^{-\frac{1}{2}}$: and in regard $a^2 - z^2$ can be divided by $a + z$ once, I divide $a^2 - z^2$ by $a + z$, and it quotes $a - z$, but then to compensate that, I increase the Index of the Surd $\sqrt{a + z}^{-\frac{1}{2}}$, by Unity: and so the Expression $y = \sqrt{a^2 - z^2} \times \sqrt{a + z}^{-\frac{1}{2}}$ is reduced to this $\sqrt{a - z} \times \sqrt{a + z}^{-\frac{1}{2} + 1} = \sqrt{a - z} \times \sqrt{a + z}^{\frac{1}{2}} = \sqrt{a^2 - z^2}$, and so $\lambda = \frac{3}{2}$. Which Alteration being made upon the Form of the Ordinate, the Area is found by this Proposition. And so in other Cases.

* Of the
Quadrature of
Curves.

* Art. 139.
of this Ex-
plication.

136. At Art. 30 *. our Author puts another Case, viz. that the Ordinate of a Curve proposed, is a rational irreducible Fraction, having it's Denominator compounded of two or more Terms. In which Case, the Denominator of the Fraction must be resolved into it's prime or in-composite Divisors¹: this done, either there must be some prime Divisor that has no Fellow, the same with it; or then each prime Divisor must have at least one Fellow equal to it. If the first be the Case, it's an Evidence the Curve cannot be squared (the Demonstration of which shall be given by and by *:) but when the last happens, you must reject one prime Divisor of each Set or Kind: this done, the Divisor, which remains; or the Product contained under all the remaining Divisors, when there are more left, is to be put for R, in the general Expression of the Ordinate, and -2 for $\lambda - 1$ or R^{-2} for $R^{\lambda-1}$, and the Ordinate proposed, must be reduced to a Fraction having R^2 for it's Denominator: except it should happen, that the Product contained under the remaining prime Divisors, be a true Square, or Cube, or Biquadrate, &c. in which Case, the corresponding Square-root, or Cube-root, or Biquadratic-root, is to be put for R, and the Indexes made negative, viz. -2 , or -3 , or -4 , &c. put for λ respectively; or $R^{-3} = R^{\lambda-1}$, $R^{-4} = R^{\lambda-1}$, $R^{-5} = R^{\lambda-1}$, &c. and then the Ordinate is to be reduced to a Fraction having R^3 , or R^4 , or R^5 , &c. for it's Denominator. All which, with the Reason of it will be very plain by the following Examples.

137. Let $\frac{z^5 + z^4 - 8z^3}{z^4 + z^3 - 5z^2 - z + 8z - 4}$ be an Ordinate of a Curve, which is a rational irreducible Fraction; and the Example proposed by the Author. The prime Divisors of the Denominator are $z - 1$, $z - 1$, $z - 1$, one Set; $z + 2$, $z + 2$ another: here there is no prime Divisor wanting a Fellow. Now 'tis evident that the Ordinate proposed is equivalent to $\frac{1}{z-1 \times z+2} \times \frac{z^5 + z^4 - 8z^3}{z-1 \times z-1 \times z+2}$, where the Denominator of the first Factor is made up of the several prime Divisors, which, according to the Rule, are to be rejected: and the Denominator of the second Factor, viz. $z-1 \times z-1 \times z+2$, i. e. $z^3 - 3z + 2$ is to be put for R, and $\frac{1}{z-1 \times z-1 \times z+2} = \frac{1}{z^3 - 3z + 2}$ or $\frac{1}{R^2}$ for $R^{\lambda-1}$, i. e. $z^3 - 3z + 2$ for $R^{\lambda-1}$. Wherefore, the proposed Ordinate must be reduced to the Denominator $z-1 \times z-1 \times z+2$

¹ For doing this see *Arith. Universalis*, cap. de Inventione Divisorum.

$= z^3 - 3z + 2$. To do which, without altering the Value of the Expression $\frac{z^5 + z^4 - 8z^3}{z^5 + z^4 - 5z^3 - z^2 + 8z - 4}$, you must multiply the Numerator $z^5 + z^4 - 8z^3$ by such Divisor (or the Product of such Divisors, if there were more than one) as is supernumerary in the Denominator $z - 1 \times z - 1 \times z + 2$ or $z^3 - 3z^2 + 2$ above those in the Denominator $z - 1 \times z + 2$, which were to be rejected: here there is only one such supernumerary Divisor, viz. $z - 1$: and so the Ordinate proposed stands in this Form $\frac{z^6 - 9z^4 + 8z^3}{z^3 - 3z + 2}$, or $z^3 \times \frac{z^3 - 3z + 2}{z^3 - 3z + 2} \times 8 - 9z + z^3$.

For $\frac{z^5 + z^4 - 8z^3}{z^5 + z^4 - 5z^3 - z^2 + 8z - 4} = \frac{1}{z - 1 \times z + 2} \times \frac{z^5 + z^4 - 8z^3}{z - 1 \times z - 1 \times z + 2} = \frac{z - 1}{z - 1 \times z - 1 \times z + 2} \times \frac{z^5 + z^4 - 8z^3}{z - 1 \times z - 1 \times z + 2} = \frac{z^6 - 9z^4 + 8z^3}{z^3 - 3z + 2}$
 $= z^3 \times \frac{z^3 - 3z + 2}{z^3 - 3z + 2} \times 8 - 9z + z^3$, or, to bring it exactly to the general Form of Ordinates, $z^{4-1} \times 2 - 3z + z^3$. And after the same Manner, may any other Ordinate, having the Properties mentioned, be reduced to due Form, and so be compared with the general Form of Ordinates, viz. $z^{\theta-1} R^{\lambda-1} \times a + bz^n + cz^{2n} + \&c.$ or $z^{\theta-1} \times (e + fz^n + gz^{2n} + \&c.)^{\lambda-1} \times a + bz^n + cz^{2n} + \&c.$ and by a due Substitution of Values, the Area found. Particularly, in the present Example, we will have $\theta = 4 . \eta = 1 . \lambda = -1 . r = (\frac{\theta}{\eta} =) 4 . s = (r + \lambda =) 3 . t = (s + \lambda =) 2 . v = (t + \lambda =) 1 . e = 2 . f = -3 . g = 0 . b = 1 . a = 8 . b = -9 . c = 0 . d = 1$. Which Values being inserted in the Series for finding the Areas of Curves, viz.

$z^{\theta} R^{\lambda} \times \frac{1}{re} + \frac{1}{r+1} \times \frac{b - sfA}{e} z^n + \frac{1}{r+2} \times \frac{c - s + 1 \times fB - tgA}{e} z^{2n} + \&c,$ it comes out $z^4 \times 2 - 3z + z^3$ $\times \frac{8}{8} + \frac{-9+9}{10} z + \frac{0}{12} z^2 + \frac{1-1}{14} z^3 + 0$
 $= \frac{z^4}{2-3z+z^3}$: for since the second, third and fourth Terms are each equal to nothing, i.e. $B=0=C=D$, and the Factors $2-3z+z^3$ and $8-9z+z^3$ only Quadrinomials, it will be evident, by considering the Progreffion of the Series, that all the subsequent Terms

in infinitum, must vanish, and therefore the Area of the Curve is

$$\frac{z^4}{2-3z+z^3}$$

N. B. It is to be observed in this Example that the Factors $2-3z+z^3$ and $8-9z+z^3$, belonging to the Expression of the Ordinate, are to be considered as Quadrinomials, and not Trinomials: for the Name of the Quantity under the Vinculum, depends upon the Number of Terms, in which the Indexes of the Powers of z , constitute an arithmetical Progression: and thus these two Factors are to be considered as standing in this Form $2-3z+0z^2+z^3$ and $8-9z+0z^2+z^3$: for the Exponents of the Powers of z , viz. 0, 1, 2, 3, &c. in the general Form of the Ordinate $z^{\theta-1} \times e + fz^{\eta} + gz^{2\eta} + bz^{3\eta} + \&c. \lambda^{-1} \times a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.$ constitute an arithmetical Progression.

138. But let us next suppose the Ordinate of a Curve to be $\frac{2+3x}{x^5+3x^6+3x^7+x^8}$, a rational irreducible Fraction: the prime Divisors of whose Denominator are $1+x$, $1+x$, $1+x$ one sort; and x , x , x , x , x another, and therefore no prime Divisor without a Fellow: reject one prime Divisor of each sort, and the Quotient is $1+x \times 1+x \times x \times x \times x \times x = x^4 + 2x^5 + x^6$ a true Square, having x^2+x^3 for it's Root; therefore, I put x^2+x^3 for R , x^2+x^3 for R^{λ} ; but x^2+x^3 for $R^{\lambda-1}$, viz. by reducing the Ordinate to the Denominator x^2+x^3 . The Manner of which Reduction, and bringing the proposed Ordinate to due Form, appears thus.

$$\begin{aligned} \frac{2+3x}{x^5+3x^6+3x^7+x^8} &= \frac{2+3x}{1+x \times 1+x \times 1+x \times x \times x \times x \times x} = \frac{1}{1+x \times x} \times \\ &\frac{2+3x}{1+x \times 1+x \times x \times x \times x \times x} = \frac{1}{1+x \times x} \times \frac{2+3x}{1+x \times x^2 \times 1+x \times x^2} = \frac{x}{1+x \times x^2} \\ &\times \frac{2+3x}{1+x \times x^2 \times 1+x \times x^2} = \frac{x \times 2+3x}{1+x \times x^2} = \frac{x \times 2+3x}{x^6 \times 1+x} = x^{-5} \times 1+x^{-3} \end{aligned}$$

$\times 2+3x$ or $x^{-4-1} \times 1+x^{-2-1} \times 2+3x$, which is equal in Value to the proposed Ordinate $\frac{2+3x}{x^5+3x^6+3x^7+x^8}$; but brought to due Form, so as to be capable of being compared with the general Form of Ordinates, viz. $z^{\theta-1} R^{\lambda-1} \times a + bz^{\eta} + cz^{2\eta} + \&c.$: by comparing of which, we shall have $\theta = -4$, $\eta = 1$, $\lambda = -2$, $r = -4$, &c. Here we need proceed no further in the Comparison, since r is a negative

negative Integer: for this is an Evidence that the Area arising from this Form of the Ordinate is infinite*. Wherefore according to what was shewn at Art. 127* I convert the Ordinate $x^{-5} \times 1 + x^{-3} \times 2 + 3x^{-1}$ into the other Form, in which the Exponents of the Powers of x under the Vinculums, are negative: and it becomes $x^{-7} \times 1 + x^{-5} \times 3 + 2x^{-3}$ or $x^{-6-1} \times 1 + x^{-1-1} \times 3 + 2x^{-1}$: by comparing which with the general Form, we have $\theta = -6$. $\eta = -1$. $\lambda = -2$. $r = 6$. $s = 4$. $t = 2$. $v = 0$. $e = 1$. $f = 1$. $g = 0$, &c. $a = 3$. $b = 2$. $c = 0$, &c. Hence, by inserting these Values in the Series for Areas, we find $x^{-6} \times 1 + x^{-4} \times 3 + 2x^{-2} \times -\frac{1}{2} = -\frac{1}{2x^4} + \frac{3}{2x^2} + 2x^0$ for the Area of the Curve whose Ordinate was propos'd, all the Terms save the first vanishing. And you are to proceed in all other Cases of the like Nature.

139. But it was said above*, that, when the Denominator of a rational irreducible Fraction, expressing the Ordinate of a Curve, viz. in Terms made up of the Absciss and known Quantities, is resolved into it's constituent prime Divisors¹, if there be any of these prime Divisors without a Fellow equal to it, the Curve is not quadrable: the Demonstration of which I put off to this Place. And now in order to demonstrate it: I suppose it is easily understood, that, if any Fluent, be a rational Fraction, including one flowing Quantity only x , having it's Fluxion $\dot{x} = 1$, the Fluxion will be a rational Fraction: and if any fluxionary Expression, including only one flowing Quantity x and it's Fluxion $\dot{x} = 1$, be a rational Fraction, it's Fluent, if finite, must be a rational Fraction also; for the Fluxion of a Surd can never be a rational Quantity; nor the Fluxion of a rational Integer, be a Fraction. Which Things are easily deduced from what was shewn on Prop. 1.

Wherefore let $\frac{N}{D}$ be a rational irreducible Fraction expressing the Area of a Curve, which therefore is quadrable: then $\frac{\dot{N}D - N\dot{D}}{D^2}$ is the Fraction of the curvilinear Area $\frac{N}{D}$, and consequently the Value of the Ordinate when the Fluxion of the Absciss is Unity; and must be a rational fractional Quantity. Now in the Expression $\frac{\dot{N}D - N\dot{D}}{D^2}$, every prime Divisor of the Denominator D^2 must be capable of dividing it

* Art. 133. of this Explication.
* Of this Explication.

* Art. 30. of the Quadrature of Curves.
Art. 136. of this Explication.

¹ By prime Divisors here and in what follows in this Art. is meant such as include the Absciss.

some

The Quadrature of CURVES explained.

some even Number of Times, since it is a Square, and of dividing D half that Number of Times, and D just once less than half the Number of Times, by Lemma third. Let P represent any such prime Divisor: then if P divide D^2 but twice, it won't divide the Numerator at all: because P divides ND the first Part of the Numerator, since it divides D once; but it won't divide the other Part of the Numerator, since it can't divide D, by Lemma 3, nor N by Hyp.; N and D being prime to each other, and so not ND; therefore it's plain, that P won't divide the Numerator ND — ND at all, when it divides the Denominator D^2 but twice. Again, if P divide D^2 four Times, it can divide ND — ND only once: for it divides ND twice and — ND but once, for the same Reason as before, therefore it can divide ND — ND only once, when it divides D^2 four Times: and thus by the same way of reasoning, whatever Number of Times any prime Divisor divides the Denominator D^2 , it will divide the Numerator ND — ND just once less than half that Number of Times. Whence it plainly follows that when $\frac{ND - ND}{D^2}$ is in it's lowest Terms, there can be no prime Divisor in the Denominator without a Fellow equal to it. Therefore if the Ordinate of a Curve be a rational irreducible Fraction, which contains any prime Divisor in it's Denominator, without a Fellow equal to it, the Curve cannot be squared. Q. E. O.

140. If the Ordinate of a Curve be a rational irreducible Fraction, whose Denominator, consisting of several Terms, is not a true Square; and some prime Divisor of that Denominator, be not contain'd at least three Times in it, the Curve is not quadrable: as easily appears from what has been said. Which is still a further Limitation, beyond that mentioned by Sir Isaac. Moreover, let $a, b, c, d, \&c.$ be the prime Divisors contained in the Denominator thrice above; and let $a^2, b^2, c^2, d^2, \&c.$ be the highest Powers of $a, b, c, d, \&c.$ which divide the Denominator of the Fraction in it's lowest Terms, then the Curve is not quadrable, unless when the Denominator of the Fraction in it's lowest Terms being multiplied by $a^{2-2}b^{2-2}c^{2-2}d^{2-2}, \&c.$ makes a complete Square: for the Multiplication of the Numerator and Denominator by this last Quantity, must bring back the Expression of the Ordinate to it's original Form: as may easily appear from what has been said: and therefore if the Curve be quadrable, the Ordinate will be

be brought back to this Form $\frac{ND-ND}{D^2}$: in which the Denominator is a true Square.

141. At Art. 31*, Sir Isaac directs how we are to manage, when the Ordinate is an irreducible Fraction, whose Denominator is made up of a rational Factor, which he calls Q, and an irreducible surd Factor, denoted by R, that the proposed Ordinate may be brought to due Form. The Operation contains no Difficulty in it: and it amounts to this. After you have thrown by one prime Divisor of each Magnitude or Sort, contained in the Root R, see whether the Quantity Q, multiplied by the Divisors of R, which remain, either make up R, or any positive Power of R, whose Index is an Integer, denoted by m: then, if that be the Case, R^{n-1} , in the general Expression of the Ordinate, becomes R^{n-m} , viz. by reducing the proposed fractional Ordinate, to the Denominator R^{n+m} : and this Reduction will be always made thus: multiply the Numerator of the proposed Fraction, by the Divisors of the Root R, which remain, after one of each Magnitude or Kind is thrown by; and instead of the Denominator $Q \times R^n$, write R^{n+m} , and it is done. That the

* Of the Quadrature of Curves.

Thing may be perfectly evident, let $\frac{N}{Q \times R^n}$ represent the proposed fractional Ordinate, and suppose $a.b.c$ and $a.a.b.c.c.c$ to denote the component prime Divisors of Q and R respectively, so that the Ordinate may be represented thus $\frac{N}{abc \times aabccc^n}$, where, after throwing by one prime Divisor of each Sort, of R, viz. $a.b.c$ and multiplying $Q = abc$ by all the remaining Divisors of R, viz. $a.c.c.c$, the Product is $aabccc = R$. Therefore, it's evident, if N be multiplied also by the remaining prime Divisors $a.c.c.c$, the Fraction $\frac{N}{abc \times aabccc^n}$ will be converted into this equivalent one $\frac{acc \times N}{aabccc \times aabccc^n}$

or $\frac{acc \times N}{R^1 \times R^n} = \frac{acc \times N}{R^{n+1}}$. After the same manner, if we suppose $\frac{N}{Q \times R^n} = \frac{N}{a^3b^2c^4 \times a^2bc^3^n}$, where, after throwing by $a.b.c$, one of each of the prime Divisors of R, and multiplying $Q = a^3b^2c^4$, by the Product of the remaining ones, viz. ac^2 , $a^4b^2c^6$ arises, which is the Square of R; it will appear that $\frac{N}{Q \times R^n} = \left(\frac{N}{a^3b^2c^4 \times a^2bc^3^n} \right) = \frac{acc \times N}{a^4b^2c^6 \times a^2bc^3^n} = \frac{acc \times N}{aabccc^{n+2}}$. And so of others.

Thus

Thus let $\frac{3q^3 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{q^3 - x^3 \times q^3 + q^2x - qx^2 - x^3}$ be the Ordinate of a Curve; the prime Divisors of $q^3 + q^2x - qx^2 - x^3$, are $q + x$, $q - x$. after throwing by $q + x$ and $q - x$ multiply the rational Factor $q - x$, by the remaining prime Divisor $q + x$, the Product is $q^3 + q^2x - qx^2 - x^3$; therefore you transform the Ordinate by multiplying the Numerator by $q + x$ and making the Denominator $q^3 + q^2x - qx^2 - x^3$, whereby it stands thus

$$\frac{q + x \times 3q^3 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{q^3 + q^2x - qx^2 - x^3} = \frac{3q^4 + 2q^3x + 8q^2x^2 + 8q^3x^3 - 7q^2x^4 - 6qx^5}{q^3 + q^2x - qx^2 - x^3}$$

$$= x^{1-1} \times \frac{3q^4 + 2q^3x + 8q^2x^2 + 8q^3x^3 - 7q^2x^4 - 6qx^5}{q^3 + q^2x - qx^2 - x^3}$$

by comparing which, with the general Ordinates $z^{\lambda-1} \times R^{\lambda-1} \times a + bz^{\lambda} + cz^{2\lambda} + \&c$; and then substituting the particular Values, in the Series for Areas, the Area of the Curve will be found to be

$$\frac{3q^3x + 3q^2x^2}{q^3 + q^2x - qx^2 - x^3}$$

142. Our Author, in this Proposition, exhibits a general Theorem or Canon, by which the Area of any Curve, whose Ordinate is of this Form $z^{\lambda-1} R^{\lambda-1} \times a + bz^{\lambda} + cz^{2\lambda} + dz^{3\lambda}$, &c. may be determined; and that either perfectly true, viz. when the curvilinear Area admits of a geometrical Comparison with right-lined Figures; or else so near the Truth, that the Difference may still be made less and less without End, viz. when the Curve cannot be squared. But whereas, it may be of considerable Use, to know when a Curve may or may not be squared: i. e. when it's Area may be expressed by a finite Number of Terms, or cannot be so expressed: therefore I shall here lay down some general Rules respecting this Matter: and for that End shall distinguish some Classes of Curves that come under the general Form mentioned in this Proposition; or may be easily reduced to it. And they are these following:

- 1°. $az^{\lambda-1} = y$.
 - 2°. $az^{\lambda} + bz^{\lambda} + cz^{\lambda} + dz^{\lambda}$, &c. $= y$.
 - 3°. $az^{\lambda-1} \times e + fz^{\lambda} = y$.
 - 4°. $az^{\lambda-1} \times e + fz^{\lambda} + gz^{2\lambda} = y$.
 - 5°. $z^{\lambda-1} \times e + fz^{\lambda} = y$.
 - 6°. $z^{\lambda-1} \times e + fz^{\lambda} = y \times a + bz^{\lambda} + cz^{2\lambda} = y$.
 - 7°. $z^{\lambda-1} \times e + fz^{\lambda} = y \times a + bz^{\lambda} + cz^{2\lambda} \&c. = y$.
 - 8°. $z^{\lambda-1} \times e + fz^{\lambda} + gz^{2\lambda} = y \times a + bz^{\lambda} + cz^{2\lambda} \&c. = y$.
- Where the Symbols are to be understood in the same Sense, they are taken by our Author, viz. z denotes the Absciss, and y the corresponding Ordinate:

nate: $a . b . c . d . e . f . g . \theta . \eta . \lambda$ also $p . q . r . s$ are constant Quantities; integral or fractional, positive or negative.

143. All Curves of this Form $ax^{\theta-1} = y$ may be squared, unless when it is $\theta = 0$: and the Area is $\frac{a}{\theta}x^{\theta}$.

For if the Ordinate $ax^{\theta-1}$ or $x^{\theta-1} \times 1^{\lambda-1} \times a$ be compared with the general Form of Ordinates $x^{\theta-1}R^{\lambda-1} \times a + bz^{\eta}$, &c. you find $f=0=g=b$, &c. $b=0=c$, &c. $R=(e=)1$, $a=a$, &c. Therefore insert these Values into the general Expression for the Area, belonging to this fifth Proposition: and it will easily appear, that all the Terms of

the Series after the first $\frac{1}{\eta}$, vanishing, because of the Letters b, c, d ,

&c. f, g, h , &c. being equal to nothing: the Area $x^{\theta}R^{\lambda} \times \frac{1}{\eta}$ becomes

$\frac{a}{\theta}x^{\theta}$. Which therefore always exhibits the Area exactly, and gives it finite, unless when $\theta = 0$: for then it becomes $\frac{a}{0}$ infinite. Q. E. O.

See also Art. 52*.

144. By this Rule all simple parabolical Curves, and hyperbolical Spaces lying betwixt the Curve and Asymptotes, unless in the Case of the Apollonian Hyperbola, are squared.

* Of this
Explica-
tion.

Ex. 1. Suppose it be $pz = y^2$, or $p^{\frac{1}{2}}z^{\frac{1}{2}} = y$: here, because $a = \sqrt{p}$, and $\theta - 1 = \frac{1}{2}$ or $\theta = \frac{3}{2}$, the Area is $\frac{2}{3}p^{\frac{1}{2}}z^{\frac{3}{2}}$: or by inserting y for $\sqrt{pz}$, $\frac{2}{3}yz$: which gives the Area of the common Parabola: being $\frac{2}{3}$ of the Rectangle under the Absciss and Ordinate.

Ex. 2. Let it be $yz^2 - a = 0$ or $az^{-2} = y$: here $\theta - 1 = -2$ or $\theta = -1$, therefore the Area is $-az^{-1}$ or $-\frac{a}{z}$: which denotes an hyperbolical Space lying betwixt the Hyperbola and Asymptote infinitely produced below the Ordinate. But if it had been $zy = a$ or $y = az^{-1}$, where $\theta - 1 = -1$ or $\theta = 0$, the Area comes out $\frac{a}{0}$ infinitely great: which is the Case of the common Hyperbola: and what must be done here, shall be shewn afterwards*.

145. All Curves of this Form $ax^p + bx^q + cx^r + dx^s$, &c. $= y$, when the Number of Terms of the Value of y , is finite, may be squared, unless some of the Exponents $p . q . r$, &c. be -1 : and the Area is $\frac{a}{p+1}x^{p+1} + \frac{b}{q+1}x^{q+1} + \frac{c}{r+1}x^{r+1}$ &c.

* Art. 190.
of this Ex-
plication.

Q

For

For the complex Ordinate $ax^p + bx^q$, &c. is to be considered as compounded of the several simple Ordinates ax^p , bx^q , cx^r , &c.: and therefore the Area of the Curve is compounded of the several Areas belonging to the simple Ordinates, which being assignable by the last Rule, the Area according thereto, will be $\frac{a}{p+1}x^{p+1} + \frac{b}{q+1}x^{q+1} + \frac{c}{r+1}x^{r+1}$ &c. Which is always finite, unless p or q , or r , &c. be equal to -1 : for then the Area belonging to the Term, where that happens, becomes infinite by the last Rule. Therefore, &c. Q.E.O.

Ex. 1. Suppose $2 - 3x + 4x^2 - 5x^3 = y$ be an Equation to a Species of Parabola; then, by inserting 2, -3 , 4, -5 for a , b , c , d respectively; and 0, 1, 2, 3 for p , q , r , s , the Area of the Curve is $2x - \frac{3}{2}x^2 + \frac{4}{3}x^3 - \frac{5}{4}x^4$.

Ex. 2. Let $x^2 + x^{-3} - x^{-4} = y$ express the Relation of the Absciss and Ordinate of a Curve: where $a=1$, $b=1$, $c=-1$, $p=2$, $q=-3$, $r=-4$: therefore by a proper Substitution of Values the Area will be $\frac{1}{3}x^3 - \frac{1}{2}x^{-2} + \frac{1}{3}x^{-3}$.

146. Cor. Hence it follows, if $a + bx^n + cx^{2n} + dx^{3n}$, &c. in *inf.* be a converging Series, approaching continually to the Value of the Ordinate of a Curve, then $ax + \frac{b}{n+1}x^{n+1} + \frac{c}{2n+1}x^{2n+1} + \frac{d}{3n+1}x^{3n+1}$ &c. in *inf.* will be an infinite converging Series, approaching continually to the Value of the corresponding Area.

And by this means, the Areas of all Sorts of Curves, the Relation of whose Absciss and Ordinate is expressed by any algebraical Equation howsoever affected, may be determined by an infinite converging Series: when they cannot otherwise be squared, by the Methods shewn in this Treatise of Quadratures. For by means of such an Equation, the Value of the Ordinate may always be expressed by an infinite Series of simple Terms, containing the Powers of the Absciss x and known Quantities only, when it cannot be expressed in finite Terms: and that by the Methods taught by our Author elsewhere¹: and thence by what has been just now said, the Area may be determined by an Approximation. Thus if $y^3 + a^2y + axy - 2a^3 - x^3 = 0$ be an Equation expressing the Relation of the Absciss and Ordinate of a Curve: by resolving the Equation according to his Method, by an infinite Series, it is $y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. in *inf.* whence,

¹ See his Treatise of Analysis by Equations of an infinite Number of Terms: likewise his Method of Fluxions and infinite Series, Chap. 1. published Anno 1736.

by what has been just now said, we shall have $ax - \frac{1}{2}x^2 + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$, &c. an approximate Value of the Area.

Likewise if we have $\sqrt{rr - zz} = y$, an Equation to the Circle: by extracting the square Root of $rr - zz$, according to his binomial Theorem, it is $r - \frac{z^2}{2r} - \frac{z^4}{8r^3} - \frac{z^6}{16r^5} - \frac{5z^8}{128r^7} - \&c.$ in $inf. = y$: wherefore the circular Area adjacent to the Absciss z , which begins at the Center, is $rz - \frac{z^3}{6r} - \frac{z^5}{40r^3} - \frac{z^7}{112r^5} - \frac{5z^9}{1152r^7} - \&c.$ r being the Radius.

Thus let ADE be a Quadrant of a Circle (see Fig.) $AE = r$. $AB = z$, and let BC be perpendicular to AB. Put $r = 1$. $z = 0,1$. then the Area



ABCD is $0,1 - \frac{0,001}{6} - \frac{0,00001}{40} - \frac{0,0000001}{112} - \frac{0,000000005}{1152}$ very nearly :

But if it be $AB = z = 1$, the Area of the Quadrant is $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152}$ nearly.

147. All Curves of this Form $ax^{\theta-1} \times e^{\pm f x^{\eta}} = y$, may always be squared, when s , that is $\frac{\theta}{\eta} + \lambda$, is either equal to nothing, or a negative Integer, as $-1, -2, -3$, &c. unless when r , that is $\frac{\theta}{\eta}$, is either nothing, or a negative Integer, not greater than s , at the same Time: in which Case the Area is infinitely great. And the Area of the Curve is $\frac{ax^{\theta}}{\eta} \times e^{\pm f x^{\eta}} \lambda \times$

$$\frac{1}{r^s} - \frac{f}{r \times r + 1 \times e^2} x^{\eta} + \frac{s \times s + 1 \times f^2}{r \times r + 1 \times r + 2 \times e^3} x^{2\eta} - \frac{s \times s + 1 \times s + 2 \times f^3}{r \times r + 1 \times r + 2 \times r + 3 \times e^4} x^{3\eta} + \&c.$$

Where the Number of Terms of the Series is one, when $s = 0$: 2, when $s = -1$: 3, when $s = -2$, &c. i. e. one more than the negative Units contain'd in s . In other Cases the Curve is not quadrable. But it must be observed, that both Expressions of the Ordinate must be tried by the Rule; as well that which has the Index of z under the Vinculum negative; as that, where it is positive.

For by considering the Series for Areas in general, exhibited by our Author in this Proposition*: and applying it to the Case of a simple binomial Curve, such as we are now speaking of: where it is $b = 0 = c = d$, &c. likewise $g = 0 = h = i$, &c. in infinitum,

* Art. 26. of the Quadrature of Curves.

tum, it will evidently be reduced to this Shape $z^0 \times e + f z^n)^{\lambda} \times$

$$\frac{1}{re} + \frac{-sfA}{r+1 \times e} z^n + \frac{-s+1 \times f^2 B}{r+2 \times e} z^{2n} + \frac{-s+2 \times f^3 C}{r+3 \times e} z^{3n}, \&c. :$$

Or, which comes to the same, $z^0 \times e + f z^n)^{\lambda} \times \frac{1}{r} \times \frac{z^0}{e} -$

$$\frac{sf \times \frac{1}{r} \times \frac{z^0}{e}}{r \times r+1 \times e^2} + \frac{s \times s+1 \times f^2 \times \frac{1}{r} \times \frac{z^0}{e}}{r \times r+1 \times r+2 \times e^3} - \frac{s \times s+1 \times s+2 \times f^3 \times \frac{1}{r} \times \frac{z^0}{e}}{r \times r+1 \times r+2 \times r+3 \times e^4} \times \frac{z^0}{e^4} + \&c.$$

* Of this
Explica-
tion.

as appears by Art. 125 *: for the whole Series, in this Case, is reduc'd to the first horizontal Row, as it is exprest towards the End of that Art. Which, by taking the common Factor $\frac{1}{r} \times \frac{z^0}{e}$, out of all the Terms, and prefixing it to the common Multiplier z^0 , makes the Series mentioned, viz. $\frac{az^0}{r} \times e + f z^n)^{\lambda} \times$

$$\frac{1}{re} - \frac{sf}{r \times r+1 \times e^2} z^n + \frac{s \times s+1 \times f^2}{r \times r+1 \times r+2 \times e^3} z^{2n} - \frac{s \times s+1 \times s+2 \times f^3}{r \times r+1 \times r+2 \times r+3 \times e^4} z^{3n} \&c.$$

From which Series, expressing the Area of a simple binomial Curve in general, it appears that if it be $s=0$, the second and all the subsequent Terms will vanish, because of the common Factor $s=0$ found in them all: if it be $s+1=0$ or $s=-1$, all the Terms after the second must vanish; because of the common Factor $s+1=0$: if it be $s=-2$ or $s+2=0$, all the Terms after the third must vanish, and so on continually: wherefore if s be nothing, or a negative Integer, the Series will terminate; and the Number of Terms, will be one more than the negative Units contain'd in $s = \frac{0}{r} + \lambda$: but if r be equal to nothing, or any negative Integer not greater than s , the Denominators of some one or more of the Terms preceding that Term, which should be the last, will vanish, and so make one or more Terms infinite: thus e. g. if it were $s=-2$ or $s+2=0$, there would be three Terms; but if, at the same Time, it be $r=-2$, or $r=-1$, or $r=0$, i. e. $r+2=0$, or $r+1=0$, or $r=0$, one or more of the preceding Denominators must contain a Factor equal to nothing, and thereby make the Term or Terms, to which it or they belong, infinite: and the Reasoning, from the Nature and Progression of the Series, is evidently the same, whenever r is nothing, or a negative Integer not greater than s .

Moreover, it is evident, that in no other Case, except when s is either nothing or a negative Integer, can the Series terminate: i. e. in

no

no other Case can the Curve be squared: these things being supposed to be applied to both Expressions of the Ordinate; as well that wherein the Index η is negative; as that wherein it is affirmative; which will appear by the Examples. Therefore, &c.

Ex. 1. Let $\frac{4x}{27 - 108x + 144x^2 - 64x^3} = y^2$ express the Relation of the Absciss and Ordinate of a Curve: by extracting the square Root, it is

$\frac{2x^{\frac{1}{2}}}{\sqrt{27 - 108x + 144x^2 - 64x^3}} = y$: and because the Quantity under the radical Sign in the Denominator, is a true Cube, whose cube Root is $3 - 4x$, therefore I reduce it to this Shape $\frac{2x^{\frac{1}{2}}}{(3 - 4x)^{\frac{1}{2}}} = y$, which is a simple binomial Curve: When it is brought to Form, it is either

$2x^{\frac{1}{2}-1} \times 3 - 4x^{\frac{1}{2}-1}$ or $2x^{0-1} \times -4 + 3x^{-1-\frac{1}{2}-1}$. In this last Case, by comparing with the general Form of Ordinates for binomial Curves $ax^{\theta-1} \times e + f x^{\lambda-1}$, we have $\theta = 0$. $\eta = -1$. $r = (\frac{\theta}{\eta}) = 0$, whence, without proceeding further, I conclude the Area, arising from the second Form of the Ordinate, to be infinite: which is always the Case when $\theta = 0$. But I cannot thence conclude that the propos'd Curve cannot be squared, until I apply the Rule to the first Form of the Ordinate, viz. $2x^{\frac{1}{2}-1} \times 3 - 4x^{\frac{1}{2}-1}$: where we find $\theta = \frac{1}{2}$. $\eta = 1$. $\lambda = -\frac{1}{2}$. $r = \frac{1}{2}$. $s = 2$, which being positive, shews that the Series for the Area, arising from this Form of the Ordinate, runs out into an infinite Series: and now I conclude that the proposed Curve cannot be squared.

Ex. 2^d. Suppose it to be $2x^{-\frac{1}{2}} \times 3 - 4x^{\frac{1}{2}} = y$: this Ordinate may, when brought to due Form, stand thus $2x^{\frac{1}{2}-1} \times 3 - 4x^{\frac{1}{2}-1}$ or $2x^{-1-1} \times -4 + 3x^{-1-\frac{1}{2}-1}$. From the first, we have $\theta = \frac{1}{2}$. $\eta = 1$. $\lambda = -\frac{1}{2}$. $r = \frac{1}{2}$. $s = 0$: whence I conclude the Curve may be squared: and since moreover it is $e = 3$. $f = -4$. $a = 2$; the Area, which according to the Rule, is thus generally expressed $\frac{ax^{\theta}}{\eta} \times e + f x^{\lambda} \times \frac{1}{r}$, will be, by Substitution of Values, $2x^{\frac{1}{2}} \times 3 - 4x^{\frac{1}{2}} \times \frac{1}{\frac{1}{2}} = \frac{4}{3} \sqrt{\frac{x}{3-4x}}$: although, if you try the other Form of the Ordinate $2x^{-1-1} \times -4 + 3x^{-1-\frac{1}{2}-1}$, you would find the Expression for the Area to run out into an infinite Series.

Ex.

Ex. 3^d. Let $c^4y - a^4z - 2c^2yz^2 + yz^4 = 0$ be an Equation, containing the Relation of the Absciss and Ordinate of a Curve: by taking the Value of y , it is $y = \frac{a^4z}{c^4 - 2c^2z^2 + z^4}$: where the Denominator being

* Art.
136, 137.
of this Ex-
plication.

a true Square, I make it stand thus * $\frac{a^4z}{c^2 - z^2}$, or $a^4z \times \overline{c^2 - z^2}^{-1}$: which, being reduced to due Form, is either $a^4z^{2-1} \times \overline{cc - z^2}^{-1-1}$ or yet $a^4z^{-2-1} \times \overline{-1 + ccz^{-2}}^{-1-1}$. From the first Manner of expressing, we have $\theta = 2$. $\eta = 2$. $\lambda = -1$. $r = 1$. $s = 0$: whence I conclude the Area arising from this Form of the Ordinate to be finite and assignable: and since it is moreover $e = cc$. $f = -1$. $a = a^4$: the Area will be $\frac{a^4z^2}{2c^4 - 2ccz^2}$. Again, by comparing the second Form of the propos'd Ordinate, viz. $a^4z^{-2-1} \times \overline{-1 + ccz^{-2}}^{-1-1}$, with the general Form, we have $\theta = -2$. $\eta = -2$. $\lambda = -1$. $r = 1$. $s = 0$: whence it appears that the Area, arising from this other Form of the Ordinate, may be exactly determined likewise: and since it is moreover $e = -1$. $f = cc$. $a = a^4$: by a proper Substitution and Reduction, the Area will come out $\frac{a^4}{2c^2 - 2z^2}$: By this means we obtain two Areas, perfectly assignable, and both belonging to the same Curve: Of which we gave an Example formerly *.

* Art.
128, 129.
of this Ex-
plication.

Ex. 4th. Let $ax^{2\eta-1}\sqrt{e + fz^\eta}$ be the Ordinate of a Curve: it may stand thus $ax^{2\eta-1} \times \overline{e + fz^\eta}^{\frac{1}{2}-1}$; or thus $ax^{\frac{1}{2}\eta-1} \times \overline{f + ez^{-\eta}}^{\frac{1}{2}-1}$. From the first, it is $\theta = 2\eta$. $\eta = \eta$. $\lambda = \frac{1}{2}$. $r = 2$. $s = \frac{7}{2}$: whence the Series for the Area, arising from this Form of the Ordinate runs out infinitely. Wherefore I try the other Form $ax^{\frac{1}{2}\eta-1} \times \overline{f + ez^{-\eta}}^{\frac{1}{2}-1}$: here I find $\theta = \frac{1}{2}\eta$. $\eta = -\eta$. $\lambda = \frac{3}{2}$. $r = -\frac{1}{2}$. $s = -1$: whence I conclude that the Curve may be squared: and since it is moreover $e = f$. $f = e$. $a = a$, the Area by a proper Substitution of Values in the Series for the Areas of binomial Curves, will be $\frac{ax^{\frac{1}{2}\eta}}{\frac{1}{2}\eta} \times \overline{f + ez^{-\eta}}^{\frac{1}{2}}$ $\times \frac{1}{-\frac{1}{2}f} - \frac{-e}{\frac{1}{2}\eta f} z^{-\eta} =$ (by a proper Reduction) $a \times \overline{e + fz^\eta}^{\frac{1}{2}} \times \frac{-4e + 6fz^\eta}{15\eta f^2}$.

148. Cor. From what has been said it appears, that, when we consider simple binomial Curves that are capable of being squared: some are of that Nature that one Area only can be exactly found, as in the second and fourth Examples: which Area is sometimes deduced from

from that Form of the Ordinate, where η , the Index of z , is positive, as in the second Example; sometimes from the other Form of the Ordinate, where η is negative, as in the fourth Example. But there are other Curves of such a Nature that both Areas may be exactly found, as in Example third. This last Sort we may call *doubly Quadrable*. Therefore it may be of use to have a Rule ready at hand, whereby we may at first View determine, when any propos'd binomial Curve may, or may not be squared at all: and when it is doubly Quadrable.

Rule. To find when a simple binomial Curve may, or may not be squared at all: and when it is doubly Quadrable.

149. Having brought the Ordinate of the Curve proposed, to stand in due Form, viz. thus $ax^{\theta-1} \times e + fz^{\lambda-1}$: being that Form, where η , the Index of z under the Vinculum, is suppos'd to be affirmative; θ and λ being either affirmative or negative.

If $\frac{\theta}{\eta} + \lambda$ be nothing, or a negative Integer; and at the same Time $\frac{\theta}{\eta}$ be nothing, nor a negative Integer as small or smaller than it; or if $\frac{\theta}{\eta} + 1$ be nothing, or a negative Integer; and at the same Time $\frac{\theta}{\eta} + 1 - \lambda$ be not nothing, nor a negative Integer as small or smaller than it: then if either of these happen, the Curve may be squared: if neither of these happen, the Curve cannot be squared: if both of them happen, the Curve is doubly quadrable.

The Demonstration of this will be evident from what has been formerly said in Art. 147 *: if it be considered that the Ordinate $ax^{\theta-1} \times e + fz^{\lambda-1}$ may be also expressed otherwise thus $ax^{\theta+\lambda\eta-\eta-1} \times$

* Of this
Explica-
tion.

$f + ez^{-\eta}\lambda^{-1}$: for from the first, we have $r = \frac{\theta}{\eta}$. $s = \frac{\theta}{\eta} + \lambda$: and from the second, $r = -\frac{\theta}{\eta} - \lambda + 1$. $s = -\frac{\theta}{\eta} + 1$.

150. Hitherto we have been capable of laying down general Rules, by which, we may always determine, when any Curve, of the Kinds mentioned, is proposed; whether it may be squared, or not, and that immediately from the Properties of the Exponents. But with respect to trinomial Curves, viz. whose Ordinates come under this Form $ax^{\theta-1} \times e + fz^{\eta} + gz^{2\eta}\lambda^{-1}$, all we can do by means of the Properties and Relations of the Symbols θ , η , λ , which belong to the Exponents, is to shew some Limitations, without which no such Curve can be squared. And whereas there are infinite different Relations, which

• Art. 26.
of the
Quadrature of
Curves.
Art. 124.
of this Explanation.

which subsisting among all the Symbols $\theta . \eta . \lambda . e . f . g$ taken together, make the Curve quadrable, we can shew how some of these Relations may be investigated. Finally, I shall lay down a Method or Rule, by which we can find, without much trouble, whether any proposed simple trinomial Curve may, or may not be squared at all. In order to which it will easily appear, by considering the general Series for Areas, exhibited by our Author in this Proposition*, and applying it to the Case of a simple trinomial Curve, that it must put on this Shape

$$x^\theta \times e + f x^\eta + g x^{2\eta} \cdot \lambda \times \frac{1}{r e} - \frac{f A}{r+1 \times e} x^\eta - \frac{\overline{s+1} \times f B + t g A}{r+2 \times e} x^{2\eta} - \frac{\overline{s+2} \times f C + \overline{t+1} \times g B}{r+3 \times e} x^{3\eta} \\ - \frac{\overline{s+3} \times f D + \overline{t+2} \times g C}{r+4 \times e} x^{4\eta} - \frac{\overline{s+4} \times f E + \overline{t+3} \times g D}{r+5 \times e} x^{5\eta} - \&c. \text{ the other}$$

Parts of the Numerators of each complex Term, vanishing, because $b=0=c=d$, &c. and $b=0=i$, &c. *in infinitum*. From which we may deduce what follows.

151. Any Curve of this Form $ax^{\theta-1} \times e + f x^\eta + g x^{2\eta} \lambda^{-1} = y$, being proposed, set down the several Terms in order, of which it's Area is composed, by substituting the particular Values, in place of the Symbols, in the general Theorem or Series for the Area of trinomial Curves, expressed in the preceding Article: and if you come to two Terms¹, immediately succeeding one another, both separately taken, equal to nothing, the Curve may be squared: And the Area is just equal to the preceding Part of the Series: provided always that none of the preceding Terms become infinitely great, by r becoming 0 or any negative Integer less than the Number of significant Terms: which would make the Area infinitely great. If there be but two significant Terms, it must neither be $r=0$, nor $r=-1$: if there be three Terms preceding the two Terms which break off the Series, r must not be 0, -1 nor -2 , and so on: for there is no quadrable Curve of this Form, which can have but one Term, the Area being necessarily infinite.

The Demonstration of this will be evident, if it be considered, that there are only two capital Letters, in each complex-Term of the fore-

* By Terms here are meant Terms of the last Factor, constituting the Series $\frac{1}{r e} - \frac{f A}{r+1 \times e} x^\eta - \frac{\overline{s+1} \times f B + t g A}{r+2 \times e} x^{2\eta}$, &c.

going

going Series for the Area of a trinomial Curve, after you pass the second Term, *in inf.* which two capital Letters always denote the Coefficients of the two immediately preceding Terms: and these Capitals are each a Factor of one of the two Parts of which the Numerator of the Term consists: therefore, if any two Terms immediately succeeding one another, vanish; and consequently their Coefficients, denoted by the two capital Letters, contain'd in the Term immediately following them, that Term must also vanish: and so also the next Term after it, for the like Reason: and therefore all the subsequent Terms *in inf.* by the same way of reasoning. Whence if two immediately succeeding Terms vanish, all the subsequent Terms must also vanish: and therefore the Series must terminate with the Term immediately preceding them; and so exhibit the full Area of the Curve, and shew it to be quadrable: unless r by being equal to nothing; or to any negative Integer, less than the Number of significant Terms do make the Denominator of some Term to vanish: it's Numerator being supposed not to vanish: for in such a Case one or more of the preceding Terms must be infinitely great: as easily appears by considering the Series, and what was formerly said.

Moreover, that there can be no quadrable Curve of this Form, whose Area consists of no more but the first Term, *viz.* $\frac{1}{r} a \times x^0 \times (e + f x^n + g x^{2n})^\lambda$, will appear thus. When all the Terms, but the first vanish, we will have the third Term $-\frac{s+1 \times fB + t gA}{r+2 \times e} x^{2n} = 0$, that is, because the second Term also, and consequently it's Coefficient denoted by B, vanishes, $t gA = 0$: which can only happen here, when it is $t = 0$: but seeing the second Term $-\frac{s fA}{r+1 \times e} x^n = 0$: which cannot be in this Case, but when it is $s = 0$: therefore when all the Terms of the Series except the first vanish, it must be $t = (0 =) s$: but $t = s - \lambda$, therefore it is also $\lambda = 0$: again, $r = s - \lambda$, therefore it is $r = 0$: consequently the first Term of the Series, and therefore the Area of the Curve is infinitely great. Which Things were to be shewn.

N. B. The Rule contain'd in this Article, may be applied to both Series's, which denote the Areas of the Curve; both to that, where the Index n is affirmative; and that where it is negative. And by so doing, if you come at two immediately succeeding Terms, equal each to nothing, in both Series's, the Curve is doubly quadrable. Moreover it

R

must

must be remarked that r may possibly be equal to a negative Integer less than the Number of significant Terms, and yet the Area be finite, and the Curve quadrable: if so be the Numerator of that Term vanish, whose Denominator vanishes. Thus if it be $r = -2$, the Denominator of the third Term, viz. $r + 2 \times e$ vanishes, which makes that Term and consequently the Area infinitely great, provided the Numerator $s + 1 \times fB + tgA$ don't vanish at the same Time: but if that Numerator be nothing, then the third Term becomes nothing, and therefore the Area is finite, and the Curve will be quadrable upon the Conditions mentioned in this Art. And it must be observed, that the like Exception respecting the Infinity of Areas arising from the Value of r being nothing or a negative Integer, must be understood with the like Limitation in all the subsequent Rules contained in this Section.

152. In the preceding Article, I have demonstrated that, when any two immediately succeeding Terms of either of the two Series's, for the Area of a simple trinomial Curve, vanish, and thereby break off the Series, the Curve is quadrable: and the Area made up of all the Terms preceding these. I shall next shew how one may investigate certain Laws or Conditions, arising from the Properties and Relations of the Quantities, which enter the Expression of the Ordinate: which Laws or Conditions obtaining, the Curve may be squared.

153. Put any two next adjoining Terms, beginning with the third and fourth *, of the Series expressing the Area of any simple trinomial Curve, each equal to nothing: and thence will arise a certain Relation, or certain Relations among the Quantities: which subsisting, the Curve may squared.

*Art. 151.
of this Ex-
plication.

Suppose the third and fourth Terms equal to nothing: i. e. — $\frac{s+1 \times fB + tgA}{r+2 \times e} x^{2n} = 0$ and — $\frac{s+2 \times fC + t+1 \times gB}{r+3 \times e} x^{3n} = 0$: then,

because $C = (-\frac{s+1 \times fB + tgA}{r+2 \times e}) 0$, therefore, from the second

Equation, by inserting 0 for C, and dividing by $\frac{x^{3n}}{r+3 \times e}$ we will have

$s+1 \times gB = 0$: i. e. because neither g nor B is nothing, $s+1=0$ or $s=-1$: whence $s=(t-\lambda=-) -1-\lambda$ and $r=(s-\lambda=-) -1-2\lambda$: insert these Values of s and t in place of them, in the

first Equation — $\frac{s+1 \times fB + tgA}{r+2 \times e} x^{2n} = 0$, and divide it by — $\frac{x^{2n}}{r+2 \times e}$ and it becomes — $\lambda fB - gA = 0$: insert for B it's Value, viz. —

$$-\frac{fA}{r+1 \times e} = -\frac{\lambda+1 \times fA}{2\lambda \times e}, \text{ and you will have } \frac{\lambda+1 \times fA}{2e} - gA =$$

0: i. e. by reducing, $\lambda = \frac{2eg}{f^2} - 1$.

$$\text{Again } \frac{\theta}{\eta} = (r =) -1 - 2\lambda: \text{ therefore } \frac{\theta}{\eta} = 1 - \frac{4eg}{f^2}.$$

Whence it follows, that, if it be $\lambda = \frac{2eg}{f^2} - 1$, and $\frac{\theta}{\eta} = 1 - \frac{4eg}{f^2}$; the third and fourth Terms of the Series, expressing the Value of the Area, both vanish: and therefore the Curve, that has such an Ordinate, as that $\lambda = \frac{2eg}{f^2} - 1$, and $\frac{\theta}{\eta} = 1 - \frac{4eg}{f^2}$, may be squared: provided that $1 - \frac{4eg}{f^2}$, which is the Value of $\frac{\theta}{\eta}$ or r , be neither 0; nor -1 : as appears by the last Article.

Example. Let the Ordinate of a Curve be $\frac{a}{2x^9 + 3x^{13} + 3x^{15}}$: reduce it to Form and it stands thus, $ax^{-5-1} \times 2 + 3x^3 + 3x^{6-1}$:

where $\lambda = \frac{1}{3}$. $\theta = -5$. $\eta = 3$. $r (= \frac{\theta}{\eta}) = -\frac{5}{3}$. $e = 2$. $f = 3$.

$g = 3$: therefore $\lambda (= \frac{1}{3}) = \frac{2eg}{f^2} - 1$: again $r (= -\frac{5}{3}) = 1 - \frac{4eg}{f^2}$:

wherefore the Curve may be squared, and accordingly, by inserting the particular Values of the indefinite Symbols, in the Series which expresses the Value of the Area of a Trinomial *, you'll find the Area

* Art. 150. of this Explication.

of the Curve proposed, $\frac{0.33}{x^2} - \frac{0.1}{x^3} \times a\sqrt{2 + 3x^3 + 3x^6}$: all the Terms after the first two, of the Series vanishing: for $r = (r + 2\lambda =) -1$.

154. Now, by the very same way of reasoning, you may investigate and discover other Properties and Relations of the several Quantities $r (= \frac{\theta}{\eta})$. λ . e . f . and g : which subsisting, a simple trinomial Curve, whose Ordinate is of this Form $ax^{\theta-1} \times e + fx^{\eta} + gx^{2\eta} \lambda^{-1}$, may be squared: viz. by putting the fourth and fifth Terms; the fifth and sixth Terms; the sixth and seventh Terms, and so on, equal to nothing: but the Investigation becomes still more and more tedious, and perplex'd: because it involves you still in higher Equations: and then after all, you shall not be able to discover a general Rule, by which you might always know when any simple trinomial Curve, may or may not be squared: for 'tis evident that all the Rules thus discovered, extend only to particular Cases; by shewing some of the infinitely many Relations, which these Quantities bear, in order to make the Curve quadrable. And I suppose it may be easily gathered from what was said

* Art 147.
of this Ex-
plication.

formerly, that this Method of Investigation may be applied to both Expressions of the Ordinate *.

From what has been said in the four preceding Articles, we may easily deduce several things, serving as Limitations or Conditions, without which no trinomial Curve can be squared at all. Having therefore brought any propos'd Ordinate of a Curve of this Sort, to stand in this Form $ax^{\theta-1} \times e + \sqrt{fx^{\eta} + gx^{2\eta}}^{\lambda-1}$: where η , the Exponent of x under the Vinculum, must be positive: Then,

155. 1°. If both $\frac{\theta}{\eta} + 2\lambda$ and $-\frac{\theta}{\eta} + 2$ be nothing; Fractions of any Kind, or positive Integers, the Curve cannot be squared: for these are the two Values of the Symbol t , according to the two different Ways of expressing the Ordinate: and it is evident from what has been demonstrated, that the Series expressing the Area cannot terminate, unless t be a negative Integer: because no two Terms immediately succeeding one another can vanish, but when t is a negative Integer: as appears from Art. 153 *.

* Of this
Explica-
tion.

156. 2°. If both $\frac{\theta}{\eta}$ and $-\frac{\theta}{\eta} - 2\lambda + 2$ be nothing or -1 , the Curve cannot be squared: for these are the two Values of r , in the two different Expressions of the Ordinate: therefore, in this Case the Curve cannot be squared, the Area being infinitely great by Art. 151 *.

* Of this
Explica-
tion.

157. 3°. If it be $\lambda = 0$, the Curve cannot be squared: for in that Case $r = (t - 2\lambda) = t$: but the Curve cannot be squared unless t be a negative Integer by Art. 155 *: now if t being a negative Integer, the Series terminate, the last significant Term of the Series becomes infinitely great: because it's Denominator vanishes, when $r = t$, and t a negative Integer: and therefore the Area is infinitely great: which will appear by considering the Series at Art. 150 *: and what has been demonstrating in Art. 151 *.

* Of this
Explica-
tion.

* Of this
Explica-
tion.

And now, from the Consideration of all that has been said about trinomial Curves, we deduce this

R U L E

158. To know when any simple trinomial Curve, whose Ordinate is of this Form $ax^{\theta-1} \times e + \sqrt{fx^{\eta} + gx^{2\eta}}^{\lambda-1}$, may, or may not be squared; and when it is doubly quadrable.

Any such Curve being propos'd, set down it's Ordinate in that Form where η is affirmative: apply to it the Limitations mentioned in Art.

The Quadrature of CURVES explained.

125

Art. 155. 156, 157 *: and if any of these Limitations shew that it cannot be squared, you proceed no further. But if none of them determine this: then set down likewise the other Form of the Ordinate, viz. that where η is negative: seek the two Values of t , belonging to the two different Expressions of the Ordinate; one or other, or both of them must be negative Integers, from the Limitations already mentioned: if only one of the Values of t be a negative Integer, you need only set down the Series for the Area, belonging to that Form of the Ordinate, according to which t is a negative Integer: but if both Values of t be negative Integers, you must set down both Series's for the Value of the Area: that where η is positive, and the other also where it is negative. Then, if one of the Values of t only be a negative Integer, it must be $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. substitute the particular Values of the Letters, through all the different Terms of the corresponding Series for the Area, in order from the first, until you come to that Term of the Series, which immediately precedes the Term, where you have $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. and if that Term be not nothing, the Curve cannot be squared at all: if it be nothing the Curve is quadrable: (provided still r be not equal to 0 nor any negative Integer so small or smaller than t) and the Area is equal to the preceding Terms.

* Of this
Explica-
tion.

But if both Values of t be negative Integers, you must try both Series's for the Area, after the Manner just now mentioned: and if the Term of the Series's immediately preceding that Term, where you have $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. be equal to nothing, in neither Series, the Curve is not quadrable: but if that Term be nothing, in either of the Series's, the Curve is quadrable, and you have the Area: if it be so in them both, the Curve is doubly quadrable, and you have both Areas: it being suppos'd still as before, that the Value or Values of r be not 0; nor any negative Integer or Integers so small or smaller than the Value or Values of t respectively. There needs nothing be said for demonstrating the Justness of this Rule or Solution, after what has been already shewn: unless perhaps it should be thought necessary to demonstrate, that if the Term of the Series for the Area, which immediately precedes that Term where it is $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. be not nothing, the Terms of the Series will go on *in inf.* so as that it can never terminate: nor consequently the Area of the Curve, denoted by that Series, be exactly found. Which appears thus.

159. Suppose it were $t = -2$ or $t + 2 = 0$: (See the Series at Art. 150 *.) the Term where $t + 2$ is found, is the fifth. Now I say

* Of this
Explica-
tion.

say if the fourth Term, viz. $-\frac{t+2 \times fC + t+1 \times gB}{r+3 \times e} z^3$ be not equal to nothing, the Series will never terminate: for if it terminate the fourth Term, or some Term after the fourth must be the last significant Term of the Series: it cannot be the fourth Term; for then, since the next two Terms vanish, viz. $-\frac{t+3 \times fD + t+2 \times gC}{r+4 \times e} z^3$ and $-\frac{t+4 \times fE + t+3 \times gD}{r+5 \times e} z^4$ we have $-\frac{t+4 \times fE + t+3 \times gD}{r+5 \times e} z^4 = 0$: but it is also $E = 0$: for E denotes the Coefficient of the preceding Term, which is nothing: therefore we will have $\frac{t+3 \times gD}{r+5 \times e} z^4 = 0$, consequent-

ly $t+3 \times gD = 0$: but g is not nothing, from the Nature of the Trinomial; and D is not nothing by the Hyp., for it is the Coefficient of the fourth Term, therefore it must be $t+3 = 0$: but that is contrary to the Hyp., for it was suppos'd that $t+2 = 0$. Now by the very same way of reasoning the fifth cannot be the last significant Term, nor the sixth, nor any subsequent Term whatsoever in the Series, still for the same Reason, therefore the Series will never terminate. And it's evident, the Reasoning is exactly the same if we suppose $t = -1$, or $t = -3$, or $t = -4$, &c. i. e. universally, whatever negative Integer be the Value of t . Therefore, &c. Q. E. D.

160. Cor. If the Series exhibiting the Value of the Area of a simple trinomial Curve, do terminate, the Area will at least contain two Terms of the Series, multiplied into $z^{\theta} R^{\lambda}$, viz. when it is $t = -1$: the Area will be expressed by 3, 4, 5, &c. Terms multiplied into $z^{\theta} R^{\lambda}$, when it is $t = -2$, $t = -3$, $t = -4$, &c. respectively, i. e. the Number of Terms will be one more than the Number of negative Units contain'd in the Value of t .

Now I shall subjoin a few Examples of the Application of the preceding Rule.

Ex. 1. Let $\frac{d}{z} \sqrt{2 - 3z^2 + 4z^4} = y$ express the Relation of the Absciss and Ordinate of a Curve: when it is reduced to Form, according to the Rule, it stands thus $dz^{0-1} \times 2 - 3z^2 + 4z^4$: here $\theta = 0$. $\eta = 2$, $\lambda = \frac{1}{2}$. therefore $\frac{\theta}{\eta} = 0$; and $\frac{\theta}{\eta} + 2\lambda = 3$; and $-\frac{\theta}{\eta} + 2 = 2$: therefore the Curve is of that Number that cannot be squared by Art. 155 *. The same may be concluded from Art. 156 *, since it is $\frac{\theta}{\eta} = 0$ and $-\frac{\theta}{\eta} - 2\lambda + 2 = -1$.

* Of this
Explica-
tion.

Ex.

Ex. 2. Suppose $ax^{-\frac{1}{2}} \times \sqrt{1-2x+3x^2}^{-\frac{1}{2}} = y$: reduce it to Form according to the Rule, and it stands thus $ax^{\frac{1}{2}-1} \times \sqrt{1-2x+3x^2}^{-\frac{1}{2}-1}$: where we have $\frac{\theta}{\eta} + 2\lambda = -1$ and $-\frac{\theta}{\eta} + 2 = \frac{1}{2}$. further $\frac{\theta}{\eta} = \frac{5}{2}$ and $-\frac{\theta}{\eta} - 2\lambda + 2 = 0$: and since it is not $\lambda = 0$: hence it appears that this Curve comes not under the Limitations contain'd in Art. 155, 156, 157 *, so as that we can conclude it not to be quadrable. * Of this Explication. Therefore since it is $\frac{\theta}{\eta} + 2\lambda = -1$, and $\frac{\theta}{\eta} = \frac{5}{2}$: which are the Values of t and r belonging to that Form of the Ordinate, where η is positive: I set down the Series for the Area where the Value of η is positive: which by a proper Substitution of Values (see Art. 150 *) * Of this Explication. stands thus $x^{\frac{1}{2}} \times \sqrt{1-2x+3x^2}^{-\frac{1}{2}} \times 2a - \frac{2a}{3}x + 2ax^2, &c.$ Hence then I conclude the Curve cannot be squared: for since it is $t = -1$, the third Term must have been nothing, in order to make the Series terminate: and the other Value of t arising from the other Expression of the Ordinate being $\frac{1}{2}$, the other Form of the Series for the Area, must run out infinitely.

N. B. The same thing might have been concluded from Art. 153 *. * Of this Explication.

Ex. 3. Let it be $\frac{a}{\sqrt{x^6-2x^7+3x^8}} = y$: reduce it to the Form, where the Value of η is positive: and it stands thus $ax^{-2-1} \times \sqrt{1-2x+3x^2}^{\frac{1}{2}-1}$: where it is $\theta = -2$. $\eta = 1$. $\lambda = \frac{1}{2}$: whence $\frac{\theta}{\eta} + 2\lambda = -1$: $-\frac{\theta}{\eta} + 2 = 4$, and $\frac{\theta}{\eta} = -2$: therefore I cannot conclude from Art. 155 -157 *, that the Curve cannot be squared, only I know it is not doubly quadrable. Wherefore, agreeably to the Rule, since the Value of t , belonging to the Form of the Ordinate, having η positive, is -1 , and that the corresponding Value of r is -2 , a negative Integer not so small as t ; I set down the several Terms of the corresponding Series for the Area, according to the proper Values of the general Symbols: and thence I obtain $x^{-2} \times \sqrt{1-2x+3x^2}^{\frac{1}{2}} \times -\frac{1}{2}a - \frac{3a}{2}x - 0$ where, because the third Term, which immediately precedes that one, where we have $t + 1 = 0$, vanishes, the Curve is quadrable: and the full Area is, what I have just now set down; or $-\frac{3}{2x} + \frac{1}{2x^2} \times a\sqrt{1-2x+3x^2}$.

161. Schol. It appears from what has been said about simple trinomial Curves, that Things of a like Nature might be demonstrated with respect to simple Quadrinomials, *viz.* whose Ordinates are thus generally expressed $ax^{\theta-1} \times e + fz^{\eta} + gx^{\lambda-1} + bx^{\lambda-1}$; that is we might shew certain Conditions, without which no quadrinomial Curve can be squared at all, arising from the Relations and Properties of the Exponents $\theta . \eta . \lambda$, analogous to what has been laid down in Art. 155—
 • Of this 157 * concerning Trinomials: likewise, how to investigate certain
 Explica- Relations of all the Symbols $\theta . \eta . \lambda . e . f . g . b$: which obtaining, the
 tion. Curve is quadrable: like to what was done in Art. 153 *: and finally,
 • Of this Curve is quadrable: like to what was done in Art. 153 *: and finally,
 Explica- how a general Rule might be laid down; by which it might be known,
 tion. when any simple quadrinomial Curve may or may not be squared at all:
 • Of this analogous to that laid down in Art. 158 *. And so likewise with re-
 Explica- spect to Curves, whose Ordinates contain still more complicate radical
 tion. Expressions. But this would lead us beyond our Design: it being sufficient to have shewn these several things with respect to Trinomials: and thereby pointed out the Way for doing the same in these others, whose Ordinates are more complex.

162. A Curve of this Form $x^{\theta-1} \times e + fz^{\eta} \times a + bx^{\lambda} = y$ may be squared.

1°. When it is $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$; and the first Term of the Series for the Area multiplied into the common Factor $x^{\theta}R^{\lambda}$, *i. e.* $x^{\theta} \times e + fz^{\eta} \times \frac{1}{re}$ exhibits the full Area.

2°. When $s (= \frac{\theta}{\eta} + \lambda)$ is any negative Integer, as $-1, -2, -3$, &c. unless $r (= \frac{\theta}{\eta})$ be either 0 or a negative Integer not greater than s (in which Case the Area is infinitely great) and the Area of the Curve consists of such a Number of Terms of the Series, multiplied into the common Factor $x^{\theta}R^{\lambda}$, as exceeds the Number of negative Units contain'd in s , by one.

Which two Rules are to be applied to both Forms of the Ordinate: and if they agree to either, the Curve is quadrable. if to neither, the Curve cannot be squared: and if they agree to both, the Curve is doubly quadrable.

These two Rules will easily be demonstrated, from the Nature of the Series expressing the Area of such a Curve: which Series, by reducing the general one, contain'd in our Author's fifth Proposition,

to the present Case, will be thus expressed $z^0 \times e + f z^1)^{\lambda} \times$

$$\frac{\frac{1}{r}a}{r} + \frac{\frac{1}{r}b - sfA}{r+1 \times e} z^1 - \frac{\frac{1}{r+1} \times fB}{r+2 \times e} z^{2\eta} - \frac{\frac{1}{r+2} \times fC}{r+3 \times e} z^{3\eta} - \frac{\frac{1}{r+3} \times fD}{r+4 \times e} z^{4\eta}, \&c.$$

From the Nature of the Progression of this Series, it is evident, that, if any one Term, after you pass the first, vanish, all that follow it *in inf.* must vanish likewise: because the capital Letter, which is a Factor in each Term, denotes the Coefficient of the immediately preceding Term.

Now let us suppose the second Term $\frac{\frac{1}{r}b - sfA}{r+1 \times e} z^{\eta} = 0$: thence we have $\frac{1}{r}b - sfA = 0$: from which by inserting $\frac{\theta}{\eta} + \lambda$ for s , and

$\frac{1}{r}a$ for A , and reducing, we obtain this Equation $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$. Therefore conversely if it be $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$, the second Term, and consequently all the subsequent ones, of the Series, vanish.

2°. If it be $s = -1$ or $s+1 = 0$, the third Term, and consequently all the subsequent Terms, vanish: if it be $s = -2$, or $s+2 = 0$, the fourth Term, and all following it, vanish; and so on. Therefore since no Term after the second can vanish, but upon supposing $s+1 = 0$ or $s+2 = 0$, &c. and since the other things, with respect to the Values of r , which make the Area infinitely great, easily appear from what has been said formerly, the several things that were to be demonstrated, are evident.

Ex. 1. Suppose $\frac{2x - 3x^{\frac{3}{2}}}{\sqrt{3 - 4x^{\frac{1}{2}}}} = y$: reduce it to Form, and it may stand

thus, $z^{2-1} \times 3 - 4z^{\frac{1}{2}-1} \times 2 - 3z^{\frac{3}{2}}:$ which Form falls under the Consideration of this Article: and by comparing it with the general Form of Ordinates of this Class, $z^{\theta-1} \times e + f z^{\eta})^{\lambda-1} \times a + b z^{\eta}$, we find $\theta = 2$. $\eta = \frac{1}{2}$. $\lambda = \frac{1}{2}$. $r = (\frac{\theta}{\eta}) = 4$. $s = 4\frac{1}{2}$: whence it appears that we cannot determine the Curve to be quadrable, by the second Part of the Rule; from this Expression of the Ordinate, where the Exponent of z under the Vinculum is positive: for it ought to have been s equal to some negative Integer for that purpose. Therefore I try it by the first Part of the Rule, to see whether it be $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$: and since it is $a = 2$. $b = -3$. $e = 3$. $f = -4$,

S

I find

I find $\frac{b^2}{a^2} - 1 = (\frac{1}{3} - 1 = -\frac{2}{3} = -\frac{2}{3})^{\frac{2}{3}}$, therefore I conclude the Curve may be squared, all the Terms of the Series after the first vanishing. Accordingly, by a due Substitution, the Area will be found to be $\frac{x^3}{3} \sqrt{3 - 4x^2}$.

Ex. 2. Let it be $\frac{4bx + 2ax^3}{\sqrt{4a^3 + 3ax^2}} = y$, this Ordinate, reduced to Form, stands thus, $x^{3-1} \times 4a^3 + 3ax^2)^{\frac{1}{2}-1} \times 4b^3 + 2ax^2$; or thus, $x^{3-1} \times 3a + 4a^3x^{-2})^{\frac{1}{2}-1} \times 2a + 4b^3x^{-2}$. In the first Form we have $\theta = 2$. $\eta = 2$. $\lambda = \frac{1}{2}$. $r = 1$. $s = \frac{1}{2}$. $e = 4a^3$. $f = 3a$. $a = 4b^3$. $b = 2a$: where since neither s is a negative Integer nor $\frac{\lambda\eta}{\theta} = \frac{b^2}{af} - 1$: I proceed to the other Form where η is negative, there I find $\theta = 3$. $\eta = -2$. $\lambda = \frac{1}{2}$. $r = -\frac{3}{2}$. $s = -1$, thence I conclude the Curve is quadrable, and the Area consists of two Terms of the Series multiplied into the common Factor $x^0 R^\lambda$: and since in this Case it is $e = 3a$. $f = 4a^3$. $a = 2a$. $b = 4b^3$, by substituting these Values into the general The-

orem for the Area, viz. $x^0 \times e + f x^\eta)^{\lambda} \times \frac{\frac{1}{\eta} a}{re} + \frac{\frac{1}{\eta} b - fA}{r+1 \times e} x^\eta - \frac{\frac{1}{\eta+1} \times fB}{r+2 \times e} x^{2\eta}$

&c. you'll obtain $\frac{4b^3}{3a} - \frac{10a^2}{27} + \frac{2x^2}{9} \sqrt{4a^3 + 3ax^2}$ for the full Area, after a proper Reduction.

163. A Curve of this Form $x^{\theta-1} \times e + f x^\eta)^{\lambda-1} \times a + b x^\eta + c x^{2\eta} = y$ may be squared.

1°. When it is $\frac{r^2 + r}{s+1} = \frac{rbef - saf^2}{ce^2}$: provided it be not $r = 0$, nor yet $r = -1$ (in which Case the Area will be infinitely great) and the Series for the Area terminates with the second Term.

2°. When s is equal to any negative Integer greater than -1 : provided r be not 0 nor any negative Integer so small or smaller than s : (in which Cases the Area becomes infinitely great) and the Series expressing the Area, will consist of as many Terms and one more, multiplied into $x^0 R^\lambda$, as s contains negative Units.

The Curve is quadrable, if these Conditions or Properties agree to the Ordinate in either of it's Forms: it is doubly quadrable, if they agree to both: but if none of them agree to either, the Curve cannot be squared at all.

These Things will easily appear, by considering the Series for the Areas of Curves, exhibited by our Author: when it is reduced to the present

present Case: for it will stand thus $z^0 \times e + f z^n \times \frac{1}{r} + \frac{1}{r+1} \times \frac{1}{r} z^{2n}$

$$+ \frac{\frac{1}{r} - s + 1 \times fB}{r+2 \times e} z^{2n} - \frac{s+2 \times fC}{r+3 \times e} z^{3n} - \frac{s+3 \times fD}{r+4 \times e} z^{4n}, \text{ \&c. where}$$

it is evident, if any Term after the second, become nothing, all the subsequent Terms in *inf.* must vanish: as appears by what was said in

the preceding Art. Now if the third Term $\frac{\frac{1}{r} - s + 1 \times fB}{r+2 \times e} z^{2n}$ be put

equal to nothing, you'll thence obtain $\frac{r^2 + r}{s+1} = \frac{r b e f - s a f^2}{c e^2}$: whence the first Part of the Rule appears.

Again, if it be $s = -2$ or $s + 2 = 0$, the fourth Term vanishes: if it be $s = -3$, or $s + 3 = 0$, the fifth Term vanishes, and so on.

But it is evident, from what has been just now said, that the third Term of the Series cannot vanish, unless it be $\frac{r^2 + r}{s+1} = \frac{r b e f - s a f^2}{c e^2}$: and if it don't vanish, none of the subsequent Terms can vanish, unless s be a negative Integer greater than -1 : therefore if none of these happen, the Series will run out infinitely: consequently, since the other things, respecting the Value of r , are plain by Inspection of the Series: the several things belonging to this Rule, are abundantly evident.

164. When a Curve is of this Form $z^{s-1} \times e + f z^n)^{s-1} \times a + b z^n + c z^{2n} + d z^{3n} + \text{\&c.} = y$, where the last Factor $a + b z^n + c z^{2n}$, \&c. consists of four, or five, or more Terms: it is quadrable

1°. When $\frac{1}{n} d = s + 2 \times fC$; or $\frac{1}{n} e = s + 3 \times fD$, \&c. respectively.

2°. When s is equal to a negative Integer not less than 3 or 4, \&c. respectively: provided still that no Term become infinitely great by r being equal to nothing, or any negative Integer so small or smaller than s . It being suppos'd, you apply these two Rules to both Forms of the Series.

And if neither of these happen the Curve is not quadrable. Which things are evident enough from what hath been already said.

165. Finally, when a Curve of this Form $z^{s-1} \times e + f z^n + g z^{2n})^{s-1} \times a + b z^n + c z^{2n} + \text{\&c.} = y$ is proposed, where the Quantity under the Vinculum of the Root is a Trinomial; and the last Factor $a + b z^n$, \&c. consists of two, or three, or more Terms: proceed thus.

Set down in order the several Terms of the Series expressing the Area, by substituting the particular Values answering to the general Symbols in the Theorem exhibited by our Author in this fifth Prop.; and after you have set down as many Terms of that Series as there are Terms in the last Factor of the Expression of the Ordinate, viz. $a + bx^n + cx^{2n}$, &c.

1°. If the last two of these Terms be each equal to nothing, the Curve is quadrable: and the Terms preceding these last two, multiplied into the common Factor $z^t R^\lambda$, make up the Value of the Area.

2°. If the last one of these Terms only be found equal to nothing, and the Symbol t be any negative Integer (among which 0 is included) equal to the Number of Terms contained in the Factor $a + bx^n + cx^{2n}$, &c. diminished by 2, the Curve is quadrable; and the Area is equal to the preceding Terms multiplied into the common Factor $z^t R^\lambda$.

3°. If neither of the two preceding Cases happen, then see whether t be a negative Integer as great or greater than the Number of Terms less one, contain'd in the Factor $a + bx^n + cx^{2n}$, &c. for if it be not, the Curve cannot be squared: if it be, continue to write down the Terms of the Series for the Area, until you arrive at that Term which immediately precedes the one where it is $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. and if such Term be not nothing the Curve is not quadrable: but if it be nothing the Curve is quadrable; and you have got the Value of the Area: for all the subsequent Terms *in inf.* vanish: it being still supposed in the Cases where I have determined the Curve to be quadrable, that the Symbol r be not nothing, nor any negative Integer so small as to render any Term from the first to the last mentioned, infinitely great, in the Manner formerly explained; which will appear as you proceed in writing down the Forms of the Series: and moreover, that the above Rules be applied to both Terms of the Series; both to that in which the Powers of z are positive; and also to that in which they are negative.

These things will easily appear to any one that considers the Progression of the Series for expressing the Areas of Curves; together with what hath been said already upon this Subject. And this is all I shall say with respect to such Curves, to determine when they are quadrable, and when they are not: because although there might be other particular Rules given for this Purpose, yet they are too particular and intricate to be of any considerable Use.

166. Cor. It easily follows from what has been said with respect to binomial Curves, thus universally expressed $z^{t-1} \times e + \sqrt{fz^n}^{\lambda-1} \times$
 $a +$

$a + bx^n + cx^{2n} + \&c. = y$, that they may be squared when $s (= \frac{0}{n} + \lambda)$ is equal to any negative Integer (among which 0 is included) not less than the greatest Repetition of n among the Exponents of z in the last Factor $a + bx^n + cx^{2n} + \&c.$ which is always one less than the Number of regular Terms contain'd in that Factor: *i. e.* if there be none but the first Term a ; the Curve may always be squared, when s is 0, or -1 , or -2 , $\&c.$: when there are two Terms, *viz.* $a + bx^n$, it may be done if s be -1 , or -2 , or -3 , $\&c.$: if there are three Terms, as $a + bx^n + cx^{2n}$, or $a + cx^{2n}$: it can be done when s is -2 , or -3 , or -4 , $\&c.$ and so for others: provided still that $r (= \frac{0}{n})$ be not 0, nor any negative Integer so small, or smaller than s .

In simple Binomials, where it is $z^{0-1} \times e + f z^{n-1} \times a z^{0n}$, or $a z^{0-1} \times e + f z^{n-1} = y$, the Curve can never be squared in any other Case *: and in others, it very seldom happens *: supposing still that both Forms or Expressions of the Ordinate be tried by this Rule.

* Art. 147.
of this Ex-
plication.
* Art. 163.
of this Ex-
plication.

S E C T. V.

Of the Position and Limits of the Areas of Curves.

167. **T**HE Areas of Curves, discovered by the Method of Quadratures, laid down by our Author in this Treatise, may have various Positions and Limits, according to the different Natures and Properties of the Curves, whose Areas are thus determined. Therefore that the young Geometrician may not be at a loss with respect to this Affair, I shall explain by some Examples, how this Variety happens: and then shew, by what means, the Area of a Curve, adjacent to any Portion of the Absciss, may always be found: when once the Area of the Curve in general is discovered, by the Methods taught by our Author.

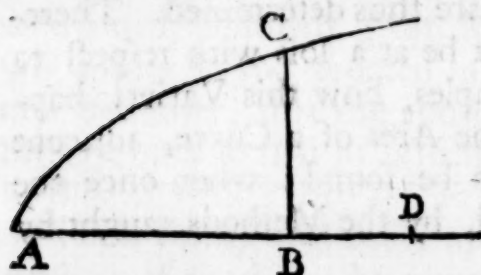
168. In order to which, suppose $az^n = y$ be an Equation, expressing the Relation of the Absciss, and perpendicular Ordinate of a

Curve. The Area, found by the Method of Quadratures, is $\frac{az^{n+1} + 1}{n+1}$ *. And this Area, as likewise, every other curvilinear Area, found by the same Method, is bounded by the Ordinate drawn through the Extremity of the Absciss, and more or less of the Absciss and Curve-line, proceeding

* Art. 143.
of this Ex-
plication.

proceeding from the Extremities of that Ordinate: but this Area, sometimes lyes upon the nearer Side of the Ordinate; (by which I mean the same Side the Beginning of the Absciss lyes upon) at other times, it lyes upon the further Side of the Ordinate: it may happen likewise, that an Area may lye partly upon one Side, and partly upon the other, in some Curves, according to the different Terms, by which that Area is expressed. Now that you may know how far the Area extends, and upon what Side of the Ordinate it is situate, and consequently it's initial Limit, from whence it is to be computed, put the Expression of the Area equal to nothing, and the Value of z thence arising, or which agrees to that Supposition, will give you this initial Limit: *i. e.* an Ordinate drawn through the Extremity of z , taken of that Length; which is thus determin'd, will be the initial Limit of the Area: and so you'll have the Limits of the Area upon every Side.

Thus in the Example just now mentioned, where the Area is $\frac{ax^{\eta+1}}{\eta+1}$, put or suppose $\frac{ax^{\eta+1}}{\eta+1} = 0$: and if the Index $\eta + 1$ be positive, it is evident, this will happen when $z = 0$: and cannot happen in any other Case. But if the Index $\eta + 1$ be negative, the Area denoted by the Expression above, *viz.* $\frac{ax^{\eta+1}}{\eta+1}$, can never vanish, but when z is infinitely great. Hence you conclude, that in the former Case, the Area commences at the Beginning of the Absciss, where also the Curve intersects it, and so reaches from that, along the Absciss, all the way to the Ordinate. Whereas in the other Case, it reaches along the Ab-

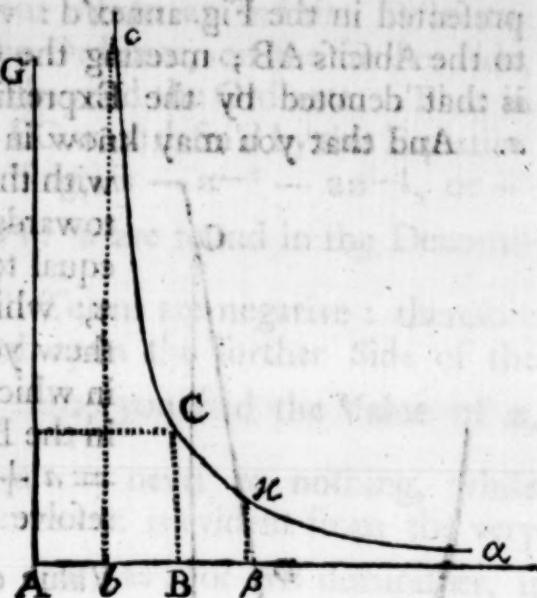


sciss infinitely produced beyond the Ordinate. Let AB be the Absciss, and BC the perpendicular Ordinate, belonging to the Curve AC and take $BD = a$: suppose likewise, in the preceding Example, $\eta = \frac{1}{2}$: so that the Property of the Curve be expressed

by the Equation $ax^{\frac{1}{2}} = y$; by inserting $\frac{1}{2}$ for η , the Area is $\frac{2ax^{\frac{3}{2}}}{3} = \frac{2}{3}BD \times \sqrt{AB^3}$ or $\frac{2}{3}AB \times BC$, which is the Area ABC: since the Expression $\frac{2}{3}ax^{\frac{3}{2}}$ can never be equal to nothing, but when $z = AB$ is nothing.

But, if the Relation of AB to BC be expressed by this Equation $x^2y = a$ or $y = ax^{-2}$ (see the Fig. in the next Page) the Area, by inserting -2 for η , becomes

becomes $-ax^{-1}$ or $-\frac{a}{x}$: which denotes the Area $BC\alpha$ lying upon the further Side of the Ordinate BC , and extending along the Line $B\alpha$ infinitely produced: for if you put $-\frac{a}{z} = 0$, z must be infinitely great, since a is a given Quantity: which shews that the initial Limit of the Area removes to an infinite Distance. This Area comes out negative, in regard it is situate upon the further Side of the Ordinate BC , and wholly above the Absciss AB produc'd: for since the Area ABC upon the nearer Side of the Ordinate, and above the Absciss is positive, the Area $BC\alpha$ ought to be represented negative, from the opposite Nature of positive and negative Quantities. Thus if we suppose $a = 1$, and $z = AB = 1$ also, we have $-\frac{a}{z} = -1$ for the Area $BC\alpha$: whose positive Value therefore is $1 = AB^2$ or the Square AC , for in this Case $AB = BC$.



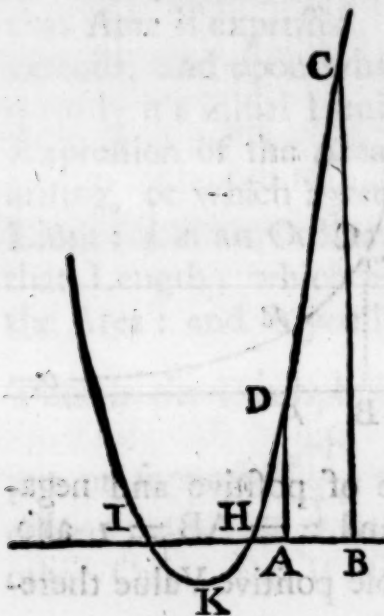
169. After the same Manner, and for the like Reason, it will be found, that the Area of every Curve, which consists of any Number of simple Terms, in every one of whose Numerators there is contain'd some positive Power of z , will have it's initial Limit, where the Absciss begins: And every Area whose Expression contains some positive Power of z in the Denominator of every Term, will be situate upon the further Side of the Ordinate, and extend itself along the Absciss infinitely produc'd beyond the Ordinate.

Let $a + bz + cz^2 = y$ be an Equation, containing the Relation of the Absciss and Ordinate of a Curve. The Area, by Art. 145 *, is $az + \frac{1}{2}bz^2 + \frac{1}{3}cz^3$: where the positive Powers of z run through the Numerators of all the Terms: therefore the initial Limit of that Area is at the Beginning of the Absciss: and the Area is bounded by an Ordinate drawn through the Beginning of the Absciss, the Ordinate belonging to the Absciss z , and the Absciss and Curve-line intercepted betwixt these two Ordinates.

* Of this
Explica-
tion.

170. I have said, you must draw an Ordinate to the Curve at the Beginning of the Absciss in this Example, in order to have the initial Limit of the Area: because the Curve does not meet with the Absciss at the Beginning, as is evident: for since $y = a + bz + cz^2$; when the

the Absciss $z = 0$, we have $y = a$: which shews the Ordinate at the Beginning of the Absciss, to be equal to the given Quantity a , as is represented in the Fig. annex'd: where, if you draw AD perpendicular to the Absciss AB; meeting the Curve in the Point D, the Area ABCD is that denoted by the Expression $ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$; and $AD = a$. And that you may know in every Case, where the Curve meets



with the Base or Absciss produc'd, if need be, towards both Hands, suppose the Value of y equal to nothing, and the Value or Values of z , which arise from that Supposition, will shew you the Point or Points of the Absciss, in which the Curve-line meets with it. Thus in the Example just now mentioned, where $y = a + bz + cz^2$, put $cz^2 + bz + a = 0$: resolve the Equation, and you'll find a double

Value of z , viz. $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$, both which Values are negative, and shew that if the Absciss AB be produc'd backwards, and the Points H and I be taken such, that $AH = \frac{b - \sqrt{b^2 - 4ac}}{2c}$; and $AI = \frac{b + \sqrt{b^2 - 4ac}}{2c}$, the

Curve CD continued, will meet with the Absciss produc'd, in the Points H and I: and in no other Points, in regard z has no other Value in the Equation $a + bz + cz^2 = 0$, i. e. when y is equal to nothing.

But if the given Quantities a , b and c be such that it be $b^2 = 4ac$, then the two Points H and I fall into one, because $\sqrt{b^2 - 4ac}$ becomes nothing: and so there will be but one Point of the Absciss, in which the Curve will meet with it, distant from the Beginning of the Absciss, produc'd backwards, by the given Quantity $\frac{b}{2c}$.

Again, if $4ac$ be greater than b^2 , the Expression $\sqrt{b^2 - 4ac}$ is impossible or imaginary, and so there will be no possible Value of z , or the Roots of the Equation $cz^2 + bz + a = 0$, are imaginary: and this shews that the Curve can no where meet with the Absciss, in this Case. And by the same Method, you may always find whether any Curve proposed meets with it's Absciss or not; and if it do, in what Point or Points of the Absciss it meets with it. The Rule is this.

171. Put the Value of the Ordinate, expressed by given Quantities and the Absciss z , equal to nothing: and the Values of z thence arising, will be the Distances of these Points of Concurrence from the Beginning of the Absciss: and if all the Roots of the Equation be impossible, there is no Point of Concurrence.

172. But

172. But to return to the Limits of Areas. If the positive Powers of z run through the Denominators of all the Terms of the Expression of the Area, the initial Limit runs off to an infinite Distance; and so the Area extends itself from the Ordinate, on the further Side, all along the Absciss infinitely produc'd beyond the Ordinate. Thus in the Fig. p. 135, if the Relation of AB to BC were defin'd by this Equation $z^{-2} + z^{-\frac{1}{2}} = y$: the Area thence arising, is $-z^{-1} - 2z^{-\frac{1}{2}}$, or $-\frac{1}{z} - \frac{2}{z^{\frac{1}{2}}}$ *; where the positive Powers of z are found in the Denomi-

* Art. 145. of this Explication.

nators of the several Terms; and both Terms are negative: therefore it is the Area BC α , infinitely extended upon the further Side of the Ordinate: for if you put $-\frac{1}{z} - \frac{2}{z^{\frac{1}{2}}} = 0$, you find the Value of z ,

infinite: which shews that that Area can never be nothing, while the Absciss AB is of a finite Length: but it is evident from the very Nature of the Expression $-\frac{1}{z} - \frac{2}{z^{\frac{1}{2}}}$, that, as z or AB diminishes, it

increases; and as AB increases, it continually diminishes: therefore, it must lye beyond the Ordinate BC: and it also lyes wholly above the Absciss, as will appear by the preceding Art. For if you put $\frac{1}{z^2} + \frac{1}{z^{\frac{1}{2}}} (= y) = 0$, z can have no finite Value. From both which

Considerations and Circumstances it appears, why it is represented negative. Let AB ($= z$) = 4, then $-\frac{1}{z} - \frac{2}{z^{\frac{1}{2}}} = -\frac{1}{4} - \frac{2}{2} = -1\frac{1}{4}$,

which being made positive, we have $1\frac{1}{4}$ for the Value of the Area BC α .

173. But sometimes the Area of a Curve will lye partly above, and partly below the Absciss or Base: which happens, when the Curve decussates or crosses the Base, and passes to the opposite Side, as is represented by the Figures in p. 136, 140 and 141. In order to discover whether there be any Points of Decussation; and if they be, where they are; do this. Find the Points of the Absciss, where the Curve meets with it, by Art. 171 *. And if there be no Point of Concurrence, it is evident, there can be no Point of Decussation: in which Case the Area must lye wholly upon one Side of the Absciss. But if you find Points of Concurrence, then you may know whether any of these be a Point of Decussation, thus: add to, and subtract from the given Length of the Absciss at the Point of Concurrence, a very small Quantity, which call p : and instead of z , in the Value of y , insert $z + p$, and then $z - p$: then neglecting all the Terms, where p rises higher than the Root, or other lowest Power; observe whether the Values of

* Of this Explication.

T

y,

y , which remain, be, the one affirmative, the other negative: if they be, there is a Point of Decussation, at that Point of Concurrence: if not, but that both Values be affirmative, or both negative, there is no Point of Decussation, in that Place: and if one of the Values involve any impossible Supposition, the Curve is reflected back towards the Parts from whence it came.

The Reason of this Rule will easily appear, by considering, that, when there is a Point of Decussation, the Values of the Ordinate towards either hand of that Point, must be affected with opposite Signs: and the Powers of p above the Root, or above the lowest Power, may be neglected; because p is suppos'd to be diminished infinitely. (See Art. 17 *. and Note upon it.) The Thing will be further illustrated by Examples.

* Of the
Quadrature of
Curves.

Ex. 1. Let $AB=z$, and $BC=y$, be the Absciss and Ordinate of the Curve CDE (see the Fig. in p. 141.) and suppose $\frac{1}{z^2} - \frac{1}{z^{\frac{3}{2}}} = y$ express the

Relation of the Absciss and Ordinate: then putting $\frac{1}{z^2} - \frac{1}{z^{\frac{3}{2}}} = 0$, we have

$z = \text{infinite}$; and $z = 1$: so that at the Extremity of the Absciss 1, there is a Point of Concurrence, wherefore I take $z + p$, and then $z - p$, and insert these for z , in the Value of y , viz. $\frac{1}{z^2} - \frac{1}{z^{\frac{3}{2}}}$: thence

you obtain 1^o $\frac{1}{(z+p)^2} - \frac{1}{(z+p)^{\frac{3}{2}}} = y$, i. e. by inserting the Value of z at that Point, viz. 1, $\frac{1}{(1+p)^2} - \frac{1}{(1+p)^{\frac{3}{2}}} = y$, where $\frac{1}{(1+p)^2} - \frac{1}{(1+p)^{\frac{3}{2}}}$ is plainly negative: since $1 + p$ is greater than 1.

Again, by inserting $1 - p$ for z you'll have 2^o $y = \frac{1}{(1-p)^2} - \frac{1}{(1-p)^{\frac{3}{2}}}$ positive, because $1 - p$ is less than 1: wherefore if you take $AD = 1$ the Point D is a Point of Decussation.

Ex. 2^d. In the Curve CDHI, (see the Fig. in p. 136.) whose Absciss is $AB=z$, and Ordinate $BC=y$: let it be $y = a + bz + cz^2$, by putting $a + bz + cz^2 = 0$, you obtain $z = \frac{-b + \sqrt{b^2 - 4ac}}{2c}$ and $z = \frac{-b - \sqrt{b^2 - 4ac}}{2c}$, denoting two Points of Concurrence as before. For the first, take $\frac{-b + \sqrt{b^2 - 4ac}}{2c} + p$, and then $\frac{-b + \sqrt{b^2 - 4ac}}{2c} - p$, or by substituting q for $\frac{-b + \sqrt{b^2 - 4ac}}{2c}$, (for Brevity's sake) $q + p$ and $q - p$: insert these

these for z in the Equation $y = a + bz + cz^2$: thence you have 1°

$$y = \frac{cq^2}{bq} + \frac{2cq}{b}p + cp^2$$

$$2^{\circ}y = \frac{cq^2}{bq} - \frac{2cq}{b}p + cp^2$$

Neglect or throw away what is common in these two Values of y ; what remains is $2cqp + bp$ in the first; and $-2cqp - bp$ in the second: which are, the one affirmative; the other negative: therefore if you produce the Absciss AB backwards to the Point H, and take

$$AH = \frac{b - \sqrt{b^2 - 4ac}}{2c}, \text{ the Point H will be a Point of Decussation.}$$

And after the same Manner, if you produce it still further, in the same

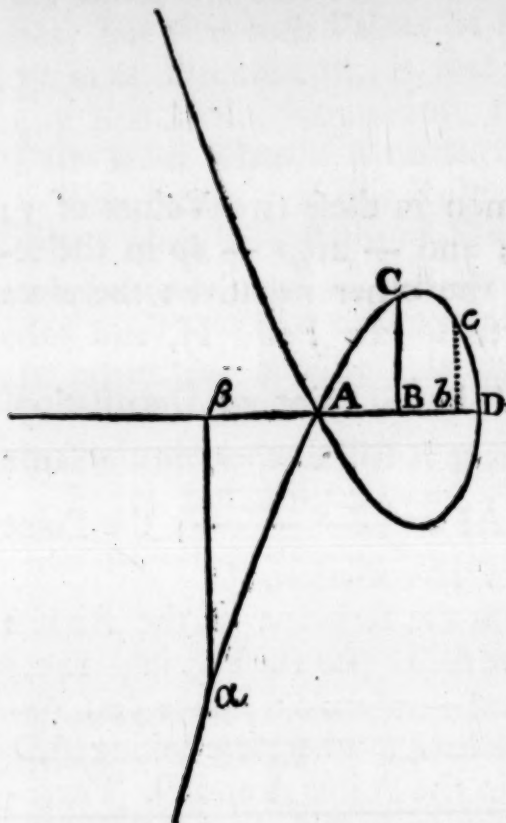
$$\text{Direction, to the Point I, so that it be } AI = \frac{b + \sqrt{b^2 - 4ac}}{2c}, \text{ the Point}$$

I would be found to be another Point of Decussation.

Ex. 3. Suppose $4z^2 - z^3 = y^2$ expresses the Relation of the Absciss $AB = z$ and Ordinate $BC = y$, in the Curve ACD (see the Fig. in p. 140.) put the Value of y equal to nothing, or $4z^2 - z^3 = 0$: hence you obtain two Values of z , viz. $z = 0$; and $z = 4$: wherefore taking $AD = 4$, the Curve meets with the Absciss in the Points A and D. Therefore to find whether the Curve decussates the Base in the Point A, where $z = 0$, put $+p$, and then $-p$ instead of z in one and the same Value of y ; suppose the positive Value of y , viz. $+\sqrt{4z^2 - z^3}$ or $z\sqrt{4 - z}$: (for it is $y = \pm\sqrt{4z^2 - z^3} = \pm z\sqrt{4 - z}$) hence you'll have $+p\sqrt{4 - p}$ and $-p\sqrt{4 + p}$ for the two Values of y ; of which the first being positive, the other negative, shews the Curve decussates the Base at the Point A.

Again, for the other Point of Concurrence, viz. D; where you have $z = 4$, by inserting $4 + p$ and then $4 - p$ in place of z in the Equation $y = z\sqrt{4 - z}$, you have 1° $y = 4 + p\sqrt{-p}$, and 2° $y = 4 - p\sqrt{p}$: the first of which Values being impossible, hence you conclude that the Curve proceeds no further that way, but returns back towards the Beginning of the Absciss, making either a Cuspis, or continued Arch: which last it doth in this Case, forming another half upon the lower Side of the Base AB, exactly similar and equal to the half above: since the Property of the Curve is represented by the Equation $y^2 = 4z^2 - z^3$, and consequently $y = \pm\sqrt{4z^2 - z^3}$: which

shews that the Ordinate, at any Point of the Absciss, meets the Curve on both Sides, at the same Distance, as is represented by the Fig. annex'd, being the *modated Parabola*, which constitutes the sixty-eighth Species of Curves of the second Order, according to our Author's Enumeration of Lines of the third Order.



And after the like Manner, in any other Case, it will appear whether the Curve proceed forward or not, at any Point of Concurrence with the Absciss: as for other Circumstances, which may occur, with respect to the Points of Concurrence and Decussation, the Reflection and Continuation of Curves, they must be left to the young Geometrician's own Sagacity: because the insisting upon these things in this Place would carry us too far away from our original Design¹.

174. If the Curve decussate the Base or Absciss betwixt the initial Limit of the Area and the Ordinate: whereby one Part of the curvilinear Area lyes above the Base, and another Part below, the Area which arises by the Method of Quadratures, will exhibit the Difference of the two Parts of the Area above and below the Base, lying betwixt the initial Limit and the Ordinate belonging to any given Absciss.

Thus in the Curve CDE (see the Fig. in the opposite Page) whose Absciss is $AB = z$, and perpendicular Ordinate $BC = y$, suppose it be $z^{-2} - z^{-1} = y$: it was shewn above *, that the Curve will decussate it's Base AB, in one Point only, at the Distance 1 from A the Beginning of the Absciss: therefore, if we take $AD = 1$, D is the Point, at which the Curve crosses the Base, and afterwards extends itself infinitely below it: Now

the Area of the Curve is $-\frac{1}{z} + \frac{2}{z^{\frac{1}{2}}}$ *: which Area extends itself, upon the further Side of the Ordinate BC, along the Absciss infinitely produc'd towards α ; because of the positive Powers of z , in the Denominators of the Terms *: but that Expression denotes the Difference

* Art. 168, 169. of this Explanation.

¹ Some Things of this sort, respecting the Symptoms of Curves, will occur afterwards in the Notes upon the last Scholium.

rence

it be positive, it denotes the Excess of $DE\alpha$ above CBD ; but if it amount to nothing, it denotes that the Areas CBD , $DE\alpha$ are equal.

Suppose $AB = \frac{1}{9}$, then $-\frac{1}{z} + \frac{2}{z^{\frac{1}{2}}} = -9 + 6 = -3$, therefore in that Case CBD would exceed $DE\alpha$ by 3, but if it be $AB = \frac{4}{9}$, then $-\frac{1}{z} + \frac{2}{z^{\frac{1}{2}}} = -\frac{9}{4} + 3 = \frac{3}{4}$ positive, which shews that $DE\alpha$ exceeds CBD by so much. Lastly, if it were $AB = \frac{1}{4}$, then $-\frac{1}{z} + \frac{2}{z^{\frac{1}{2}}} = -4 + 4 = 0$: in which Case CBD and $DE\alpha$ are exactly equal.

176. But if you would have the Sum of both Areas CBD and $DE\alpha$; find the Value of the Area $DE\alpha$ by itself, *viz.* by supposing the Point B the Foot of the Ordinate, to fall in with the Point of Decussation D : and so having obtain'd the Area $DE\alpha$, lying all upon one Side of the Base: and having formerly found the Difference of the two Areas CBD and $DE\alpha$, above and below the Base, you easily obtain the Area CBD also by itself, and consequently the Sum of both. Thus when B and D coincide $AB = z = 1$ *: therefore the Area $DE\alpha = (-\frac{1}{z} + \frac{2}{\sqrt{z}}) 1$: but if $AB (=z) = \frac{1}{9}$ as formerly, then $-\frac{1}{z} + \frac{2}{\sqrt{z}} = -3$, which is the Difference of the two Areas, the Difference then being made positive is 3: which is the Excess of the Part lying above the Base, above the Part lying below. Therefore the Part lying above is $3 + 1 = 4$, and the Sum of both is $4 + 1 = 5$: and so in other Cases.

177. Cor. Hence it follows, that, if the Absciss AB , be of such a Length, as to make the Areas CBD and $DE\alpha$ above and below the Base, equal, and you take the Absciss Ab less than AB , and draw the Ordinate bc : and then suppose $Ab = z$, the Expression $-\frac{1}{z} + \frac{2}{z^{\frac{1}{2}}}$ will exhibit the Value of the Area $BCcb$, for that is the same with $cDb - DE\alpha$. And thus the Ordinate BC , so drawn, will be an initial Limit, from whence the Area is computed, when the Absciss Ab is less than AB . And thus there are two initial Limits in this Curve: the one at an infinite Distance from A , the Beginning of the Absciss, towards α ; the other is an Ordinate drawn from that Point of the Absciss; which is distant from A , by $\frac{1}{4}AD = \frac{1}{4}$. These initial Limits, both arise, by putting the Expression of the Area, *viz.* $-\frac{1}{z} + \frac{2}{z^{\frac{1}{2}}} = 0$:

$=0$: for that will happen, both when z is infinitely great, and also when $z = \frac{1}{4}$: as will appear by resolving the Equation.

178. By considering of this, you'll know how to apply any like Case, that may happen: where, by putting the Expression for the Area equal to nothing, as directed above, in order to find it's initial Limit, you obtain different Values of z , or Roots of the Equation: for these different Values of z , give you so many different initial Limits, from any one of which, the Area may be computed. The Reason of which is evident: because the curvilinear Area lying betwixt any two initial Limits amounts to nothing, in the algebraical Notation; the Parts above and below the Base, which are opposite Quantities, exactly compensating one another.

But although the curvilinear Area may be thus computed from any one of the initial Limits mentioned: yet it is most simple and natural to compute the Area from the highest initial Limit. Thus when BC (see the Fig. in p. 141.) is an initial Limit: if you take $Ab = z$ less than AB, e.g. $\frac{1}{9}$, the Area thence arising, viz. $-\frac{1}{z} + \frac{2}{\sqrt{z}} = -3$, is the Area BCcb; as well as the Difference betwixt Dbc and DEa.

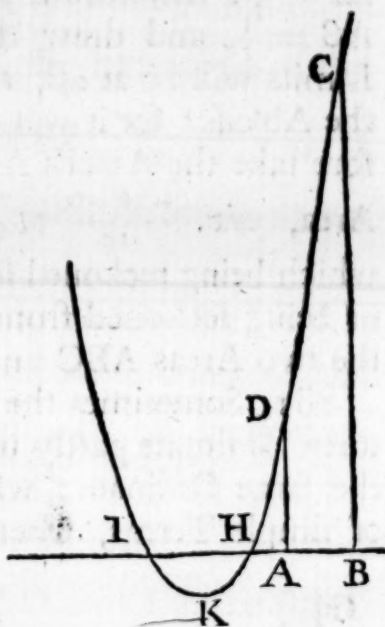
179. Let it be $y = a + bx + cx^2$ (see the Fig. annexed) as before, the Area is $ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$: put that Expression of the Area $ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3 = 0$: thence you obtain, 1°. z

$$= 0. \quad 2^\circ. z = \frac{-3b + \sqrt{9b^2 - 48ac}}{4c}. \quad 3^\circ. z = \frac{-3b - \sqrt{9b^2 - 48ac}}{4c}.$$

Whence you have three initial Limits: one at the Point A, where the Absciss begins: the other two beyond the Beginning of the Absciss AB, viz. produced towards H, as is denoted by the negative Values of z . The two last evidently fall into one, if so be $16ac = 3b^2$: for then the

common radical Part $\sqrt{9b^2 - 48ac} = 0$, in the two negative Values of z : so that, in that Case, they both become $z = -\frac{3b}{4c}$. Let us suppose $a = 1. b = 4. c = 3$: then $16ac = 48, = 3b^2$, therefore $\sqrt{9b^2 - 48ac} = 0$: and so $z = \frac{-3b + \sqrt{9b^2 - 48ac}}{4c} = -1$; and $z = \frac{-3b - \sqrt{9b^2 - 48ac}}{4c} = -1$ likewise. In this Case

the



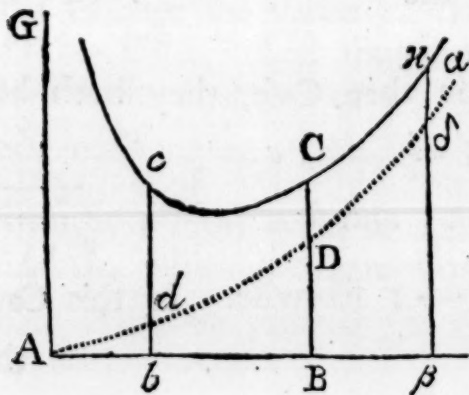
the Points of Decussation H and I will be such, that $AH = -\frac{1}{2}$ and $AI = -1$: as will easily appear, by substituting these Values of the Symbols a, b , and c in place of them, in the Equations above, for determining these Points (see Ex. 2. Art. 173 *). Hence if you take $AI = -1$, *i. e.* produce the Absciss AB backwards to I, so that $AI = 1$, the Points A and I will be the only initial Limits: and the Area $ADH = HKI$ *. To find the Area ADH, substitute $-\frac{1}{2}$ for z in the general Expression of the Area, *viz.* $ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$, and 1, 4, 3 for a, b, c ; and it becomes $-\frac{1}{2} + \frac{2}{9} - \frac{1}{27} = -\frac{4}{27}$. Again, to find the Difference betwixt the two Areas ADH and HKI, substitute -1 for z in the Expression for the Area, *viz.* $ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$, and you'll have $-1 + 2 - 1 = 0$ for the Difference.

180. Resuming Ex. 3^d. of Art. 173 *, *viz.* $4z^2 - z^3 = y^2$, belonging to the Curve represented by the Fig. in p. 140, by extracting the square Root, it is $y = z\sqrt{4-z}$: therefore the Area is $\frac{-16-6z}{15} \times 4 - z^{\frac{3}{2}}$ *. To find the

initial Limit, put $\frac{-16-6z}{15} \times 4 - z^{\frac{3}{2}} = 0$: and you find this to happen in two Cases, and no more, *viz.* 1°. when $z = 4$; 2°. when $z = -\frac{8}{3}$: therefore if you take in the Absciss AB, produc'd backwards, $A\beta = \frac{8}{3}$, and draw the perpendicular Ordinate $a\beta$: the two initial Limits will be at $a\beta$, and at the Point D, where the Curve meets with the Absciss: for it was demonstrated already that $AD = 4$ *. Wherefore take the Absciss $AB = 2 = z$: and the general Expression of the Area, *viz.* $\frac{-16-6z}{15} \times 4 - z^{\frac{3}{2}}$, becomes $\frac{-16-12}{15} \times 2^{\frac{3}{2}} = -\frac{28}{15}\sqrt{8}$: which being reckoned from the initial Limit D, denotes the Area DBC; or being reckoned from the initial Limit $a\beta$ denotes the Difference of the two Areas ABC and $A\beta a$ lying above and below the Base.

181. Sometimes the Area, which arises by the Method of Quadratures, is situate partly upon one Side, and partly upon the other Side of the same Ordinate: which happens when the Value of y is made up of simple Terms, whereof some have the positive Powers of z in the Numerators; and others have them in the Denominators.

Thus let the Relation of the Absciss $AB = z$, and Ordinate $BC = y$, be defin'd by this Equation $z^2 + \frac{1}{z^2} = y$: (see the Fig. annex'd.) The Area, by the Method of Quadratures, will be $\frac{1}{3}z^3 - \frac{1}{z}$ *: the first Part of which, $\frac{1}{3}z^3$ denotes an Area lying betwixt the Beginning



* Art. 145.
of this Ex-
plication.

ginning of the Absciss and the Ordinate; the other Part of the Expression, viz. $-\frac{1}{z}$, denotes an Area situate upon the further Side of the Ordinate, and extends itself to an infinite Distance upon that Side *.

* Art 169. of this Explication.

The Thing is explained after the same Manner as the Example in Art. 175*. For let AdD be another Curve, whose Ordinate $BD = z^2$, the first Part of the Expression $z^2 + \frac{1}{z^2} = BC$; then, it is evident, the re-

* Of this Explication.

maining Part of the Ordinate BC , viz. $DC = \frac{1}{z^2}$, the other Part of the Expression $z^2 + \frac{1}{z^2}$.

Now the Area belonging to the Ordinate $BD = z^2$, viz. $\frac{1}{3}z^3$, is ABD ; and the Area belonging to the other Ordinate $DC = \frac{1}{z^2}$, viz. $-\frac{1}{z}$, is $CD\alpha$, lying upon the further Side of the Ordinate BC , and extending itself infinitely betwixt the two Curve-lines $cC\alpha$ and $AdD\alpha$, after the Manner formerly explained in that Art. 175*.

* Of this Explication.

182. If we shall seek for the initial Limit of the Area in this Case, viz. by putting the Area $\frac{1}{3}z^3 - \frac{1}{z} = 0$, we find only one Value of

z , viz. $z = \sqrt[4]{3}$: Take the Absciss $AB = \sqrt[4]{3}$, draw through the Point B , the Ordinate BDC , substitute $\sqrt[4]{3}$ for z in the Expression of the Area, and you'll have the Area $\frac{1}{3}z^3 - \frac{1}{z} = \frac{1}{3}\sqrt[4]{27} - \frac{1}{\sqrt[4]{3}} = \sqrt[4]{\frac{27}{81}} - \sqrt[4]{\frac{1}{3}} = 0$. This shews that when the Ordinate BDC is drawn at the Distance $\sqrt[4]{3}$ from A , the Areas ABD and $CD\alpha$ are equal.

183. If the given Absciss $A\beta$ be greater than the Absciss AB , which reaches to the Limit, the Expression $\frac{1}{3}z^3 - \frac{1}{z}$ is positive and denotes the Area $B\beta\kappa C$: but when the Absciss Ab is less than AB , the Expression $\frac{1}{3}z^3 - \frac{1}{z}$ is negative: but being changed into it's opposite, will denote the Area $BbcC$. For when $z = A\beta$, then $\frac{1}{3}z^3 - \frac{1}{z} = A\beta\delta - \alpha\delta\kappa = ABD + B\beta\delta D - CD\alpha + CD\delta\kappa = B\beta\delta D + CD\delta\kappa$ or $B\beta\kappa C$ for $ABD = CD\alpha$. Again, when $z = Ab$, then $\frac{1}{3}z^3 - \frac{1}{z} = Abd - \alpha dc$: but $Abd = ABD - bBDd$, and $\alpha dc = \alpha DC + DCcd$: subtract the latter from the former, and you have $\frac{1}{3}z^3 - \frac{1}{z} = ABD - bBDd - \alpha DC - DCcd = -bBDd - DCcd = -BbcC$.

Let it be $A\beta = 2$, then $\frac{1}{3}z^3 - \frac{1}{z} = \frac{8}{3} - \frac{1}{2} = 2\frac{1}{6}$, the Value of the Area $B\beta\kappa C$ lying betwixt the Ordinate $\beta\kappa$ and initial Limit BC . Again
U suppose

suppose $Ab = \frac{2}{3}$, then $\frac{1}{3}z^3 - \frac{1}{z} = \frac{8}{27} - \frac{1}{2} = -\frac{1}{18}$: which being made positive, is the Value of the Area $BbcC$.

184. Sometimes no initial Limit of the Area can be found at all. This will always happen, when, putting the Expression for the Area equal to nothing, the Equation has no real Root or Value of z . Thus if the Curve be such that $z^2 - \frac{1}{z^2} = y$: then the Area, by the Me-

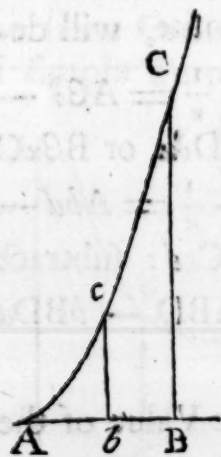
*Art. 145.
of this Ex-
plication.

thod of Quadratures, is $\frac{1}{3}z^3 + \frac{1}{z}$ *: if we put $\frac{1}{3}z^3 + \frac{1}{z} = 0$, thence we have $z^4 + 3 = 0$ or $z = \sqrt[4]{-3}$, an impossible Expression: and thus, there is not any possible or real Value of z , in the Equation $\frac{1}{3}z^3 + \frac{1}{z} = 0$, or $z^4 + 3 = 0$. Therefore no initial Limit, from whence the curvilinear Area ought to be computed, can be found. Again, suppose the Curve to be defin'd by this Equation $z\sqrt{a^2 + z^2} = y$:

*Art. 147.
of this Ex-
plication.

The Area, by the Method of Quadratures, is $\frac{a^2 + z^2}{3}\sqrt{a^2 + z^2}$ *. Put it equal to nothing, i. e. $\frac{a^2 + z^2}{3}\sqrt{a^2 + z^2} = 0$, and you find this can never happen, unless it could be $a^2 + z^2 = 0$ or $z = \sqrt{-a^2}$, which is impossible. Therefore no initial Limit can be found: and so in other Cases.

185. When this happens, you may, notwithstanding, find the Area which is adjacent to any given Part of the Absciss, and lying betwixt the two Ordinates at the two Extremities of that given Part of the Absciss, viz. thus: subtract the Area belonging to the lesser Absciss, arising by the Method of Quadratures, from the Area belonging to the greater Absciss, the Difference is the Value of the Area adjacent to the Difference of the Abscisses.



Let the Curve AC (see Fig. annexed) whose Base is AB, be defin'd by the Equation $z\sqrt{a^2 + z^2} = y$, mention-
ed just now: the Area, by the Doctrine of Quadra-
tures is $\frac{a^2 + z^2}{3}\sqrt{a^2 + z^2}$. Wherefore, to find the cur-
vilinear Area adjacent to any given Part of the Absciss
as Bb: let $AB = 2$, $Ab = 1$: substitute $z = AB$ for
 z in the general Expression of the Area $\frac{a^2 + z^2}{3}\sqrt{a^2 + z^2}$,
and it becomes $\frac{a^2 + 4}{3}\sqrt{a^2 + 4}$: again substitute $1 =$
 Ab for z , and you'll have $\frac{a^2 + 1}{3}\sqrt{a^2 + 1}$: subtract

this

this last from the other, and it gives $\frac{a^2+4}{3}\sqrt{a^2+4} - \frac{a^2+1}{3}\sqrt{a^2+1}$, for the just Value of the Area BbcC adjacent to the given Part of the Absciss Bb.

186. Universally: call the greater Absciss $AB = z$; the lesser $Ab = \zeta$; and then the Area BbcC is $\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^2+\zeta^2}{3}\sqrt{a^2+\zeta^2}$. If $\zeta (=Ab) = 0$, then $\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^2+\zeta^2}{3}\sqrt{a^2+\zeta^2} = \frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^2}{3}$, for the Value of the Area ABC, reckoning from the Beginning of the Absciss. It comes to the same Purpose to suppose $z = 0$ in the general Expression $\frac{a^2+z^2}{3}\sqrt{a^2+z^2}$, and then subtract what arises, *viz.* $\frac{a^2}{3}\sqrt{a^2} = \frac{a^3}{3}$, from that general Expression, giving $\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^3}{3}$ for the Value of the Area ABC adjacent to the whole Absciss AB.

Let it be $a = 1 = AB$; then the Area ABC $= (\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^3}{3}) = \frac{2}{3}\sqrt{2} - \frac{1}{3}$.

187. The same Method may be used in the Case of any other Curve, whether it admit of an initial Limit for the Area, according to the Rule delivered above, or not. Yea in regard in some Curves, one may be perplex'd, and ready to mistake, you do best to follow this same Method, in the Case of any Curve proposed.

Thus in the Curve ACcD, represented by the Fig. p. 140. where we have $y = z\sqrt{4-z}$: let it be required to find the Area BbcC adjacent to Bb a given Part of the Absciss. The general Expression of the Area is $\frac{-16-6z}{15} \times 4 - z^{\frac{3}{2}}$. Calling therefore the lesser Absciss $AB = z$; and the greater Absciss $Ab = \zeta$, subtract the Area belonging to z from the Area belonging to ζ , and you'll have $\frac{-16-6\zeta}{15} \times 4 - \zeta^{\frac{3}{2}} + \frac{16+6z}{15} \times 4 - z^{\frac{3}{2}}$, for the Value of the Area BbcC. And if $z (=AB) = 0$ you'll have $\frac{-16-6\zeta}{15} \times 4 - \zeta^{\frac{3}{2}} + \frac{16}{15} \times 8 = \frac{128-16+6\zeta \times 4 - \zeta^{\frac{3}{2}}}{15}$, for the true Value of the Area Abc. Let it be $AB = z = 2$, and $Ab = \zeta = 3$, then $\frac{-16-6\zeta}{15} \times 4 - \zeta^{\frac{3}{2}} + \frac{16+6z}{15} \times 4 - z^{\frac{3}{2}} = -\frac{34}{15} + \frac{28}{15}\sqrt{8} = \text{Area BbcC}$. Again, putting $z = AB = 0$, you'll have (by retaining the same Value of $Ab = \zeta$) $\frac{128-16+6\zeta \times 4 - \zeta^{\frac{3}{2}}}{15} = \frac{128-34 \times 1}{15} = \frac{94}{15}$ for

the Value of the Area Abc , reckoning from the Beginning of the Absciss at A.

* Of this
Explica-
tion.

188. Thus likewise, when the Curve is doubly quadrable: *e. g.* let it be $y = \frac{a}{1+z^2}$ (see the Fig. at Art. 128 *.) It was there demonstrated that the Area of the Curve was either $\frac{az}{1+z^2}$; or yet $\frac{-a}{1+z}$: either of which Expressions will serve to find the Area $BbcC$ adjacent to any given Part of the Absciss as Bb . Calling the lesser Absciss $Ab = \zeta$, the first Form will give you $\frac{az}{1+z^2} - \frac{a\zeta}{1+\zeta^2}$: the second Form will give you $\frac{-a}{1+z} + \frac{a}{1+\zeta}$: either of which exhibits the Value of the Area $BbcC$. Let it be $AB = z = 2$; $Ab = \zeta = 1$, then $\frac{az}{1+z^2} - \frac{a\zeta}{1+\zeta^2} = \frac{2a}{3} - \frac{a}{2} = \frac{1}{6}a$ for the Area $BbcC$. Again $\frac{-a}{1+z} + \frac{a}{1+\zeta} = \frac{-a}{3} + \frac{a}{2} = \frac{1}{6}a$, the same as before. In the first Expression, if you put $\zeta = Ab = 0$, in order to find the Area ABC adjacent to the whole Absciss $AB = 2$, then $\frac{az}{1+z^2} - \frac{a\zeta}{1+\zeta^2}$ becomes $\frac{az}{1+z^2} = \frac{2a}{3} = ABC$. Again, the second Expression, *viz.* $\frac{-a}{1+z} + \frac{a}{1+\zeta}$ becomes $\frac{-a}{1+z} + a = \frac{-a}{3} + a = \frac{2a}{3}$, the same.

189. Let the Nature of the Curve be defin'd by this Equation $z^2 + \frac{1}{z^2} = y$, see the Fig. p. 144. The general Expression of the Area was found

* Art 181.
of this Ex-
plication.

to be $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta}$ *. Call $A\beta = z$ and $Ab = \zeta$, then the Area $b\beta\kappa c = \frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta}$: suppose $A\beta = z = 2$. $Ab = \zeta = \frac{2}{3}$: then $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta} = \frac{8}{3} - \frac{1}{2} - \frac{8}{81} + \frac{3}{2} = 3\frac{4}{9}$ the Value of the Area $b\beta\kappa c$. Again, suppose $\zeta = \sqrt[3]{3}$, and $z = 2$, as before: then $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta} = \frac{8}{3} - \frac{1}{2} - \frac{\sqrt[3]{27}}{3} + \frac{1}{\sqrt[3]{3}} = \frac{8}{3} - \frac{1}{2} = 2\frac{1}{6}$: as

* Art. 183.
of this Ex-
plication.

was found by means of the initial Limit before *.

But, if you either suppose $Ab = \zeta = 0$, or yet $A\beta = z = \text{infinite}$: then the Expression $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta}$, in both Cases, becomes infinitely great: for when $\zeta = 0$, the Term $\frac{1}{\zeta}$ is infinite: again when $z = \text{infinite}$, the Term $\frac{1}{3}z^3$ is infinite. The first shews that the Area upon the nearer Side of any Ordinate, comprehended betwixt it and AG perpendicular

pendicular to AB, is infinitely great: the other signifies that the Area situate upon the further Side of the Ordinate, and extending along the Absciss infinitely produc'd, is also infinitely great. Therefore in this and the like Cases, we must be contented to find the curvilinear Area, adjacent to any middle Part of the Absciss, as has been just now shewn.

190. And this naturally leads us to speak of the hyperbolical Space contain'd betwixt the Curve and Assymptote in the Case of the *Apolonian* Hyperbola, whose Assymptotes are at right Angles to each other: the Consideration of which was put off to this Place.

Suppose $cCxa$ to be such an Hyperbola; (see the Fig. p. 135.) AB, AG it's Assymptotes: the Equation to it is $xy=a$, or $y=\frac{a}{x}$, a denoting a given

superficial Space. The Area, by the Method of Quadratures, is $\frac{-ax^0}{0}$ * Art. 144. of this Explication.

= infinite: whence it is evident that we can neither find the Area on the nearer Side of any Ordinate BC, lying between it and the Assymptote AG; nor the Area CBa, lying on the further Side of the Ordinate BC, along Ba infinitely produced; these Spaces, in both Cases being infinite: nor yet can we, by the foregoing Method, find the Value of any intermediate Area, as BbcC, adjacent to any middle Part of the Absciss Bb, as long as the Point A continues to be the Beginning of the Absciss. Therefore, to remedy this, you must alter the Beginning of the Absciss: and suppose it to be at some Point in the Assymptote, as at the Point B, at a given Distance from A: which Distance, call d : draw the Ordinate BC; likewise the Ordinates βx , bc , on each Side of BC. Then you'll have $B\beta=z$, $Bb=-z$, $A\beta=d+z$, and $Ab=d-z$: and so the Ordinate $\beta x=\frac{a}{d+z}$, and $bc=\frac{a}{d-z}$. Reduce the Ordinate $\frac{a}{d+z}$ to Form, and it

stands thus $z^{1-1} \times d+z^{0-1} \times a$: which belongs to a Curve that is not quadrable, as appears by Art. 149 *. Wherefore we must content ourselves with an Approximation. Accordingly, by substituting 1 for θ , η , r , s and f ; and d , a for e , a respectively, in the general Series for the Area of a simple Binomial, viz. $z^1 \times e + fz^\eta$ ^{* Of this Explication.}

$$\frac{1}{re} + \frac{-fA}{r+1 \times e} z^\eta + \frac{-s+1 \times fB}{r+2 \times e} z^{2\eta} + \frac{-s+2 \times fC}{r+3 \times e} z^{3\eta} + \frac{-s+3 \times fD}{r+4 \times e} z^{4\eta} \&c.$$

$$\text{you'll have } z \times d + z^0 \times \frac{a}{d} - \frac{a}{2d} z + \frac{a}{3d} z^2 - \frac{a}{4d} z^3 + \frac{a}{5d} z^4 - \&c. \\ = \frac{az}{d} \times 1 - \frac{z}{2d} + \frac{z^2}{3d^2} - \frac{z^3}{4d^3} + \frac{z^4}{5d^4} - \&c. \text{ in } \text{inf.} \text{ which denotes the}$$

Area

* Art. 169.
of this Ex-
plication.

Area $B\beta C$, because of the positive Powers of z contain'd in the Numerators of the several Terms *.

After the same manner, if you take $y = \frac{a}{d-z}$, you'll find, by making the like Substitution in the general Theorem, expressing the Areas of binomial Curves, $\frac{az}{d} \times 1 + \frac{z}{2d} + \frac{z^2}{3d^2} + \frac{z^3}{4d^3} + \frac{z^4}{5d^4} + \mathcal{E}c. \text{ in } \text{inf.}$ for the Value of the Area BbC , for the like Reason.

191. Whence it follows, that, if $B\beta = Bb$, then by adding both Series's together, you'll have $\frac{2az}{d} \times 1 + \frac{z^2}{3d^2} + \frac{z^4}{5d^4} + \frac{z^6}{7d^6} + \mathcal{E}c. \text{ in } \text{inf.}$ for the Value of the Area $b\beta C$, made up of both the preceding Areas: which Series converges double as fast as the preceding ones. And thus you may, most conveniently, find an approximate Value of any hyperbolic Space, such as $b\beta C$, adjoining to any intermediate Part $b\beta$ of the Assymptote, viz. by supposing the Beginning of the Absciss z to be at the Point B, exactly in the Middle betwixt b and β : for by this means the Series made up of the two others, as above, converges faster, than by supposing the Beginning of the Absciss to be at one of the Extremities of the given Part of the Assymptote; or by throwing the Beginning of the Absciss without the Part $b\beta$, and working as in

* Of this
Explica-
tion.

Art. 186 *: which may likewise be done here.

Ex. Let $Ab = 0,9$, $A\beta = 1,1$: and you would have the Area $b\beta C$. Take $bB = 0,1 = B\beta = z$, and consequently $AB = d = 1$: then substitute 0,1 for z and 1 for d , in the preceding Series, and you'll

have $0,2a \times 1 + \frac{0,01}{3} + \frac{0,0001}{5} + \frac{0,000001}{7} + \mathcal{E}c. = a \times 0,2 + 0,000666666, \mathcal{E}c. + 0,000004 + 0,000000028, \mathcal{E}c. + \mathcal{E}c. = 0,20067069, \mathcal{E}c. \times BC$ for an approximate Value of the Area $b\beta C$: and since the Progression of the Series is manifest, the Value of the Area, may be carried to any Degree of Exactness you please. Now although the Series's above, would converge very slowly, if it were $z = d$; and not at all, if it were z greater than d *, yet it is evident, by the above expedient, z must always be less than d : for $Bb = B\beta = z$ must always be less than $AB = d$.

* Art. 28.
of the
Quadrature
of
Curves.

And thus we have shewn at full Length, what is to be done in order to find the curvilinear Area adjacent to any middle Part of the Assymptote, in the Case of the *Apollonian* Hyperbola, as we promis'd above *.

* Art. 144.
of this Ex-
plication.

192. And as it has been shewn, in the Case of the Hyperbola, how you may change the Beginning of the Absciss; it may not be improper

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per here to observe, that the same Expedient may be used, in the Case of any other Curve whatsoever: which may be of considerable Use, upon Occasions, for extricating one out of several Difficulties that may occur in this Doctrine of Curve-lines: particularly, by this means, you may express one and the same curvilinear Area, by infinite different Ways: since every different Beginning of an Absciss, which may be infinitely varied, will afford you a different Expression for the Area. And thus likewise you may get free of an hyperbolical Term, which may enter at any Time into the Expression of the Ordinate of a Curve. I call an hyperbolical Term, that which contains the first Power or Root of the Absciss z in it's Denominator: which always denotes an hyperbolical Ordinate, applied to the Assymptote. Let $z + \frac{z}{z} = y$

be an Equation to a Curve: then because of the hyperbolical Term $\frac{z}{z}$ the Area becomes infinite, viz. $\frac{1}{2}z - \frac{2z^2}{3}$: but if you suppose the Absciss z to be increas'd or diminish'd by any given Length, e.g. to be diminish'd by 1, by taking a new Absciss $\zeta = z - 1$ or $1 + \zeta = z$: and substituting $1 + \zeta$ for z in the Equation to the Curve: hence you obtain $y = 1 + \zeta + \frac{z}{1+\zeta}$: and therefore since the Area corresponding to $1 + \zeta$ is $\zeta + \frac{1}{2}\zeta^2$, and that corresponding to the Part $\frac{z}{1+\zeta}$ is $2\zeta - \frac{2\zeta^2}{2} + \frac{2\zeta^3}{3} - \frac{2\zeta^4}{4} + \&c. \text{ in inf.}$ after the Manner already demonstrated: hence by joining these Areas together, you'll have $3\zeta - \frac{\zeta^2}{2} + \frac{2\zeta^3}{3} - \frac{1}{2}\zeta^4 + \frac{2}{5}\zeta^5 - \&c. \text{ in inf.}$ for an approximate Value of the Area of the Curve proposed.

193. And so I shall finish this Section, respecting the Position and Limits of the Areas of Curves, being afraid I have insisted too long upon it already, when I shall have remarked that, if the Curve decussate it's Base, betwixt the two Extremities of that Part of it, to which the Area to be found, adjoins, then, by observing the Rule delivered above, viz. by subtracting the Area belonging to the shorter Absciss from the Area belonging to the longer Absciss, found by the Method of Quadratures, the thing which arises is the Difference between that Part of the intervening Area, which lyes above the Base; and that Part, which lyes below, the latter being subtracted from the former. Thus if it be $AB = z$ and $A\beta = \zeta$ (see the Fig. p.141.): the Area belonging to the lesser Absciss $AB = z$, is $-\frac{1}{z} + \frac{z}{z^{\frac{1}{2}}}$: and that belonging to the

larger Absciss $A\beta = \zeta$, is $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{3}{2}}}$ therefore $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{3}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{3}{2}}}$ denotes the Area $DBC - D\beta\kappa$.

Let $AB = z = \frac{4}{9}$ and $A\beta = \zeta = \frac{2}{4}$, then $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{3}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{3}{2}}} = -\frac{4}{9} + \frac{4}{3} + \frac{9}{4} - 3 = \frac{5}{36}$ the Difference arising by subtracting the Area $D\beta\kappa$ from Dbc .

Again, let it be $AB = z = \frac{81}{100}$, and $A\beta = \zeta = 4$: then $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{3}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{3}{2}}} = -\frac{1}{4} + 1 + \frac{100}{81} - \frac{20}{9} = -\frac{77}{324}$ the Difference arising by subtracting the Area $D\beta\kappa$ below the Base from the Area DBC above the Base: which comes out negative, in regard, $D\beta\kappa$ is, in this Case, greater than DBC .

Finally, let it be $AB = z = \frac{4}{9}$, and $A\beta = 4$: then $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{3}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{3}{2}}} = -\frac{1}{4} + 1 + \frac{9}{4} - 3 = 0$: which shews that the Parts of the Area above and below the Base are equal.

S E C T. VI.

Containing Notes on Art. 32.

DEMONSTRATION of PROP. VI.

THE Demonstration of this Prop. which our Author has omitted, is as follows.

194. By Prop. 4. and Art. III. the following Ordinates and Areas of Curves correspond.

Ordinates.	Areas.
$ \begin{array}{l} 1^o \theta ekA \begin{array}{c} \pm \theta \\ \lambda \eta \end{array} f k A z^{\eta} \begin{array}{c} \pm \theta \\ 2 \lambda \eta \end{array} g k A z^{2 \eta} \begin{array}{c} \pm \theta \\ 3 \lambda \eta \end{array} h k A z^{3 \eta} \\ \begin{array}{c} \pm \theta \\ \mu \eta \end{array} e l A z^{\eta} \begin{array}{c} \pm \theta \\ \lambda \eta \end{array} f l A z^{2 \eta} \begin{array}{c} \pm \theta \\ 2 \lambda \eta \end{array} g l A z^{3 \eta} \text{ \&c.} \\ \begin{array}{c} \pm \theta \\ 2 \mu \eta \end{array} e m A z^{2 \eta} \begin{array}{c} \pm \theta \\ \lambda \eta \end{array} f m A z^{3 \eta} \\ \begin{array}{c} \pm \theta \\ 3 \mu \eta \end{array} e n A z^{3 \eta} \end{array} $	$ \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} x z^{\theta-1} R^{\lambda-1} S^{\mu-1} \quad \left \quad A z^{\theta} R^{\lambda} S^{\mu} \right. $

	Ordinates.	Areas.
2°.	$\left. \begin{array}{l} \overline{\theta + \eta} k B z^\eta + \overline{\theta + \eta} f k B z^{2\eta} + \overline{\theta + \eta} g k B z^{3\eta} \\ \overline{\theta + \mu} e l B z^{2\eta} + \overline{\theta + \mu} f l B z^{3\eta} \text{ \&c.} \\ \overline{\theta + 2\mu} e m B z^{3\eta} \end{array} \right\} \times z^{\theta-1} R^{\lambda-1} S^{\mu-1}$	$B z^{\theta+\eta} R^\lambda S^\mu$
3°.	$\left. \begin{array}{l} \overline{\theta + 2\eta} k C z^{2\eta} + \overline{\theta + 2\eta} f k C z^{3\eta} \\ \overline{\theta + 2\mu} e l C z^{3\eta} \text{ \&c.} \end{array} \right\} \times z^{\theta-1} R^{\lambda-1} S^{\mu-1}$	$C z^{\theta+2\eta} R^\lambda S^\mu$
4°.	$\overline{\theta + 3\eta} k D z^{3\eta} \text{ \&c. } \times z^{\theta-1} R^{\lambda-1} S^{\mu-1}$	$D z^{\theta+3\eta} R^\lambda S^\mu$

Now if the Sum of these Ordinates be suppos'd equal to the Ordinate $a + b z^\eta + c z^{2\eta} + d z^{3\eta} + \text{\&c.}$ $\times z^{\theta-1} R^{\lambda-1} S^{\mu-1}$ proposed, then the Sum of all the Areas, corresponding to the several Ordinates, *i. e.* $A z^\theta R^\lambda S^\mu + B z^{\theta+\eta} R^\lambda S^\mu + C z^{\theta+2\eta} R^\lambda S^\mu + D z^{\theta+3\eta} R^\lambda S^\mu + \text{\&c.}$ or $A + B z^\eta + C z^{2\eta} + D z^{3\eta} + \text{\&c.}$ $\times z^\theta R^\lambda S^\mu$, shall be equal to the Area of the Curve, whose Ordinate is $a + b z^\eta + c z^{2\eta} + d z^{3\eta} + \text{\&c.}$ $\times z^{\theta-1} R^{\lambda-1} S^{\mu-1}$.

Wherefore, let the correspondent Terms of the Ordinates be put equal, *i. e.* such Terms as have the same Power of z in them: that by that means, we may discover the Values of $A, B, C, D, \text{\&c.}$ which are of certain Values, although not yet determined. Hence we have these Equations.

$$1^\circ. a = \theta k A.$$

$$2^\circ. b = \overline{\theta + \lambda \eta} \times f k A + \overline{\theta + \mu \eta} \times e l A + \overline{\theta + \eta} \times e k B.$$

$$3^\circ. c = \overline{\theta + 2\lambda \eta} \times g k A + \overline{\theta + \lambda \eta + \mu \eta} \times f l A + \overline{\theta + 2\mu \eta} \times e m A + \overline{\theta + \eta + \lambda \eta} \times f k B + \overline{\theta + \eta + \mu \eta} \times e l B + \overline{\theta + 2\eta} \times e k C.$$

$$4^\circ. d = \overline{\theta + 3\lambda \eta} \times b k A + \overline{\theta + 2\lambda \eta + \mu \eta} \times g l A + \overline{\theta + \lambda \eta + 2\mu \eta} \times f m A + \overline{\theta + 3\mu \eta} \times e n A + \overline{\theta + \eta + 2\lambda \eta} \times g k B + \overline{\theta + \eta + \lambda \eta + \mu \eta} \times f l B + \overline{\theta + \eta + 2\mu \eta} \times e m B + \overline{\theta + 2\eta + \lambda \eta} \times f k C + \overline{\theta + 2\eta + \mu \eta} \times e l C + \overline{\theta + 3\eta} \times e k D.$$

\&c.

X

In

The Quadrature of CURVES explained.

In which Equations, if you put or suppose $\frac{0}{n} = r$. $r + \lambda = s$. $s + \lambda = t$. $t + \lambda = v$. &c. And again $r + \mu = s'$. $s' + \mu = t'$ &c. as they are expressed by the Author himself: they will appear thus.

$$1^{\circ}. a = r\eta ekA.$$

$$2^{\circ}. b = s\eta fkA + s'\eta elA + \overline{r+1} \times ekB.$$

$$3^{\circ}. c = t\eta gkA + t'\eta flA + t''\eta emA + \overline{s+1} \times \eta fkB + \overline{s'+1} \times \eta elB + \overline{r+2} \times \eta ekC.$$

$$4^{\circ}. d = v\eta bkA + v'\eta glA + v''\eta fmA + v'''\eta enA + \overline{t+1} \times \eta gkB + \overline{t'+1} \times \eta flB + \overline{t''+1} \times \eta emB + \overline{s+2} \times \eta fkC + \overline{s'+2} \times \eta elC + \overline{r+3} \times \eta ekD.$$

&c.

From which Equations, by a due Reduction, you may find the Values of A, B, C, D, &c. which are the following, viz.

$$A = \frac{a}{r\eta ek} = \frac{\frac{1}{n}a}{r\eta ek}$$

$$B = \frac{\frac{1}{n}b - s\eta fkA - s'\eta elA}{\overline{r+1} \times ek}$$

$$C = \frac{\frac{1}{n}c - \overline{s+1} \times \eta fkB - \overline{s'+1} \times \eta elB}{\overline{r+2} \times ek}$$

$$D = \frac{\frac{1}{n}d - \overline{s+2} \times \eta fkC - \overline{s'+2} \times \eta elC - \overline{t+1} \times \eta gkB - \overline{t'+1} \times \eta flB - \overline{t''+1} \times \eta emB}{\overline{r+3} \times ek}$$

Wherefore, since, by putting the Sum of the Ordinates above mentioned, equal to the Ordinate $a + bx^n + cx^{2n} + dx^{3n} + \&c. \times x^{0-1}$ $R^{\lambda-1}S^{\mu-1}$; and the corresponding Terms of the one and of the other equal likewise, these Values of the Coefficients A, B, C, D, &c. do arise, it follows that, if you insert these Values of A, B, C, D, &c. so determin'd, in place of them, in the Sum of the Areas, viz. $A + Bx^n + Cx^{2n} + Dx^{3n} + \&c. \times x^0 R^{\lambda} S^{\mu}$; what results will be the Area of the Curve, whose Ordinate was propos'd, which will be thereby determin'd: and accordingly it will be expressed by the following Series.

2

$z^1 R^{\lambda} S^{\mu}$

$$\begin{aligned}
 & z^R S^u \times + \frac{\frac{1}{n} a}{r e k} \\
 & + \frac{\frac{1}{n} b - \frac{1}{n} f k}{r+1 \times e k} A \\
 & + \frac{\frac{1}{n} c - \frac{1}{n} f k \times \frac{1}{r+1} - \frac{1}{n} g k}{r+2 \times e k} B - \frac{\frac{1}{n} g k}{r+1 \times e k} A \\
 & + \frac{\frac{1}{n} d - \frac{1}{n} f k \times \frac{1}{r+2} - \frac{1}{n} g k \times \frac{1}{r+1} - \frac{1}{n} h k}{r+3 \times e k} C - \frac{\frac{1}{n} h k}{r+2 \times e k} B - \frac{\frac{1}{n} g k}{r+1 \times e k} A \\
 & + \frac{\frac{1}{n} e - \frac{1}{n} f k \times \frac{1}{r+3} - \frac{1}{n} g k \times \frac{1}{r+2} - \frac{1}{n} h k \times \frac{1}{r+1} - \frac{1}{n} i k}{r+4 \times e k} D - \frac{\frac{1}{n} i k}{r+3 \times e k} C - \frac{\frac{1}{n} h k}{r+2 \times e k} B - \frac{\frac{1}{n} g k}{r+1 \times e k} A \\
 & + \text{Ec.}
 \end{aligned}$$

In which Series, although the Capitals A, B, C, Ec. are retain'd, yet it is evident, they are now fully known: for $A = \frac{\frac{1}{n} a}{r e k}$ the first Term, which is wholly known: consequently the Coefficient of the second Term is wholly known, *i. e.* B is wholly known: and so on for all the rest. And as A, B, C, Ec. are the Values of the Coefficients of the first, second, third, Ec. Terms; so, when any of these Coefficients is negative, the capital Letter, which stands for it, must be negative likewise. And therefore if $a + b x^n + c x^{2n} + d x^{3n} + \text{Ec.} \times x^{R-1} S^{u-1}$ be the Ordinate of a Curve, the Series expressed above, will exhibit it's Area: and that either perfectly, *viz.* when the Series terminates; or else by Approximation, *viz.* when the Series runs out infinitely. But it must still be remember'd, that every Ordinate of a Curve may be expressed in two different Forms: for either the Exponents of the Powers of x , contain'd in R, S and $a + b x^n + c x^{2n} + \text{Ec.}$ may be positive, or negative: which was sufficiently explain'd upon the foregoing Proposition: and accordingly both Forms of the Ordinate must be tried, in the Application of this Theorem: and if the Curve be quadrable, the Series will terminate in one or other of the Cases: but if otherwise, the Series will converge, either when n is positive, or else when it is negative; and so it will exhibit an approximate Value of the Area.

195. Whatever Observations or Remarks were made upon the foregoing Prop. the like may be made upon this, with respect to the Laws or Conditions of the Termination, Continuation and Convergency of the Series : and the bringing Ordinates of Curves propos'd, which fall under this Prop. into due Form, as likewise with respect to the Position and Limits of Areas, and other things relating thereto. But that would be too large a Subject to enter upon : and therefore I shall let it alone.

The Series exhibiting the Areas of Curves falling under this Prop. may be easily continued, by considering the Law of the Progression. It is evident likewise that the Series of such Propositions, as the fifth and sixth, is infinite ; as well as that of the third and fourth, which was mentioned already *, *i. e.* we may suppose T to stand for $a + \beta z^n + \gamma z^{2n} + \text{Ec.}$ and ν for any numeral Index : and then if the Ordinate of a Curve be of this Form $z^{\theta-1} R^{\lambda-1} S^{\mu-1} T^{\nu-1} \times a + bz^n + cz^{2n} + dz^{3n} + \text{Ec.}$ you may exhibit a Series for it's Area, analogous to these in Prop. fifth and sixth, by considering what was said in Art. 117 *, and applying that to this Case, as Prop. third and fourth have been applied for demonstrating Prop. fifth and sixth : and so likewise for other Ordinates still more complex.

* Art. 117.
of this Ex-
plication.

* Of this
Explica-
tion.

S E C T. VII.

Containing Notes on Art. 33—39.

EXPLICATION of PROP. VII.

C A S E I.

196. **S**upposing the same things mentioned by the Author in this Prop. Let $R = e + fz^n$ a Binomial : I say if the Area of any one of the infinitely many Curves, whose Ordinates are thus generally express'd $z^{\theta \pm \sigma n} R^{\lambda \pm \tau}$, be given, the Areas of all these Curves will be given.

The Demonstration of this Prop. as far as it respects binomial Curves, is left by our Author to be gathered from his Demonstration of it in the Case of Trinomials : which therefore I shall here supply.

The first of the four Cases into which this Prop. is divided by him, respects the Increase or Diminution of the Index of z , *viz.* $\theta \pm \sigma n$, by any Repetition of n .

197. Let

197. Let the Area of the Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$, be given: and call it A, *i. e.* multiplying both by the Quantity p , whose Value is afterwards to be determined, we shall have $pz^{\theta-1}R^{\lambda-1}$ the

Ordinate of a Curve, whose Area is pA . But, by Prop. 3, $\theta e + \theta + \lambda \eta \times fz^n \times z^{\theta-1}R^{\lambda-1}$, is the Ordinate of a Curve, whose Area is $z^{\theta}R^{\lambda}$: subtract the former Ordinate and Area from the latter Ordinate and Area re-

spectively; and you'll have $\theta e - p + \theta + \lambda \eta \times fz^n \times z^{\theta-1}R^{\lambda-1}$, and $z^{\theta}R^{\lambda} - pA$, the Remainders; the first the Ordinate of a Curve, whereof the last is the Area: and that whatever be the Value of p , which is the Quantity by which the given Ordinate and Area, *viz.* $z^{\theta-1}R^{\lambda-1}$ and A, were multiplied. Now put $p = \theta e$ a known Quantity, and substitute the one for the other in the last mentioned Area, and thence

you'll have $\theta + \lambda \eta \times fz^n \times z^{\theta-1}R^{\lambda-1}$ and $z^{\theta}R^{\lambda} - \theta eA$, for correspondent Ordinate and Area. Again, dividing both by the given Quantity $\theta + \lambda \eta \times f$, we have $z^{\theta+1}R^{\lambda-1}$ the Ordinate of a Curve, whose Area is $\frac{z^{\theta}R^{\lambda} - \theta eA}{\theta + \lambda \eta \times f}$, which is wholly known.

That the Process may appear at one View, I shall set down the several Steps in Order.

	Ordinates.	Areas.
1 ^o .	$p \quad * \quad \times z^{\theta-1}R^{\lambda-1}$	$----- pA$
2 ^o .	$\theta e + \theta + \lambda \eta \times fz^n \times z^{\theta-1}R^{\lambda-1}$	$----- z^{\theta}R^{\lambda}$
3 ^o .	$\theta e - p + \theta + \lambda \eta \times fz^n \times z^{\theta-1}R^{\lambda-1}$	$----- z^{\theta}R^{\lambda} - pA.$
4 ^o .	$----- \theta + \lambda \eta \times fz^n \times z^{\theta-1}R^{\lambda-1}$	$----- z^{\theta}R^{\lambda} - \theta eA.$
5 ^o .	$----- z^{\theta+1}R^{\lambda-1}$	$----- \frac{z^{\theta}R^{\lambda} - \theta eA}{\theta + \lambda \eta \times f}.$

198. By the same way of reasoning, if you call this Area

$\frac{z^{\theta}R^{\lambda} - \theta eA}{\theta + \lambda \eta \times f} = B$, whose Ordinate is $z^{\theta+1}R^{\lambda-1}$, you may find the

Area of another Curve, whose Ordinate is $z^{\theta+2}R^{\lambda-1}$: by considering what is expressed in Art. 114 *. Which is done by subtracting the Ordinate $qz^{\theta+1}R^{\lambda-1}$ or $qz^n \times z^{\theta-1}R^{\lambda-1}$, and correspondent Area

* Of the
Explication.

qB from the Ordinate $\theta + \eta \times ez^n + \theta + \eta + \lambda \eta \times fz^{2n} \times z^{\theta-1}R^{\lambda-1}$, and correspondent Area $z^{\theta+2}R^{\lambda}$ respectively: which last Ordinate and Area

Area belong to the same Curve by that Art. and then putting $qz^\eta = \theta + \eta \times ez^\eta$; or $q = \theta + \eta \times e$: and substituting this Value in place of it: for thence you'll obtain $\theta + \eta + \lambda\eta \times fz^{2\eta} \times z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+1}R^\lambda - \theta + \eta \times eB$; or, by dividing both by $\theta + \eta + \lambda\eta \times f$, $z^{\theta+2\eta-1}R^{\lambda-1}$ and $\frac{z^{\theta+1}R^\lambda - \theta + \eta \times eB}{\theta + \eta + \lambda\eta \times f}$, for Ordinate and Area belonging to one and the same Curve: which Ordinate has the Exponent of z , viz. $\theta + 2\eta - 1$ greater than the Exponent of z in the Ordinate $z^{\theta-1}R^{\lambda-1}$ belonging to the given Area A, by 2η : and since B is known, being formerly found, the Area $\frac{z^{\theta+1}R^\lambda - \theta + \eta \times eB}{\theta + \eta + \lambda\eta \times f}$ is also wholly known.

And thus it is evident, that, by the like way of proceeding, you may find the Areas of all the Curves, whose Ordinates are thus generally expressed $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, σ being a positive Integer. For if $z^{\theta+\sigma\eta-1}R^{\lambda-1}$ be the Ordinate of a Curve, and A denote the Area of that Curve, then $\frac{z^{\theta+\sigma\eta}R^\lambda - \theta + \sigma\eta \times eA}{\theta + \sigma\eta + \lambda\eta \times f}$ will be the Area of another

Curve, belonging to that same Progression of Curves, which has the Index of z in it's Ordinate greater by once η than it is in the Ordinate of the other: as appears from Art. 114 *, compared with what hath been already said.

* Of this
Explica-
tion.

199. Again, by the like way of reasoning, it will appear, that if the Area of the Curve be given, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$ as formerly, which call A, then the Area of the Curve, whose Ordinate is $z^{\theta-\eta-1}R^{\lambda-1}$, may be thence found, and will be $\frac{z^{\theta-\eta}R^\lambda - \theta - \eta + \lambda\eta \times fA}{\theta - \eta \times e}$.

For you'll have Ordinates and Areas corresponding to each other as follows.

Ordinates.

Areas.

$$1^\circ. \text{-----} p \times z^{\theta-1}R^{\lambda-1} \text{-----} pA.$$

* Art. 114.
of this Ex-
plication.

$$2^\circ. \frac{\theta - \eta \times e z^{\theta-\eta} + \theta - \eta + \lambda\eta \times f \times z^{\theta-1}R^{\lambda-1}}{\theta - \eta \times e} \text{-----} z^{\theta-\eta}R^\lambda *.$$

Subtract first Step from second: put $p = \theta - \eta + \lambda\eta \times f$: and divide by $\theta - \eta \times e$.

$$3^\circ. \text{-----} z^{\theta-\eta-1}R^{\lambda-1} \text{-----} \frac{z^{\theta-\eta}R^\lambda - \theta - \eta + \lambda\eta \times fA}{\theta - \eta \times e}$$

This last Area being found, call it P, and then from the known Area P, you proceed to find the Area of the Curve next in order in this

this Progreſſion of Curves, viz. the Area of the Curve, whoſe Ordinate is $x^{\theta-2\eta-1}R^{\lambda-1}$: which by the ſame way of reaſoning, will be found to be $\frac{x^{\theta-2\eta}R^{\lambda}-\theta-2\eta+\lambda\eta \times f^P}{\theta-2\eta \times e}$: and ſo you may proceed from this

laſt Area, now given, to determine the Area of that Curve which comes next in the Order of the Progreſſion of Curves, viz. that whoſe Ordinate is $x^{\theta-3\eta-1}R^{\lambda-1}$: and ſo on continually. So that univerſally if you put P for the Area of any Curve whoſe Ordinate is $x^{\theta-\sigma\eta-1}R^{\lambda-1}$,

then $\frac{x^{\theta-\sigma\eta-\eta}R^{\lambda}-\theta-\sigma\eta-\eta+\lambda\eta \times f^P}{\theta-\sigma\eta-\eta \times e}$ ſhall be the Area of a Curve whoſe

Ordinate is $x^{\theta-\sigma\eta-\eta-1}R^{\lambda-1}$: as eaſily appears by ſubſtituting $\theta - \sigma\eta$ for θ in the firſt Part of this Art.

200. And thus it appears that the Area of a Curve, whoſe Ordinate is $x^{\theta-1}R^{\lambda-1}$, being given, the Area of every other Curve, whoſe Ordinate is thus generally expreſſed $x^{\theta \pm \sigma\eta-1}R^{\lambda-1}$, is likewise given. Wherefore by a contrary Analyſis, you may come back from the Area of any Curve, whoſe Ordinate is $x^{\theta \pm \sigma\eta-1}R^{\lambda-1}$, to the Area of the Curve, whoſe Ordinate is $x^{\theta-1}R^{\lambda-1}$. For, if you put A for the Area of any Curve, whoſe Ordinate is $x^{\theta+\sigma\eta-1}R^{\lambda-1}$: and B for the Area of the Curve whoſe Ordinate is $x^{\theta+\sigma\eta+\eta-1}R^{\lambda-1}$: then, by what has

been demonſtrated, $B = \frac{x^{\theta+\sigma\eta}R^{\lambda}-\theta+\sigma\eta \times eA}{\theta+\sigma\eta+\lambda\eta \times f}$: therefore it follows con-

trarily by reſolving the Equation, that if B be given and A ſought, it will be $A = \frac{x^{\theta+\sigma\eta}R^{\lambda}-\theta+\sigma\eta+\lambda\eta \times fB}{\theta+\sigma\eta \times e}$. And again, in the other Pro-

greſſion of Areas proceeding the contrary way, if P ſtand for the Area of a Curve, whoſe Ordinate is $x^{\theta-\sigma\eta-1}R^{\lambda-1}$; and Q for the Area of a Curve, whoſe Ordinate is $x^{\theta-\sigma\eta-\eta-1}R^{\lambda-1}$: it has been demonſtrated

that $Q = \frac{x^{\theta-\sigma\eta-\eta}R^{\lambda}-\theta-\sigma\eta-\eta+\lambda\eta \times fP}{\theta-\sigma\eta-\eta \times e}$: Wherefore, ſuppoſing Q

given, and P ſought, by reſolving the Equation, we have $P = \frac{x^{\theta-\sigma\eta-\eta}R^{\lambda}-\theta-\sigma\eta-\eta \times eQ}{\theta-\sigma\eta-\eta+\lambda\eta \times f}$. From both which it plainly follows, that,

if the Area of any one of the infinitely many Curves, whoſe Ordinates are generally expreſſed thus $x^{\theta \pm \sigma\eta-1}R^{\lambda-1}$, be given, the Areas of all the reſt are alſo given, viz. when $R = e + fz$ a Binomial.

N. B. It's the ſame thing whether the general Form of the Ordinates be expreſſed thus $x^{\theta \pm \sigma\eta-1}R^{\lambda-1}$; or thus $x^{\theta \pm \sigma\eta}R^{\lambda \pm \tau}$, as it is by the Author himſelf in the Prop. you need only ſuppoſe that θ in the laſt is equivalent to $\theta - 1$ in the firſt: and that $\lambda \pm \tau$ in the laſt is equivalent

equivalent to $\lambda - 1$ in the first; or $\lambda + \tau + 1$ in the last, the same with λ in the first.

201. It appears from what has been shewn in the three preceding Art. that if A and B be made to stand for the Areas of any two Curves next to one another, whether they be in the Progeffion running towards the one hand, or towards the other: so that the Exponent of z in the Expression of the Ordinate answering to the Area B, exceed the Exponent of z in the Ordinate corresponding to A, by $+ \eta$: then, we may make $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$ to represent any two such Ordinates universally, since θ may represent any Index positive or negative, integral or fractional. From whence, by considering what hath been said, we shall have the two following Theorems for determining the Areas of all binomial Curves, from any one Area being given.

$$202. \text{ Th. 1. } B = \frac{z^{\theta}R^{\lambda} - \theta e A}{(+\lambda\eta \times f)}$$

$$203. \text{ Th. 2. } A = \frac{z^{\theta}R^{\lambda} - \theta \div \lambda\eta \times f B}{\theta e}$$

204. And thus having demonstrated the first Case of the Prop. when $R = e + fz^{\eta}$, a Binomial; the Demonstration of the first Case of the Prop. accommodated to the Supposition of $R = e + fz^{\eta} + gz^{2\eta}$, a Trinomial, set down by the Author himself, includes no new Difficulty. The way of finding the Area of the Curve whose Ordinate is $z^{\theta+2\eta-1}R^{\lambda-1}$, from the Areas of the two Curves, whose Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$, being given, he shews himself at length: and so you have the Method of determining the Area of any Curve whose Ordinate is $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, when the Areas of the two Curves are given, whose Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$. And he tells you that the Method of proceeding, in order to find the Areas of Curves, in the Progeffion of Curves, which goes the contrary way, *viz.* those Curves whose Ordinates are thus generally expressed $z^{\theta-\sigma\eta-1}R^{\lambda-1}$, is like to that same, which he shews for the Areas of Curves in the other Progeffion. That the Truth of this may appear I shall set down the Process of the Steps in order.

	Ordinates.	Areas.
1 ^o .	----- p -----	$\times z^{\theta-1}R^{\lambda-1}$ - - pA .
2 ^o .	----- qx^{η} -----	$\times z^{\theta-1}R^{\lambda-1}$ - - qB .
3 ^o .	$\theta - \eta \times e z^{\theta-\eta-1} + \theta - \eta + \lambda\eta \times f z^{\theta-1} + \theta - \eta + 2\lambda\eta \times g z^{\theta+\eta-1}$	$\times z^{\theta-1}R^{\lambda-1}$ - - $z^{\theta-\eta}R^{\lambda}$ *
Subtract the first and second Steps from the third: put $p = \theta - \eta + \lambda\eta \times f$ and $q = \theta - \eta + 2\lambda\eta \times g$: and divide by $\theta - \eta \times e$.		
4 ^o .	----- $z^{\theta-\eta-1}R^{\lambda-1}$ -----	$\frac{z^{\theta-\eta}R^{\lambda} - \theta - \eta + \lambda\eta \times f A - \theta - \eta + 2\lambda\eta \times g B}{\theta - \eta \times e}$

*Art. 114.
of this Ex-
plication.

Which

Which last Area is that which was to be found, and is now wholly known, since A and B were given. And after the same Manner that our Author shews how to find the Area of the Curve, whose Ordinate is $z^{\theta+2\eta-1}R^{\lambda-1}$, from the Areas of two Curves being given, whose Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$; and that I have just now shewn, how from these same Areas given, to find the Area of the Curve, whose Ordinate is $z^{\theta-\eta-1}R^{\lambda-1}$, the Areas of all the Curves in both Progressions may be found. For, universally if A and B stand for the Areas of two Curves next one another in the first Progression, any where, viz. such whose Ordinates are $z^{\theta+\sigma\eta-1}R^{\lambda-1}$ and $z^{\theta+\sigma\eta+\eta-1}R^{\lambda-1}$, then
$$\frac{z^{\theta+\sigma\eta}R^{\lambda} - \frac{\theta+\sigma\eta \times eA - \theta+\sigma\eta+\lambda\eta \times fB}{\theta+\sigma\eta+2\lambda\eta \times g}}{\theta+\sigma\eta+2\lambda\eta \times g}$$
 will be the

Area of the Curve following next in that Progression, viz. whose Ordinate is $z^{\theta+\sigma\eta+2\eta-1}R^{\lambda-1}$: as easily appears by inserting $\theta+\sigma\eta$ instead of θ in our Author's Demonstration, i. e. by supposing $\theta+\sigma\eta$ here, to be equivalent to θ there.

Again, if P and Q stand for the Areas of any two Curves next to each other in the Progression of Curves going the contrary way, i. e. if they be the Areas of any two Curves, whose Ordinates are $z^{\theta-\sigma\eta-1}R^{\lambda-1}$ and $z^{\theta-\sigma\eta-\eta-1}R^{\lambda-1}$, then
$$\frac{z^{\theta-\sigma\eta-2\eta}R^{\lambda} - \frac{\theta-\sigma\eta-2\eta+\lambda\eta \times fQ - \theta-\sigma\eta-\eta+\lambda\eta \times gP}{\theta-\sigma\eta-2\eta \times e}}{\theta-\sigma\eta-2\eta \times e}$$

must be the Area of the Curve following next in order in this Progression, viz. the Curve whose Ordinate is $z^{\theta-\sigma\eta-2\eta}R^{\lambda-1}$: which easily appears by inserting $\theta-\sigma\eta-\eta$ instead of θ in the Demonstration just now given; i. e. by supposing $\theta-\sigma\eta-\eta$ here, to be equivalent to θ there. And therefore it hence appears that, if the Areas of the two Curves, whose Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$, be given, the Areas of all the Curves in both Progressions, are thence also given. And therefore by a contrary Analysis, if the Areas of any two Curves, whether both in the Progression of Curves proceeding the same way, or one in the Progression proceeding the one way; and the other in the Progression that proceeds the contrary way, be given, the Areas of the two Curves whose Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$; and also of all the others whose Ordinates are thus generally expressed, $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$, may thence be found. For let A, B, C, D, &c. denote the Areas of the Curves belonging to the Progression, whose Ordinates are thus expressed $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, where $\sigma\eta$ is positive, and A, the Area of the Curve whose Ordinate is $z^{\theta-1}R^{\lambda-1}$; and again let A, P, Q, R, &c. denote the Areas of the Curves belonging to the Progression proceeding the contrary way, viz. those whose Ordinates are thus expressed $z^{\theta-\sigma\eta-1}R^{\lambda-1}$. A being as before the Area belonging to the

Y

Curve,

Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$: which Curve stands in the Middle betwixt the two Progressions, or rather is the first of both. Then by what has been shewn, we have an Equation defining the Relation of A, B and C; another defining the Relation of B, C and D; a third defining the Relation of C, D and E, and so on: whereby we have a new Equation for every new Area. Again, we have an Equation defining the Relation of B, A and P, when we proceed now the contrary way: we have another, exhibiting the Relation of A, P and Q: another that of P, Q and R, and so on: therefore, it is evident, there will be as many distinct Equations except two, as there are Areas: consequently if any two Areas be given, all the rest may be found.

205. Although in the preceding Art. we have considered the Curves whose Ordinates are generally expressed thus $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, and those whose Ordinates are expressed thus $z^{\theta-\sigma\eta-1}R^{\lambda-1}$, distinctly: and accordingly exhibited two distinct Equations, for determining the Areas of the Curves belonging to the one, and the other: yet they are both in effect one and the same, and both virtually included in that one Equation or Theorem mentioned by the Author himself. For if A, B and C represent the Areas of three different Curves next to one another, so that A be the Area of a Curve any where belonging to the Class of Curves, whose Ordinates are thus generally expressed $z^{\theta+\sigma\eta-1}R^{\lambda-1}$; B the Area of a Curve, in whose Ordinate the Index of z is greater by $+\eta$; and C the Area of a third, in whose Ordinate the Index of z is greater by $+2\eta$, than it is in the Ordinate corresponding to the Area A; then we may make $z^{\theta-1}R^{\lambda-1}$ to represent universally the Ordinate of the first; $z^{\theta+\eta-1}R^{\lambda-1}$ the Ordinate of the second; and $z^{\theta+2\eta-1}R^{\lambda-1}$ to represent the Ordinate of the third, corresponding to the three Areas A, B, C: so that the Equation $z^{\theta}R^{\lambda}-\theta eA-\theta+\lambda\eta\times fB=\theta+2\lambda\eta\times gC$ mentioned by him, may serve for finding all the Areas in this whole Class of Curves, whose Ordinates are thus generally expressed $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, any two of them being given: which contains these three Theorems following.

$$206. \text{ Th. 1. } C = \frac{z^{\theta}R^{\lambda}-\theta eA-\theta+\lambda\eta\times fB}{\theta+2\lambda\eta\times g}$$

$$207. \text{ Th. 2. } B = \frac{z^{\theta}R^{\lambda}-\theta eA-\theta+2\lambda\eta\times gC}{\theta+\lambda\eta\times f}$$

$$208. \text{ Th. 3. } A = \frac{z^{\theta}R^{\lambda}-\theta+2\lambda\eta\times gC-\theta+\lambda\eta\times fB}{\theta e}$$

209. But, says our Author, if any of the Terms θ , $\theta+\lambda\eta$ and $\theta+2\lambda\eta$ be wanting or vanish, or break off the Series: let the Area
pA

pA be assum'd in the Beginning of the one Progression, and the Area *qB* in the Beginning of the other, and from these two Areas, will be given all the Areas in both Progressions.

From the preceding Theorems it is evident, that if θ or $\theta + \lambda\eta$ or $\theta + 2\lambda\eta$ be equal to nothing, the Series of Curves is broke off or interrupted: thus in the general Equation or Theorem $z^{\theta}R^{\lambda} - \theta eA - \overline{\theta + \lambda\eta} \times fB = \overline{\theta + 2\lambda\eta} \times gC$, if it be $\theta = 0$, *A* disappears; if it be $\theta + \lambda\eta = 0$, *B* disappears; and if it be $\theta + 2\lambda\eta = 0$, *C* disappears. But the Remedy our Author proposes in this Case, I acknowledge I don't sufficiently understand, according as he expresses it above. For suppose we call *A*, *B*, *C*, *D*, &c. the ascending Progression; and *A*, *P*, *Q*, *R*, &c. the descending Progression, signifying as above: the Meaning of what he says, would seem to be this, that when $\theta = 0$, from the Assumption of *B* and *P*, all the Areas in both Progressions on both Sides of *A* may be found: that if it were $\theta + \lambda\eta = 0$, all the Areas on both Sides of *B* may be found from the Assumption of *A* and *C*; and that if it be $\theta + 2\lambda\eta = 0$, all the Areas on both Sides of *C* may be found from the Assumption of *B* and *D*. But this is not true: for when $\theta + \lambda\eta = 0$, from the Assumption of *A* and *C*, you cannot find the Area of any other Curve in either Progression. For in this Case *A* and *C* are not sufficient for determining *B*: since the Equation above $z^{\theta}R^{\lambda} - \theta eA - \overline{\theta + \lambda\eta} \times fB = \overline{\theta + 2\lambda\eta} \times gC$ becomes $z^{\theta}R^{\lambda} - \theta eA = \lambda\eta \times gC$, where *B* disappears: but without *B* and *C*, neither *D*, nor any of the Areas above *C* can be found; and without *B* and *A* both, neither *P*, nor any of the Areas in the descending Progression can be found: as may easily appear from the Consideration of the three preceding Theorems, and what has been said formerly. However in the Case proposed, viz. when any of the Terms θ , $\theta + \lambda\eta$, $\theta + 2\lambda\eta$ is wanting, 1°. If it be $\theta = 0$, from the Assumption of *A* and *B*, all the other Areas in both Progressions may be found. 2°. If it be $\theta + \lambda\eta = 0$, from the Assumption of the same two Areas, all the others may be found. 3°. If it be $\theta + 2\lambda\eta = 0$, from the Assumption of *A* and *B*, the Areas of the Curves in the ascending Progression cannot be found: but if you assume *A* and *C*, all the Areas in both Progressions will be found, as appears from the foregoing Theorems. Yet it will not thence follow universally, that any two Areas being given, all the Areas can thence be found. For, 1°. when it is $\theta = 0$, if the two given Areas be both in the ascending Progression beyond *A*, none of the Areas in the descending Progression can be thence found: since the Equation that includes *B* and *C*, cannot include:

clude A, the Term θeA vanishing: and this evidently prevents the passing from the ascending to the descending Progression. 2°. If it be $\theta + \lambda\eta = 0$, from A and C given, none of the other Areas can be found, as was shewn already. 3°. If it be $\theta + 2\lambda\eta = 0$, from the Areas A and B given, or any two Areas in the descending Progression, none of the Areas in the ascending Progression beyond B, can be thence found: for the Equation including A and B, doth not include C, and therefore this interrupts the Passage from A and B to the other Areas in the ascending Progression. But if you except these Cases, any two Areas being given, either both in the ascending, or both in the descending; or one in the one, and another in the other, all the other Areas in both Progressions may be found by help of the preceding Theorems.

Schol. 1. In the preceding Cases, from one Area only given, in the Case of Trinomials, other Areas are found. Thus 1°. when it is $\theta = 0$, one Area above A in the ascending Progression, such as B, C, &c. being given, all the others in that Progression, except A, may be found. 2°. If it be $\theta + \lambda\eta = 0$, the Area C may be found from A, or A from C. 3°. If it be $\theta + 2\lambda\eta = 0$, from B or A, or any one Area in the descending Progression given, all the other Areas in that Progression (including also B) may be found: as may easily appear from the preceding Theorems, and what has been said.

* Of this
Explica-
tion.

Schol. 2. There are analogous Observations which hold with respect to other more complex Curves, such as Quadrinomials, Quinquinomials, &c. And particularly with respect to binomial Curves, it appears from Art. 202, 203 *, that when $\theta = 0$, from B, or any one of the Areas in the ascending Progression above B, given, neither A, nor any of the Areas in the descending Progression can be found. 2°. When $\theta + \lambda\eta = 0$, from the Area A, or any one in the descending Progression, given, neither B, nor any other Area in the ascending Progression can be thence found. But that in the first Case, the Curve whose Ordinate is $ax^{\eta-1} \times e + fx^{\eta}\lambda^{-1}$, is quadrable, it's Area being $\frac{a}{\lambda\eta f} \times e + fx^{\eta}\lambda$: and in the second Case, the Curve whose Ordinate is

$ax^{\theta-1} \times e + fx^{\eta}\lambda^{-1}$, is quadrable; and it's Area $\frac{ax^{\theta} \times e + fx^{\eta}\lambda}{\lambda\eta f}$: as appears also by Art. 149 *.

* Of this
Explica-
tion.

CASE II.

210. The second Case of the Prop. is where λ , the Exponent of R, in the Expression of the Ordinate, is to be increased, or diminished,

ed, by the continual Addition, or Subtraction of Units. This Case, as far as it respects binomial Curves, viz. when $R = e + fz^n$, is left out by our Author: and is supplied in what follows.

Suppose the Area of one such binomial Curve to be given, whose Ordinate is $z^{\theta-1}R^\lambda$, and that Area to be called A. Then $pz^{\theta-1}R^\lambda$ and pA are correspondent Ordinate and Area, p being any constant Quantity *. Further, divide, and then multiply the Ordinate $pz^{\theta-1}R^\lambda$ by R or $e + fz^n$, and it appears thus $pe + pfz^n \times z^{\theta-1}R^{\lambda-1}$ different in Appearance, but the same in Effect, with what it was before, since the Multiplication and Division by the same thing, mutually compensate

* Art. 116. of this Explanation.

each other. Then, since the Ordinate $\theta e + \theta + \lambda \eta \times fz^n \times z^{\theta-1}R^{\lambda-1}$ and Area $z^\theta R^\lambda$ answer to each other, by Prop. 3, add the former Ordinate and Area to the latter Ordinate and Area, respectively: thence

you'll have $\overline{p + \theta} \times e + \overline{p + \theta + \lambda \eta} \times fz^n \times z^{\theta-1}R^{\lambda-1}$ for the Ordinate belonging to the Area $pA + z^\theta R^\lambda$: suppose the second Term in the Value of the Ordinate equal to nothing, that is $\overline{p + \theta + \lambda \eta} \times fz^n = 0$, or $p + \theta + \lambda \eta = 0$: thence $p = -\overline{\theta + \lambda \eta}$: infer this Value of p in place of it, in the first Term of the Ordinate, and in the corresponding Area, and you'll have $-\lambda \eta e \times z^{\theta-1}R^{\lambda-1}$ and $-\overline{\theta + \lambda \eta} \times A + z^\theta R^\lambda$, or, by dividing both by $-\lambda \eta e$, $z^{\theta-1}R^{\lambda-1}$ and $\frac{\overline{\theta + \lambda \eta} \times A - z^\theta R^\lambda}{\lambda \eta e}$,

an Ordinate and Area belonging to the same Curve: which Area is wholly known, since A is given; and the Ordinate has the Index of R less by Unity, than the Index of R in the Ordinate of the given Curve. The Steps of the Process, set down distinctly, follow.

Ordinates.

Areas.

$$1^\circ. \overline{pe + pfz^n} \times z^{\theta-1}R^{\lambda-1} \text{ --- } pA.$$

$$2^\circ. \overline{\theta e + \theta + \lambda \eta} \times fz^n \times z^{\theta-1}R^{\lambda-1} \text{ --- } z^\theta R^\lambda *.$$

* Prop. 3.

Put the Sum of the second Terms equal to nothing, or

$$p = -\overline{\theta + \lambda \eta}.$$

$$3^\circ. \text{ --- } -\lambda \eta e \times z^{\theta-1}R^{\lambda-1} \text{ --- } -\overline{\theta + \lambda \eta} \times A + z^\theta R^\lambda.$$

Divide both by $-\lambda \eta e$.

$$4^\circ. \text{ --- } z^{\theta-1}R^{\lambda-1} \text{ --- } \frac{\overline{\theta + \lambda \eta} \times A - z^\theta R^\lambda}{\lambda \eta e} = B.$$

Now,

Now, after the same Manner, that, from the Area A belonging to the Ordinate $z^{\theta-1}R^{\lambda}$, given, the Area belonging to the Ordinate $z^{\theta-1}R^{\lambda-1}$, was just now found, the Area belonging to this last Ordinate, being given, you may find the Area belonging to another Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-2}$: the Method of doing it and the Demonstration differs in nothing, but inserting $\lambda - 1$, every where, for λ in the foregoing: so that if the Area just now found be called B; and the Area belonging to the Ordinate $z^{\theta-1}R^{\lambda-2}$, be denoted by C, then $C = \frac{\theta + \lambda\eta + \eta \times B - z^{\theta}R^{\lambda-1}}{\lambda\eta + \eta \times e}$. And so you may proceed to find

the Area belonging to the Ordinate $z^{\theta-1}R^{\lambda-3}$, by a like Process, viz. by inserting $\lambda - 2$ for λ , and so on continually. For in effect, the Theorem, which makes the Conclusion of the first Demonstration, viz. $B = \frac{\theta + \lambda\eta \times A - z^{\theta}R^{\lambda}}{\lambda\eta e}$, includes all possible Cases of binomial

Curves, in which the Area of any one being given, the Area of another is sought, in whose Ordinate, the Index of R, is lessened by Unity: because whatever be the Index of R, in the Ordinate belonging to the given Area; whether it be affirmative, or negative, you may suppose it to be represented by λ : and again, whatever be the Index of z without the Vinculum, it may be always represented by $\theta - 1$. Therefore also conversely, if A and B represent the Areas of any two binomial Curves, such, that the Index of R be greater by a positive Unit, in the Ordinate belonging to A than in the Ordinate belonging to B, by resolving the Equation, and considering B as given, and A as sought, you'll have $A = \frac{\lambda\eta e B + z^{\theta}R^{\lambda}}{\theta + \lambda\eta \times e}$. Which furnishes us

with the two following Theorems, serving universally for carrying on the Progression of binomial Curves, from the Area of any one being given, in whose Ordinates the Index of R is to be diminish'd or increas'd by the continual Subtraction or Addition of Units.

$$211. \text{ Th. 1. } B = \frac{\theta + \lambda\eta \times A - z^{\theta}R^{\lambda}}{\lambda\eta e}$$

$$212. \text{ Th. 2. } A = \frac{\lambda\eta e B + z^{\theta}R^{\lambda}}{\theta + \lambda\eta}$$

213. I shall illustrate what concerns the Increase and Diminution of the Index of R by Units; and of the Index of z without the Vinculum by η : for which I gave two Theorems formerly * by an Example, in the Case of a binomial Curve, which will make all evident.

* Art.
202, 203.
of this Ex-
plication.

Let

Let $\frac{z^3}{\sqrt{a^2 - z^2}} = y$, define the Relation of the Absciss and Ordinate of a Curve: and suppose I would compare the Area of this Curve, with the Area of another Curve, whose Ordinate is $\sqrt{a^2 - z^2}$, z being the common Absciss to both: which last Ordinate plainly belongs to a Circle, in which a is the Radius, and the Absciss z reaching from the Center to the Ordinate: and therefore the circular Area, with which the Area of the Curve proposed, is to be compared, is that lying betwixt the Ordinate and the Radius of the Circle parallel to it, as appears by Art. 146 *: which Area is represented by ABCD in the Fig. belonging to that Art. when $AB = z$. Now, to make this

* Of this Explication.

Comparison, let the given Ordinate, viz. $\frac{z^3}{\sqrt{a^2 - z^2}}$ be brought to the

Form $z^{3-1} \times a^2 + z^2)^{-\frac{1}{2}}$: call the Area β , and from the Assumption of β , let us first find the Area belonging to an Ordinate expressed thus $z^{3-1} \times a^2 + z^2)^{\frac{1}{2}}$ where the Exponent of the Quantity under the Vinculum, viz. $a^2 - z^2$, answering to R, is greater by Unity. Call the Area belonging to this last Ordinate, α : then, by comparing this Example with the two general Theorems just now laid down, you'll find $A = \alpha$. $B = \beta$. $\theta = 3$. $\eta = 2$. $\lambda = \frac{1}{2}$. $e = aa$. Therefore by the second Theorem you have $\alpha = \frac{a^2\beta + z^1 \times a^2 - z^2)^{\frac{1}{2}}}{4}$, for the Relation of the Areas α and β .

Again from the Area α , belonging to the Ordinate $z^{3-1} \times a^2 + z^2)^{\frac{1}{2}}$, I find the Area belonging to the Ordinate $z^{1-1} \times a^2 + z^2)^{\frac{1}{2}}$ or $\sqrt{a^2 - z^2}$: which is the former Ordinate, having the Index of z without the Vinculum, diminish'd by $2 = \eta$. This is done by Case 1. of this Prop. For calling the Area corresponding to this last circular Ordinate, viz. $\sqrt{a^2 - z^2}$, γ ; and reducing the two last Ordinates to due Form, you have $z^{3-1} \times a^2 + z^2)^{\frac{1}{2}-1}$, answering to the Area α ; and $z^{1-1} \times a^2 + z^2)^{\frac{1}{2}-1}$ answering to the Area γ : wherefore, by comparing this Example with Art. 201, 202, 203 *, you'll find $B = \alpha$ and $A = \gamma$. $\theta = 1$. $\eta = 2$. $\lambda = \frac{3}{2}$. $e = aa$. $f = -1$. And therefore

* Of this Explication.

by Art. 203 * you'll have $\gamma = \frac{z \times a^2 - z^2)^{\frac{3}{2}} + 4\alpha}{aa} =$ (by inserting the Value of α set down above) $\frac{z \times a^2 - z^2)^{\frac{3}{2}} + a^2\beta + z^1 \times a^2 - z^2)^{\frac{1}{2}}}{a^2} =$

* Of this Explication.

$z\sqrt{a^2 - z^2} + \beta$. Whence $\beta = \gamma - z\sqrt{a^2 - z^2}$, i. e. the Area of the

the

* Of this
Explica-
tion.

Art. 36.
of the
Quadra-
ture of
Curves.

the Curve proposed is equal to the circular Area, less the Rectangle contain'd under the common Absciss x and the Ordinate of the Circle, *i. e.* if you draw CG parallel to AB in the Fig. Art. 146 *, the Area CDG.

214. Our Author shews at full Length, how, from the two Areas given, belonging to the two trinomial Curves, whose Ordinates are $x^{\theta-1}R^{\lambda}$ and $x^{\theta+\eta-1}R^{\lambda}$, you may find the Area corresponding to the Ordinate $x^{\theta+\eta-1}R^{\lambda}$: and then he says by the like Means you may find the Area belonging to the Curve, whose Ordinate is $x^{\theta-1}R^{\lambda-1}$. The Manner of doing it is thus.

Let the four Ordinates, and four corresponding Areas be set down, as he hath done:

$$\begin{array}{r} \theta a e + \theta + \lambda \eta \times a f \times x^{\eta} + \theta + 2\lambda \eta \times a g \times x^{2\eta} + \theta + \eta + 2\lambda \eta \times b g \times x^{3\eta} \times x^{\theta-1} R^{\lambda-1} \quad \text{Sum of the Ord.} \\ + p e + \theta + \eta \times b e \quad + \theta + \eta + \lambda \eta \times b f \quad + q g \\ + \frac{p f}{q e} \quad + \frac{p q}{q f} \end{array}$$

$$pA + qB + ax^{\theta}R^{\lambda} + bx^{\theta+\eta}R^{\lambda} \quad \text{Sum of the Areas.}$$

Then suppose the Coefficients of all the Terms, save the first, of the complex Ordinate, equal, each to nothing: by which you'll have three distinct Equations. From the last of which, *viz.* $\theta + \eta + 2\lambda \eta \times b g + q g = 0$, find a Value of q , which is $q = -\frac{\theta + \eta + 2\lambda \eta \times b}{g}$. Then insert this Value of q in place of it, in the second Equation, which arises from the third Term, *viz.* $\theta + 2\lambda \eta \times a g + \theta + \eta + \lambda \eta \times b f + p g + q f = 0$, thence you have a Value of p , *viz.* $p = \lambda \eta \times \frac{b f}{g} - \frac{\theta + 2\lambda \eta \times a}{g}$. Again insert these Values of p and q now found, in the first Equation, arising from the second Term, which is $\theta + \lambda \eta \times a f + \theta + \eta \times b e + p f + q e = 0$: and so you'll have an Equation free of p and q , *viz.* $-\lambda \eta \times a f - 2\lambda \eta \times b e + \lambda \eta \times \frac{b f^2}{g} = 0$: which by a proper Reduction, gives you the Value of b , *viz.* $b = \frac{a f g}{f^2 - 2 g e}$. Next, substitute this Value of b in place of it, in the preceding Equations, expressing the Values of q and p , *viz.* $q = -\frac{\theta + \eta + 2\lambda \eta \times b}{g}$ and $p = \lambda \eta \times \frac{b f}{g} - \frac{\theta + 2\lambda \eta \times a}{g}$: and so you'll have $q = \frac{\theta + \eta + 2\lambda \eta \times a f / g}{2 e g - f^2}$, $p = \frac{\lambda \eta a f^2}{f^2 - 2 e g} - \frac{\theta + 2\lambda \eta \times a}{g}$. And so having found these Values of

q , p and b , arising from the Supposition of the Coefficients of the three last Terms of the complex Ordinate being equal to nothing: put these Values of q , p and b in place of them in the Sum of the Ordinates and Areas, and you'll obtain $\theta ae + \frac{\lambda \eta a f^2 e}{f^2 - 2eg} - \theta + 2\lambda \eta \times ae \times$

$z^{\theta-1} R^{\lambda-1}$ for the Sum of the Ordinates. And $\frac{\lambda \eta a f^2}{f^2 - 2eg} - \theta + 2\lambda \eta \times a \times A - \frac{\theta + \eta + 2\lambda \eta \times a f g}{f^2 - 2eg} B + a z^{\theta} R^{\lambda} + \frac{a f g}{f^2 - 2eg} z^{\theta+1} R^{\lambda}$ Sum of the Areas.

Multiply both by the given Quantity $f^2 - 2eg$, and then divide both by the given Quantity $4\lambda \eta a e e g - \lambda \eta a e f^2$ or $-\lambda \eta e a \times f^2 - 4eg$, and you'll obtain the Ordinate $z^{\theta-1} R^{\lambda-1}$, and correspondent Area

$$\frac{2\theta + 4\lambda \eta \times eg - \theta + \lambda \eta \times f^2 \times A - \theta + \eta + 2\lambda \eta \times f g \times B + f g z^{\theta+1} R^{\lambda} + f^2 - 2eg \times z^{\theta} R^{\lambda}}{-\lambda \eta e \times f^2 - 4eg}$$

Which therefore is the Area sought, and is wholly known, since A and B were given. Wherefore, if you call this Area C, and call A, B, and D the Areas corresponding to the Ordinates $z^{\theta-1} R^{\lambda}$, $z^{\theta+\eta-1} R^{\lambda}$ and $z^{\theta+1} R^{\lambda-1}$ respectively, as they are used by the Author: by considering what he hath here demonstrated in this 36th Art *. and what is demonstrated above, you'll have the two following general Theorems serving to find the Areas of all trinomial Curves whose Ordinates are thus generally expressed $z^{\theta \pm \sigma \eta} R^{\lambda \pm \tau}$, when the Areas of any two of them are given, where the Index of z without the Vinculum is greater by $+\eta$ in the Ordinates of the Curves whose Areas are represented by B and D, than in the Ordinates of those represented by A and C: but the Index of R in the Ordinates corresponding to the Areas A and B greater by Unity than the Index of R in the Ordinates of those whose Areas are C and D.

* Of the Quadrature of Curves.

$$215. \text{Th. 1. } D = \frac{f z^{\theta} R^{\lambda} + z g \times z^{\theta+1} R^{\lambda} - \theta f A - \theta + \eta + 2\lambda \eta \times z g B}{\lambda \eta \times f^2 - 4eg}$$

$$216. \text{Th. 2. } C = \frac{f^2 - 2eg \times z^{\theta} R^{\lambda} + f g \times z^{\theta+1} R^{\lambda} + \theta + 2\lambda \eta \times 2eg - \theta + \lambda \eta \times f^2 \times A - \theta + \eta + 2\lambda \eta \times f g \times B}{-\lambda \eta e \times f^2 - 4eg}$$

217. By these two Theorems, if what was said formerly *, be duly considered, and the Areas of two of the most simple trinomial Curves, be given, all others, in whose Ordinates, the Index of R under the Vinculum, is increas'd or diminish'd by any Number of Units, may be found.

* Art. 210. of this Explication.

C A S E S 3, 4.

218. And now it appears, by what means, if both the Index of z without the Vinculum is increas'd or diminish'd by the continual Ad-

Z

Art. 37, 38 of the Quadrature of Curves.

*Art. 213.
of this Ex-
plication.

dition or Subtraction of η ; and the Index of R under the Vinculum by the continual Addition or Subtraction of Unity: the Areas answering to all the Ordinates will be given, when the Area of any one of the binomial Curves is given: or the Areas of two of the trinomial Curves are given. For this is evidently done by the first and second Cases conjoin'd. An Example of this when the Curves are Binomials, was given already *. And the like may be done in the Case of Trinomials, when the Areas of any two of them are given. For the Ordinates of any two trinomial Curves may be represented thus, $z^{\theta}R^{\lambda}$ and $z^{\theta \pm \sigma}R^{\lambda \pm \tau}$: supposing then the Areas of the two Curves, to which these Ordinates belong, to be given: assume the Area belonging to the Ordinate $z^{\theta \pm \sigma}R^{\lambda \pm \tau}$, as if it were given: then from the two Areas corresponding to the two Ordinates $z^{\theta}R^{\lambda}$ and $z^{\theta \pm \sigma}R^{\lambda \pm \tau}$, you can, by Case first, find the Areas corresponding to the Ordinates $z^{\theta \pm \sigma}R^{\lambda}$ and $z^{\theta \pm \sigma}R^{\lambda \pm \tau}$ from the two last, you may find the Area belonging to the Ordinate $z^{\theta \pm \sigma}R^{\lambda \pm \tau}$ by Case second: and this Area being the other one which was given, it follows conversely that from the two Areas, belonging to the Ordinates $z^{\theta}R^{\lambda}$ and $z^{\theta \pm \sigma}R^{\lambda \pm \tau}$, given, you may find the Area belonging to the Ordinate $z^{\theta \pm \sigma}R^{\lambda}$, which was formerly assumed. And therefore the two Areas, corresponding to the two Ordinates $z^{\theta}R^{\lambda}$ and $z^{\theta \pm \sigma}R^{\lambda}$, being now both given, it is evident you can thence find the Area of any other trinomial Curve, whose Ordinate is thus indefinitely expressed $z^{\theta \pm \sigma}R^{\lambda \pm \tau}$.

As to the fourth Case of the Prop. when R is a Quadrinomial, or any other Multinomial: the Demonstration of it is made out after the like Manner as when R was a Trinomial.

DEMONSTRATION of PROP. VIII.

C A S E I.

The first Case of the Prop. is when the Index of z without the Vinculum, is increas'd or diminish'd by the continual Addition or Subtraction of η .

219. Let the Ordinates of two Curves be $pz^{\theta-1}R^{\lambda-1}S^{\mu-1}$ and $qz^{\theta+1-1}R^{\lambda-1}S^{\mu-1}$, and the corresponding Areas pA and qB , when $R = e + fz^{\eta}$ and $S = k + lz^{\eta}$, both Binomials: then you'll have as follows.

Ordinates.

Ordinates.	Areas.
1 ^o . $p \dots \dots \dots x^{\theta-1} R^{\lambda-1} S^{\mu-1}$	$\dots \dots \dots pA.$
2 ^o . $qx^n \dots \dots \dots x^{\theta-1} R^{\lambda-1} S^{\mu-1}$	$\dots \dots \dots qB.$
3 ^o . $\theta ek + \frac{\theta+\lambda\eta}{\theta+\lambda\eta+\mu\eta} \times f k x^n + \frac{\theta+\lambda\eta+\mu\eta}{\theta+\lambda\eta+\mu\eta} \times f l x^{2\eta} \dots \dots \dots x^{\theta-1} R^{\lambda-1} S^{\mu-1}$	$\dots \dots \dots z^{\theta} R^{\lambda} S^{\mu} \dots \dots \dots$
* Prop. 4.	
Suppose $p + \theta ek = 0$, and $q + \frac{\theta+\lambda\eta}{\theta+\lambda\eta+\mu\eta} \times f k + \frac{\theta+\lambda\eta+\mu\eta}{\theta+\lambda\eta+\mu\eta} \times f l = 0$, or $p = -\theta ek$, and $q = -\frac{\theta+\lambda\eta}{\theta+\lambda\eta+\mu\eta} \times f k - \frac{\theta+\lambda\eta+\mu\eta}{\theta+\lambda\eta+\mu\eta} \times f l$: and then take the Sum of the Ordinates and Areas.	
4 ^o . $\dots \dots \dots \frac{\theta+\lambda\eta+\mu\eta}{\theta+\lambda\eta+\mu\eta} \times f l x^{2\eta} x^{\theta-1} R^{\lambda-1} S^{\mu-1} \dots \dots \dots z^{\theta} R^{\lambda} S^{\mu} - \theta ekA - \frac{\theta+\lambda\eta}{\theta+\lambda\eta+\mu\eta} \times f kB - \frac{\theta+\lambda\eta+\mu\eta}{\theta+\lambda\eta+\mu\eta} \times f l$	
Divide both by the given Quantity $\theta + \lambda\eta + \mu\eta \times f l$.	
5 ^o . $\dots \dots \dots z^{\theta+2\eta-1} R^{\lambda-1} S^{\mu-1} \dots \dots \dots \frac{z^{\theta} R^{\lambda} S^{\mu} - \theta ekA - \frac{\theta+\lambda\eta}{\theta+\lambda\eta+\mu\eta} \times f kB - \frac{\theta+\lambda\eta+\mu\eta}{\theta+\lambda\eta+\mu\eta} \times f l}{\theta + \lambda\eta + \mu\eta \times f l} = C$	

Which last Area belongs to the Ordinate $z^{\theta+2\eta-1} R^{\lambda-1} S^{\mu-1}$, and is wholly known when A and B are known. And by the like way of proceeding, if you call this last Area C; from the Areas B and C you may find a fourth Area corresponding to the Ordinate $z^{\theta+3\eta-1} R^{\lambda-1} S^{\mu-1}$: and so on. And the Method of proceeding in the Progression going the contrary way, *viz.* from A and B to find the Areas corresponding to the Ordinates $z^{\theta-\eta-1} R^{\lambda-1} S^{\mu-1}$, $z^{\theta-2\eta-1} R^{\lambda-1} S^{\mu-1}$, &c. *in inf.* is after the same Sort *. And contrarily, from any other two Areas, you may, by Analysis, return back to the Areas A and B: as will easily appear by considering what was said upon the foregoing Prop. so that from any two of these Areas given, all the others will also be given.

* Art. 115. of this Explication.

220. And when R and S taken together consist of five Terms or Members, then there must be three Areas given: if R and S taken together consist of six Members, the Areas of four Curves must be given, &c. and then the Areas of all the rest may be found, after the like Manner, as hath been done, when R and S are each Binomials, and the Areas of two Curves given.

C A S E 2.

The second Case of the Prop. is when λ , the Index of R, is increas'd or diminish'd by the continual Addition or Subtraction of Unity.

Z 2

221. Let

The Quadrature of CURVES explained.

221. Let the Ordinates $px^{\theta-1}R^{\lambda}S^{\mu}$ and $qx^{\theta+1-1}R^{\lambda}S^{\mu}$ have pA and qB for their corresponding Areas: and let the Ordinates be first multiplied, and then divided by $R = e + fx^{\eta}$, as in Case second of the last Prop. by which they stand thus $pe + pfx^{\eta} \times x^{\theta-1}R^{\lambda-1}S^{\mu}$ and $qex^{\eta} + qfx^{2\eta} \times x^{\theta-1}R^{\lambda-1}S^{\mu}$: again, by Prop. 4. the Ordinate

$\theta ek + \theta + \lambda\eta \times fk + \theta + \mu\eta + \eta \times el \times x^{\eta} + \theta + \lambda\eta + \mu\eta + \eta \times flx^{2\eta} \times x^{\theta-1}R^{\lambda-1}S^{\mu}$ and Area $x^{\theta}R^{\lambda}S^{\mu+1}$ belong to one and the same Curve. Therefore take the Sum of the three Ordinates and the Sum of the three Areas, which will make an Ordinate and Area corresponding to each other: then supposing the Coefficients of the first and third Terms of the complex Ordinate, each equal to nothing, viz. $pe + \theta ek = 0$ and $\theta + \lambda\eta + \mu\eta + \eta \times fl + qf = 0$; or, by reducing, $p = -\theta k$ and $q = -\frac{\theta + \lambda\eta + \mu\eta + \eta \times l}{\lambda\eta \times fk - el}$: which Values substitute in place of p and q : divide by $\lambda\eta \times fk - el$: and thence you'll obtain the Area $\frac{x^{\theta}R^{\lambda}S^{\mu+1} - \theta kA - \frac{\theta + \lambda\eta + \mu\eta + \eta \times l}{\lambda\eta \times fk - el} \times B}{\lambda\eta \times fk - el}$, corresponding to the Ordinate $x^{\theta+1-1}R^{\lambda-1}S^{\mu}$, as follows.

Ordinates.	Areas.
1°. $pe + pfx^{\eta}$	$\times x^{\theta-1}R^{\lambda-1}S^{\mu} \dots pA$
2°. $qex^{\eta} + qfx^{2\eta}$	$\times x^{\theta-1}R^{\lambda-1}S^{\mu} \dots qB$
3°. $\theta ek + \theta + \lambda\eta \times fk + \theta + \mu\eta + \eta \times flx^{2\eta} + \theta + \mu\eta + \eta \times el$	$\times x^{\theta-1}R^{\lambda-1}S^{\mu} \dots x^{\theta}R^{\lambda}S^{\mu+1}$
4°. $\dots$	$x^{\theta+1-1}R^{\lambda-1}S^{\mu} \dots \frac{x^{\theta}R^{\lambda}S^{\mu+1} - \theta kA - \frac{\theta + \lambda\eta + \mu\eta + \eta \times l}{\lambda\eta \times fk - el} \times B}{\lambda\eta \times fk - el}$

If, instead of putting the Coefficients of the first and third Terms equal to nothing, in the complex Ordinate, you had put the Coefficients of the second and third Terms equal to nothing, you would have thence obtained the Area corresponding to the Ordinate $x^{\theta-1}R^{\lambda-1}S^{\mu}$: and calling these two Areas C and D, after the same Manner that from A and B given, the Areas C and D are found, you may from the Areas C and D, proceed to find the Areas corresponding to the Ordinates $x^{\theta-1}R^{\lambda-2}S^{\mu}$ and $x^{\theta+1-1}R^{\lambda-2}S^{\mu}$, and so on continually. And conversely if C and D be given, by Analysis, you find A and B; and thence the Areas of the Curves belonging to the Progression going

going the contrary way, *viz.* where λ is continually increased by Unity: and the rest follows as formerly.

Moreover the same may be done, by the like Means, when R or S, or both of them are more than Binomials: provided still the Number of given Areas be but two less than the Number of Members or Terms of which R and S, taken together, do consist.

C A S E 3.

222. The third Case of the Prop. is when the Index of S is increas'd or diminished by the continual Addition or Subtraction of Unity. Let pA and qB be the Areas to which the Ordinates $px^{\theta-1}R^{\lambda}S^{\mu}$ and $qx^{\theta+1-1}R^{\lambda}S^{\mu}$ correspond: when R and S are Binomials. Then from these two Areas pA and qB given, you may find other two Areas belonging to the Curves whose Ordinates are $x^{\theta-1}R^{\lambda}S^{\mu-1}$ and $x^{\theta+1-1}R^{\lambda}S^{\mu-1}$, after the like Manner that, by Case second, you found the Areas corresponding to the Ordinates $x^{\theta-1}R^{\lambda-1}S^{\mu}$ and $x^{\theta+1-1}R^{\lambda-1}S^{\mu}$: all the Difference is, that the Ordinates of the two Curves whose Areas are given, are first to be multiplied, and then divided by $S = k + lx^n$, instead of being multiplied and divided by $R = e + fz^n$: and then all the rest goes on as formerly.

C A S E 4.

223. The fourth Case of the Prop. is when the Index of x is increas'd, or diminish'd by the continual Addition or Subtraction of n , and likewise the Exponents of R and S by the continual Addition or Subtraction of Unity. In which Case the Areas answering to each of the Ordinates shall be given, upon the Conditions mentioned in the Prop. and it is done by the three former Cases jointly.

Let $x^{\theta-1}R^{\lambda}S^{\mu}$ and $x^{\theta\pm\sigma n-1}R^{\lambda\pm\tau}S^{\mu\pm\nu}$ represent the Ordinates of any two Curves whatsoever, whose Areas are given, R and S being Binomials: then assume, or suppose to be given, the Area answering to the Ordinate $x^{\theta+1-1}R^{\lambda}S^{\mu}$. From the Areas corresponding to the Ordinates $x^{\theta-1}R^{\lambda}S^{\mu}$ and $x^{\theta+1-1}R^{\lambda}S^{\mu}$, you may, by Case first, find the Areas answering to the Ordinates $x^{\theta\pm\sigma n-1}R^{\lambda}S^{\mu}$ and $x^{\theta\pm\sigma n-1}R^{\lambda}S^{\mu\pm\nu}$: from these last two Areas you can find, by Case second, other two Areas, corresponding to the Ordinates $x^{\theta\pm\sigma n-1}R^{\lambda\pm\tau}S^{\mu}$ and $x^{\theta\pm\sigma n+1-1}R^{\lambda\pm\tau}S^{\mu}$: again from these last two Areas, you may, by Case third, find the Area answering to the Ordinate $x^{\theta\pm\sigma n-1}R^{\lambda\pm\tau}S^{\mu\pm\nu}$, which was one of the Areas originally given: therefore conversely, by
Analysis,

Analysis, when the Areas belonging to the Ordinates $x^{a-1}R^bS^c$ and $x^{b \pm c n - 1}R^{a \pm 1}S^{n \pm 1}$ are given, as at first, thence may be found the Area which answers to the Ordinate $x^{a-1}R^bS^c$; which was at first assumed: and consequently it being now given, together with the Area corresponding to the Ordinate $x^{b-1}R^aS^n$, from these two Areas, now given, the Area of any other Curve in general belonging to the Class of Curves, mentioned in this Prop. may be found, when R and S are Binomials. And the like is the Reasoning in all other Cases, provided the Number of given Areas be such, as is mentioned in the Prop.

S E C T. VIII.

Containing Notes on Art. 40—51.

P R O P. IX.

Art. 40.
of the
Quadrature
of
Curves.
* Art. 38.
of this Ex-
plication.

224. **T**HE Truth contained in this Prop. viz. That the Areas of these Curves are equal whose Ordinates are reciprocally proportional to the Fluxions of their Abscisses, was at large insisted upon towards the Beginning, and was there demonstrated*.

C O R. I.

225. By means of this Corollary, we may find innumerable Curves, whose Areas shall all be equal: of which take the following Examples.

Ex. 1. Let $dx - x^2 = v^2$ be an Equation given, belonging to a Circle: and it is required to find other Curves, whose Areas may be equal to that of the Circle.

Let z and y be the Absciss and Ordinate of another Curve, having it's Area equal to that of the Circle: and suppose $dx = z^2$ express the Relation of the Abscisses.

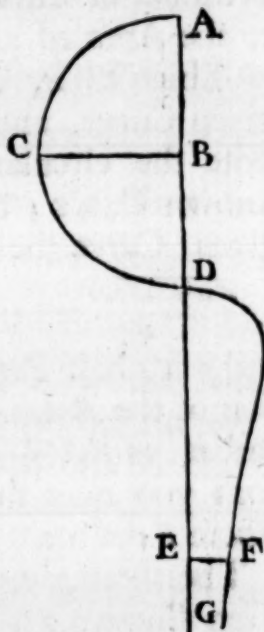
From this Equation $dx = z^2$, find the Relation of their Fluxions, by Prop. 1, which is $d\dot{x} = 2z\dot{z}$, that is by supposing $\dot{x} = 1$, $d = 2z\dot{z}$ or $\dot{z} = \frac{d}{2z}$: wherefore by this Prop. $\dot{z} = \frac{d}{2z} : \dot{x} = 1 :: v = \sqrt{dx - x^2}$:

y : whence $y = \frac{2x}{d} \sqrt{dx - x^2}$, but $dx = z^2$, by Hyp. or $x = \frac{z^2}{d}$: which Value being inserted for x , you have $y = \frac{2z^3}{d^2} \sqrt{d^2 - z^2}$ for the Property of the Curve sought, whose Area is equal to that of the Circle.

* Prop. 1. Again let it be $d^2 = xz$: then $\dot{x}z + x\dot{z} = 0$ *, or $\dot{z} = -\frac{z}{x}$: wherefore since the Ordinates and Fluxions of the Abscisses are reciprocally pro-

proportional, you'll have $-\frac{z}{x} : 1 :: \sqrt{dx - x^2} : y$, that is $y = -\frac{x}{z}\sqrt{dx - x^2}$; or (by inserting $\frac{d^2}{z}$ for x , from the given Equation $dd = xz$) $y = -\frac{d^2}{z^2}\sqrt{-d^2 + dz}$, the Equation to the Curve sought: here the curvilinear Area is situate beyond the Ordinate, extending itself along the Absciss infinitely produc'd: for z is negative, z diminishing as x increases, and contrarily: and when $x = 0$, z is infinite, but when $x = d$, $z = d$: which things easily appear from the Equations above.

Thus let ABCD be a Semicircle, the Diameter $AD = d$, the Absciss $AB = x$, and Ordinate $BC = v$: produce AD indefinitely towards G, take AE a third proportional to AB and AD, which call z : whence it is $z = \frac{dd}{x}$: draw EF perpendicular to AE, and take it equal to $\frac{d^2}{z^2}\sqrt{-d^2 + dz} = y$. Then the Ordinate EF shall describe a curvilinear Area extending itself infinitely along the right Line DG, which is the Affymptote to the Curve: so that the Area EFG adjacent to the Absciss infinitely produced beyond the Ordinate, shall be equal to the circular Area ABC adjacent to the Absciss AB; and the remaining Area DEF, lying above the Ordinate EF, and adjacent to the Part of the Absciss DE, is equal to the remaining circular Area BCD: and the whole curvilinear Area DGFD = the Semicircle ACDA. And if you take $DE = AD$, you shall have DEFD equal to the Quadrant of the Circle.



And thus you may, by assuming new Relations of x and z , find infinite other Curves, whose Areas shall be equal to the Area of the Circle.

Ex. 2. Let $\frac{aa}{d+x} = v$ be an Equation to the Hyperbola: and you would find another Curve, whose Area is equal to the hyperbolic Area: call it's Absciss and Ordinate z and y , as formerly: then take any Relation of the Abscisses, as thus, $bx = z^2$: thence by Prop. 1, $b = 2xz$ or $z = \frac{b}{2x}$: therefore $\frac{b}{2z} : 1 :: \frac{a^2}{d+x} : y$, whence $y = \frac{2a^2z}{bd + bx} = (\text{by inserting the Value of } x, \text{ viz. } \frac{z^2}{b}) \frac{2a^2z}{bd + z^2}$, which shews the Property of the Curve, whose Area is equal to the hyperbolic Area. After the like manner you may find infinite other Curves, having their Areas

Areas equal to the hyperbolical Area. And likewise any known Curve being assign'd, you may, by the like means, find as many Curves as you please, having their Areas, each, equal to the Area of the known Curve.

N. B. This Operation may be called *Transformation of Curves*: because by it you are taught a general Method of transforming any given Curve into as many other equivalent Curves, as you please: which may be of great Use in the Solution of Problems in Geometry.

226. It would be much the same, if it were required to find any Number of Curves, each of whose Areas bears any assignable Relation to the Area of any known Curve.

Thus let $x, v; z, y$ represent the Abscisses and Ordinates, of the given Curve, and of the Curve sought, as before: moreover let s denote the circular, elliptical, hyperbolical, or any other Area of a known Curve; and t the Area of the other Curve. And suppose the given Curve be defined by the Equation $a^2p - px^2 = av^2$ or $\sqrt{\frac{a^2p - px^2}{a}} = v$: which is to the Ellipse, having a the half Axe, and p it's half Parameter: let $2s + t - xv = 0$, be the assigned Relation of the Areas. Then, you assume any Relation of the Abscisses x and z , as $px^2 + az^2x^2 = pa^2$. And by means of these three Equations you may find the Value of y , *i. e.* the Property of the Curve, whose Area bears the Relation assigned to the elliptical Area.

The three given Equations are, 1°. $a^2p - px^2 = av^2$, being to the given Curve: 2°. $px^2 + az^2x^2 = pa^2$, defining the Relation of the Abscisses. 3°. $2s + t - xv = 0$, containing the given Relation of the Areas, from which y is sought.

*Art. 108.
of this Ex-
plication.

Now since $t = yz$ or $y = \frac{t}{z}$ *, you must find the Ratio of t to z or $z \times 1$ in Terms made up of z and known Quantities, which can always be done by means of the three given Equations. In order to which, from the second Equation $px^2 + az^2x^2 = pa^2$, take the Relation of the Fluxions, *viz.* $2px\dot{x} + 2az\dot{z}x^2 + 2az^2x\dot{x} = 0$, or $\dot{z} = \frac{-ax^2\dot{x} - p\dot{x}}{axx}$, *i. e.* by supposing $\dot{x} = 1$ (for we may always suppose one of the flowing Quantities to flow uniformly) $\dot{z} = \frac{-ax^2 - p}{axx}$: which Value of $\dot{z}$ you substitute for it in the Equation $y = \frac{t}{z}$, and it becomes $y = \frac{-axxi}{ax^2 + p}$, having one fluxionary Letter only: that you may get free of it likewise, from the third given Equation $2s + t - xv = 0$, take the Relation of the Fluxions, *viz.* $2\dot{s} + \dot{t} - \dot{x}v - x\dot{v} = 0$, whence

whence you have $i = (xv + xv - 2i = v + xv - 2i =$ (because $i = xv = v)$ $xv - v$: but from the first of the given Equations, viz. $a^2p - px^2 = av^2$, we have $-2pxx = 2avv$ or $v = \frac{-px}{av}$: substitute this for v in the Equation $i = xv - v$, and you have $i = \frac{-px^2}{av} - v$,

i. e. by inserting $\sqrt{\frac{a^2p - px^2}{a}}$ for v , from the first of the three given Equations, $i = \left(\frac{-px^2}{a\sqrt{\frac{a^2p - px^2}{a}}} - \sqrt{\frac{a^2p - px^2}{a}} \right) = \frac{-a^2p}{\sqrt{a^2p - px^2}}$:

wherefore, since it was $y = \frac{-axx}{ax^2 + p}$, we shall now have, by putting in $-\frac{a^2p}{\sqrt{a^2p - px^2}}$ for i , $y = \frac{a^2pxx}{p + ax^2\sqrt{a^2p - px^2}}$. Finally from the second of the three given Equations, viz. $px^2 + az^2x^2 = a^2p$, find the Value of x , which is $\sqrt{\frac{a^2p}{p + az^2}}$, insert it for x in the last Equation, and reduce, and you'll obtain, at length, $y = \frac{a^2p}{p + az^2}$ the Equation defining the Property of the Curve sought: whose Area is to the Area of the Ellipse in the Relation proposed. And after the same manner if other Relations of the Abscisses of x and z , different from the foregoing, were assumed, the Relation of the Areas continuing the same, you may find other Curves, each of whose Areas shall be in the same given Relation to the Area of the given Ellipse. And by the same way of proceeding, when any known Curve is assigned, you may find infinite other Curves, whose Areas shall have any given Relation to the Area of that Curve.

C O R. 2.

227. The Truth of this Corollary appears thus. Since by the Hyp: $z' = x$, thence by taking the Fluxions, you have $sz'^{-1}z' = \dot{x}$, i. e. by substituting $\frac{z}{x}$ for s , x for z' and $x^{-\frac{1}{2}}$ for z'^{-1} or $x^{\frac{-1}{2}}$ for z'^{-1} ,

which is deduced from the given Equation $z' = x$) $\frac{z}{x}x^{-\frac{1}{2}}\dot{z}' = \dot{x}$: hence you obtain the Ratio of the Fluxions of x and z , viz. $\dot{x} : \dot{z} :: \frac{z}{x}x^{-\frac{1}{2}} : 1$. But, by this Prop. these Curves have equal Areas, whose Ordinates are reciprocally proportional to the Fluxions of their Abscisses: wherefore, if to the three Quantities $\frac{z}{x}x^{-\frac{1}{2}}$ and

$z^{n-1} \times e + fz^n + gz^{2n} + Ec.$], you find a fourth Proportional, that shall be the Ordinate of a Curve, whose Area is equal to the Area of the Curve propos'd, it's Absciss being x : but a fourth Proportional to these three Quantities is $\frac{z^{n-1} \times e + fz^n + gz^{2n} + Ec.]{^{\frac{n}{n-1}}}}{\frac{1}{n-1}x^{\frac{n}{n-1}}}$, where if you sub-

stitute x for z , and $x^{\frac{n-1}{n}}$ for z^{n-1} , from the Equation $z^n = x$ given at the Beginning, you shall have $\frac{1}{n-1}x^{\frac{n}{n-1}} \times e + fx^n + gx^{2n} + Ec.]^{\frac{n}{n-1}}$, for the Ordinate of a Curve, which having x for it's Absciss, has it's Area equal to the Area of the propos'd Curve, whose Absciss being z has it's Ordinate $z^{n-1} \times e + fz^n + gz^{2n} + Ec.]^{\frac{n}{n-1}}$.

C O R. 3.

Art. 43.
of the
Quadrature
of
Curves.

228. This Corollary appears thus: since by the Hyp. it is $x = z^n$, thence by Prop. 1. $\dot{x} = \frac{n}{1}z^{n-1}\dot{z}$: wherefore if you put $\dot{z} = 1$, you'll have $\dot{x} = \frac{n}{1}z^{n-1}$: consequently, if to the three Terms $\frac{1}{n-1}z^{\frac{n}{n-1}}$, 1, and $z^{n-1} \times a + bz^n + cz^{2n} + Ec. \times e + fz^n + gz^{2n} + Ec.]^{\frac{n}{n-1}}$, you find a fourth Proportional, that shall be the Ordinate of the Curve whose Absciss is x , and it's Area equal to the Area of the Curve propos'd: the fourth Proportional is $\frac{z^{n-1} \times a + bz^n + cz^{2n} + Ec. \times e + fz^n + gz^{2n} + Ec.]^{\frac{n}{n-1}}}{\frac{1}{n-1}z^{\frac{n}{n-1}}}$.

i. e. (by substituting x for z^n , $x^{\frac{n-1}{n}}$ for z^{n-1} and $x^{\frac{1}{n}}$ for $z^{\frac{n}{n-1}}$, from the Equation $x = z^n$, and reducing) $\frac{1}{n-1}x^{\frac{n}{n-1}} \times a + bx^n + cx^{2n} + Ec. \times e + fx^n + gx^{2n} + Ec.]^{\frac{n}{n-1}}$.

C O R. 4.

Art. 44.
of the
Quadrature
of
Curves.

224. Since the Relation of the Abscisses, and consequently of their Fluxions is the same as before; to the three Quantities $\frac{1}{n-1}z^{\frac{n}{n-1}}$, 1, and $z^{n-1} \times a + bz^n + cz^{2n} + Ec. \times e + fz^n + gz^{2n} + Ec.]^{\frac{n}{n-1}} \times k + lx^n + mx^{2n} + Ec.]^{\frac{n}{n-1}}$, take a fourth Proportional, substitute x for z^n , $Ec.$ as in the last, and you'll obtain $\frac{1}{n-1}x^{\frac{n}{n-1}} \times a + bx^n + cx^{2n} + Ec. \times e + fx^n + gx^{2n} + Ec.]^{\frac{n}{n-1}}$

x

$\times k + lx + mx^2 + \text{Ec.}$ for the Ordinate of a Curve, whose Absciss being $x = z^{-1}$, shall have it's Area equal to that of the Curve proposed.

C O R. 5.

230. Since $x = \frac{1}{z}$, thence $\dot{x} = -z^{-2}\dot{z}$: therefore, calling the Ordinate of the Curve, whose Absciss is x , v , you'll have this Prop. Art. 45. of the Quadrature of Curves.
 $-z^{-2} : 1 :: z^{\theta-1} \times e + fz^{\eta} + gz^{2\eta} + \text{Ec.}$: v , i. e. $v = -z^{\theta+1} \times e + fz^{\eta} + gz^{2\eta} + \text{Ec.}$, or, by substituting x^{-1} for z and $x^{-\theta-1}$ for $z^{\theta+1}$ from the given Equation $x = z^{-1}$, $v = -\frac{1}{x^{\theta+1}} \times e + fx^{-\eta} + gx^{-2\eta} + \text{Ec.}$: i. e. when the Quantity under the Vinculum is a Binomial, by dividing by x^{η} , and then multiplying by it, $v = -\frac{1}{x^{\theta+1+\eta}} \times f + ex^{\eta}$: but if the Quantity under the Vinculum be a Trinomial, then by first dividing, and then multiplying by $x^{2\eta}$, it is $v = -\frac{1}{x^{\theta+1+2\eta}} \times g + fx + ex^{2\eta}$: which are the Ordinates of Curves, having $x = \frac{1}{z}$ for their Absciss, and their Areas equal to the Areas of the Curves whose Abscisses being z have their Ordinates $z^{\theta-1} \times e + fz^{\eta}$, and $z^{\theta-1} \times e + fz^{\eta} + gz^{2\eta}$ respectively.

C O R. 6.

231. The Truth of this Corollary appears the same way : for since $\dot{z} : \dot{x} :: 1 : -z^{-2}$, you'll find that $z^{\theta-1} \times e + fz^{\eta} + gz^{2\eta} + \text{Ec.}$ $\times$ Art. 46. of the Quadrature of Curves.
 $k + lx + mx^2 + \text{Ec.}$: $\frac{1}{x^{\theta+1}} \times e + fx^{-\eta} + gx^{-2\eta} + \text{Ec.}$ $\times$
 $k + lx^{-1} + mx^{-2} + \text{Ec.}$: $-z^{-2} : 1$, by considering that $x = \frac{1}{z}$. And when the Quantities under the Vinculums are Binomials, the general Expression of the Ordinate $-\frac{1}{x^{\theta+1}} \times e + fx^{-\eta} + gx^{-2\eta} + \text{Ec.}$ $\times$
 $k + lx^{-1} + mx^{-2} + \text{Ec.}$, by dividing the first Factor by $x^{\eta+1}$; and multiplying the second Factor by x^{η} and the last Factor by $x^{2\eta}$, which Operations compensate each other, becomes $-\frac{1}{x^{\theta+1+\eta+2\eta}} \times f + ex^{\eta} \times l + kx^{\eta}$: and if there be three Members

under the former Vinculum and two under the latter; by dividing the first Factor by $x^{2\eta\lambda+\eta\mu}$; and multiplying the second Factor by $x^{2\eta\lambda}$, and the last by $x^{\eta\mu}$, the Ordinate appears thus — $\frac{g + f\sqrt{x^{2\eta} + ex^{2\eta}}}{x^{2\eta+1+2\eta\lambda+\eta\mu}} \times$

$\sqrt{x^{2\eta} + ex^{2\eta}} \times \sqrt{1 + kx^{\eta\lambda}}$. And so in other Cases.

Art. 47.
of the
Quadrature
of
Curves.

232. The Ordinates belonging to the Curves whose Absciss is x , in the two last Corollaries, come out negative in the Operation, whereby they are discovered: although the negative Sign be not prefixt by the Author: which arises from this, that the Fluxion of x is negative; or x diminishes as z increases, and contrarily: whence it happens, that, if the Area of the Curve having z for it's Absciss, lye upon the nearer Side of the Ordinate, and adjacent to the Absciss, the Area of the other Curve, having x for it's Absciss, and whose Area is equal to that of the former, will be situate along the Absciss produc'd upon the further Side of it's Ordinate, and contrarily, as is observed by him in this Art. For if the Area of the Curve, having z for it's Absciss, be adjacent to the Absciss, and so increase as the Absciss z increases; the Area of the other Curve, that it may increase at the same Time, must be situate beyond it's Ordinate, for the Increase of z is the Diminution of x . But if the Area of the Curve, whose Absciss is z , lye upon the further Side of the Ordinate, the equal Area of the other will lye upon the nearer Side of the Ordinate; for the same Reason.

* Of this
Explication.

See Art. 225*.

C O R. 7.

Art. 48.
of the
Quadrature
of
Curves.

233. The Truth of this Corollary appears thus. It is supposed that

$$1^{\circ}. s = \frac{y-d}{x}, \quad 2^{\circ}. x = \frac{1}{s} z, \quad 3^{\circ}. \lambda = \frac{y-d}{x+d+\beta n}$$

Find the Relation of the Fluxions x and z , from the second of these

Equations, viz. $z : x :: (1 : z^{1-1} :: z : z^1 ::) \frac{1}{s} x^{\frac{1}{s}} : sx$. Therefore,

since the Areas of the Curves are equal by the Hyp. it is $\frac{1}{s} x^{\frac{1}{s}} : sx ::$

$v : y$; whence you have $y = \frac{1}{s} x^{\frac{1}{s}} v$: from thence $y^2 =$

$\frac{1}{s^2} x^{\frac{2}{s}} v^2$, and $y^n = \frac{1}{s^n} x^{\frac{n}{s}} v^n$; &c. Again $z^2 = \frac{\beta}{s} x^{\frac{\beta}{s}}$, and

$z^d = \frac{1}{s} x^{\frac{1}{s}}$, &c. Wherefore in the Equation to the given Curve, viz.

$y^2 x e + f y^n z^2 + g y^{2n} z^{2d} + h y^{3n} z^{3d} + \&c. = z^2 x k + l y^n z^d + m y^{2n} z^{2d} + \&c.$

by inserting in place of y^2 , y^n ; z^2 , z^d , these Values just now found,

it will stand thus

$$\frac{1}{s^2} x^{\frac{2}{s}} v^2 x e + f \frac{1}{s^n} x^{\frac{n}{s}} v^n \frac{\beta}{s} x^{\frac{\beta}{s}} + g \frac{1}{s^{2n}} x^{\frac{2n}{s}} v^{2n} \frac{1}{s} x^{\frac{1}{s}} + \&c. = \frac{\beta}{s} x^{\frac{\beta}{s}} x k + l \frac{1}{s^n} x^{\frac{n}{s}} v^n \frac{1}{s} x^{\frac{1}{s}} + m \frac{1}{s^{2n}} x^{\frac{2n}{s}} v^{2n} \frac{1}{s} x^{\frac{1}{s}} + \&c.$$

$$= \frac{\beta}{s} \frac{\beta}{x} \times k + l \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \frac{\beta}{s} \frac{\beta}{x} + m \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \frac{\beta}{s} \frac{\beta}{x} + \mathcal{E}c.$$

But $\frac{\beta}{s} \frac{\beta}{x} = -\frac{\beta}{s}$, from the first of the original Equations, wherefore, by throwing out repugnant Expressions, the Equation is

$$\frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \times e + f \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + g \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + \mathcal{E}c. = \frac{\beta}{s} \frac{\beta}{x} \times k + l \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + m \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + \mathcal{E}c.$$

Divide both Sides of the Equation by $\frac{\beta}{s} \frac{\beta}{x} \times k + l \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + m \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + \mathcal{E}c.$

and moreover by $x^{\frac{\beta}{s}}$, and you'll have it standing thus $\frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \times$

$$\frac{e + f \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + g \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + \mathcal{E}c.}{\frac{\beta}{s} \frac{\beta}{x} \times k + l \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + m \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + \mathcal{E}c.} = x^{\frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v}}$$

But $\frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} = -\frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} = -\frac{1}{\lambda}$, from the first and third of the original Equations. Wherefore by inserting these Values, and raising both Sides of the Equation to the Power whose Exponent is λ , we shall have the Equation fully reduced, standing thus $\frac{1}{\lambda} v^{\lambda} \times$

$$\frac{e + f \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + g \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + h \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + \mathcal{E}c.}{\frac{\beta}{s} \frac{\beta}{x} \times k + l \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + m \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} + \mathcal{E}c.}^{-\lambda} = x.$$

Which expresses the Relation betwixt the Absciss and Ordinate, in the Curve having x and v for it's Absciss and Ordinate, and that by an Equation not affected: and whose Area is equal to the Area of the Curve propos'd, whose Absciss and Ordinate are z and y . By means of which the Absciss x is determin'd, when the Ordinate v is given.

C O R. 8.

234. In this Corollary it is supposed that, 1°. $s = \frac{\gamma}{\gamma}$. 2°. $x = \frac{\gamma}{\gamma}$. Art. 49. of the Quadrature of Curves.

3°. $\mu = \frac{\alpha\delta + \beta\gamma}{\gamma - \delta}$. 4°. $\nu = \frac{\alpha\delta + \gamma\eta}{\gamma - \delta}$.

Whence the Relation of the Ordinates y and v , is the same as in the preceding Corollary: since, by the second Equation, the Relation of the Abscisses, and consequently of their Fluxions, is the same: so that the Values of y^{α} , y^{β} , z^{β} , z^{δ} may be thence transferred. We

have moreover in this Corollary $z^{\gamma} = \frac{\gamma}{s} \frac{\gamma}{x}$. Wherefore, by inserting these Values in the given Equation

$$y^{\alpha} \times e + f y^{\beta} z^{\delta} + g y^{\gamma} z^{\delta} + \mathcal{E}c. = z^{\beta} \times k + l y^{\gamma} z^{\delta} + m y^{\gamma} z^{\delta} + \mathcal{E}c. + z^{\gamma} \times$$

$p + q y^{\gamma} z^{\delta} + r y^{\gamma} z^{\delta} + \mathcal{E}c.$ we shall have

$$\frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \times e + f \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \frac{\beta}{s} \frac{\beta}{x} + \mathcal{E}c. = \frac{\beta}{s} \frac{\beta}{x} \times$$

$$k + l \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \frac{\beta}{s} \frac{\beta}{x} + \mathcal{E}c. + \frac{\gamma}{s} \frac{\gamma}{x} \times p + q \frac{\beta}{s} \frac{\beta}{x} \frac{\beta}{v} \frac{\beta}{s} \frac{\beta}{x} + \mathcal{E}c.$$

But,

But, as in the last Corollary, $\frac{y-y+a}{y} = 0$: and moreover, in this, $\frac{y-y+a}{y} = \mu$, and $\frac{y-y+a}{y} = v$: as appears from the first and fourth original Equations compared, therefore, by throwing out Repugnancies, dividing both Sides of the Equation by $\frac{y-y+a}{y} \times \frac{y-y+a}{y}$, and substituting μ and v for their Values, we shall have $v^2 \times e + f v^2 + g v^2 + \mathcal{E}c.$
 $= s^2 x^2 \times k + l v^2 + m v^2 + \mathcal{E}c. + s x \times p + q v^2 + r v^2 + \mathcal{E}c.$
 The Equation sought, expressing the Relation of the Absciss and Ordinate x and v , belonging to a Curve whose Area is equal to the Area of the other Curve proposed, whose Absciss and Ordinate are x and y : and that less affected.

C O R. 9.

Art. 50.
of the
Quadrature
of
Curves.

235. This Corollary is demonstrated after the same Manner as Corollaries fifth and sixth. Here it is supposed that, 1°. $\theta = \lambda$. 2°. $x = e x' + f x'^2 + g x'^3 + \mathcal{E}c.$ 3°. $\sigma = \frac{\tau}{\pi}$. 4°. $\vartheta = \frac{\lambda - \pi}{\pi}$. From the second, by Prop. 1, we have the Relation of the Fluxions of x and z , which is expressed by this Proportion $\dot{x} : \dot{z} :: \pi \times$
 $e x' + f x'^2 + g x'^3 + \mathcal{E}c. \pi^{-1} \times v e z'^{-1} + v + \pi \times f x'^{-1} + v + 2 \pi \times g x'^{-2} + \mathcal{E}c.$
 $: 1 :$ but $\dot{x} : \dot{z} :: y : v$, by this Prop. Therefore the two last Ratios of these two Proportions are equal: consequently, if you reduce them into an Equation, and in place of y , insert it's Value from the Corollary itself, you'll have

$$v = \frac{\pi x^{\theta-1} \times v + \pi \times f x'^2 + \pi \times g x'^3 + \mathcal{E}c. \times e + f x' + g x'^2 + \mathcal{E}c. \times a + b \times e x' + f x'^2 + g x'^3 + \mathcal{E}c.}{\pi \times e x' + f x'^2 + g x'^3 + \mathcal{E}c. \pi^{-1} \times v e z'^{-1} + v + \pi \times f x'^{-1} + v + 2 \pi \times g x'^{-2} + \mathcal{E}c.}$$

Which, by dividing the first Factor of the Numerator by the first Factor of the Denominator, the second by the third, and the third by the second, will stand thus $v = z^{\theta-1} \times z^{-\pi+1} \times z^{-\pi} \times$
 $e + f x' + g x'^2 + \mathcal{E}c. \lambda - \pi \times a + b \times e x' + f x'^2 + g x'^3 + \mathcal{E}c. \pi^{-1}.$

That is by multiplying the Factors, and inserting x for $e x' + f x'^2 + g x'^3 + \mathcal{E}c.$ $v = z^{\theta-\pi} \times z^{\pi-\lambda} \times a + b x^{\frac{\tau}{\pi}}.$

That is, (because $\lambda = \theta$ and $\frac{\tau}{\pi} = \sigma$, and $\frac{\lambda - \pi}{\pi} = \vartheta$ from the original Equations) $v = x^{\vartheta} \times a + b x^{\sigma}.$

Which

Which therefore exhibits the Property of a Curve having x for it's Absciss, and equal in Area to the Curve proposed, whose Absciss is z : the Relation of x and z being that mentioned above.

236. Sir Isaac desires to remark three different Cases, or Suppositions, upon which the former of the two Ordinates in this Corollary becomes more simple. The first is when $\lambda = 1$: for then the Or-

dinate is $\pi x^{0-1} \times e + \sqrt{\pi} \times f x^1 + \sqrt{\pi} \times 2\eta \times g x^{2-1} + \&c. \times a + b \times e x^1 + f x^{2-1} + \&c. \sqrt{\pi}$.

The second Supposition is, if $\tau = 1$, and the Root denoted by the Index ω be capable of being extracted: which would evidently make the Expression more simple.

The third Case mentioned, is when $\omega = -1$ and $\lambda = 1 = \tau = \sigma = \pi$: for then the former Ordinate will put on this Form

$\frac{x^{0-1} \times e + \sqrt{\pi} \times f x^1 + \sqrt{\pi} \times 2\eta \times g x^{2-1} + \&c.}{a + b \times e x^1 + f x^{2-1} + g x^{3-1} + \&c.}$: and the latter this $\frac{1}{a + b x}$.

And besides these three Cases, there are a Variety of others, upon Supposition of which, the proposed Ordinate will become more simple.

C O R. 10.

237. By some means or another an Error has crept into this Corollary, both in the Edition published by the Author himself, at the End of his Optics; and that published by Mr. Jones, with other original Pieces of Sir Isaac's.

The most simple Correction is by altering the Sign of the Index ν , belonging to S in the last Factor of the Expression of the Ordinate proposed: so that, instead of $\pi S^r + \phi R^s \times R^{\lambda-1} S^{\mu-1} \times a S^{\nu} + b R^{\tau} \sqrt{\pi}$, it be made $\pi S^r + \phi R^s \times R^{\lambda-1} S^{\mu-1} \times a S^{-\nu} R^{\tau} \sqrt{\pi}$, and the other things remain the same. Or the Correction may be made thus: supposing the Expression of the given Ordinate to be made the same, instead of $\frac{\mu-\nu\omega}{\lambda} = \frac{\nu}{\tau} = \frac{\phi}{\pi}$, put $\frac{\mu+\nu\omega}{\lambda} = -\frac{\nu}{\tau} = \frac{\phi}{\pi}$, that is, you change the Sign of ν in these Equations, into it's opposite. But without one or other of these Corrections, the Curve whose Ordinate is propos'd, cannot be transform'd into the other that is mentioned. Therefore I shall first demonstrate the Truth of the Corollary supposing the Corrections, which I have mentioned to be made: and then shew that the Corollary will not hold as the things are expressed in the printed Copies.

238. For

238. For $ex^2 + fz^{2n-1} + gz^{2n} + \text{Ec. } vx^{2n-1} + v + \eta \times fz^{2n-1} + v + 2\eta \times gz^{2n-1} + \text{Ec. } k + lz^n + mz^{2n} + \text{Ec. and } \eta lz^{2n-1} + 2\eta mz^{2n-1} + \text{Ec.}$ write R , r , S , and s respectively, and every Curve whose Ordinate is $\pi Sr + \phi Rj \times R^{\lambda-1} S^{\mu-1} \times aS^{-v} + bR^{\tau}$, if it be $\frac{\mu-v}{\lambda} = \frac{v}{\tau} = \frac{\phi}{\pi}$, $\frac{\tau}{\pi} = \sigma$, $\frac{\lambda-\pi}{\pi} = \vartheta$, and $R^{\pi} S^{\phi} = x$, passes into another one equal to it in Area, whose Ordinate is $x^{\vartheta} \times a + bx^{\sigma}$.

For since $x = R^{\pi} S^{\phi}$, we have by Prop. 1, $x = \pi R^{\pi-1} R S^{\phi} + \phi R^{\pi} S^{\phi-1} S$: but it's evident that $r = \frac{R}{x}$ and $s = \frac{S}{x}$; or $R = rz$ and $S = sz$: by inserting of which Values of R and S in the preceding Equation, you'll have $x = \pi R^{\pi-1} rz S^{\phi} + \phi R^{\pi} S^{\phi-1} sz$, whence $x : z : \pi R^{\pi-1} r S^{\phi} + \phi R^{\pi} S^{\phi-1} s : 1$. Wherefore by this Prop. if the given Ordinate $\pi Sr + \phi Rj \times R^{\lambda-1} S^{\mu-1} \times aS^{-v} + bR^{\tau}$ be divided by $\pi R^{\pi-1} r S^{\phi} + \phi R^{\pi} S^{\phi-1} s$, we shall have $R^{\lambda-\pi} S^{\mu-\phi} \times aS^{-v} + bR^{\tau}$, for the Ordinate of another Curve, equal in Area to the former, and having x for it's Absciss. But, because $x = R^{\pi} S^{\phi}$ or $R^{\tau} = x^{\frac{\tau}{\pi}} S^{-\frac{\tau}{\phi}}$, hence the Ordinate may stand thus, $R^{\lambda-\pi} S^{\mu-\phi} \times aS^{-v} + bx^{\frac{\tau}{\pi}} S^{-\frac{\tau}{\phi}}$, or, because $\frac{v}{\tau} = \frac{\phi}{\pi}$, i. e. $S^{-\frac{\tau}{\phi}} = S^{-v}$, it will stand thus $R^{\lambda-\pi} S^{\mu-\phi-v} \times a + bx^{\frac{\tau}{\pi}}$. Where if, instead of $\mu - v$ you put in $\frac{\lambda\phi}{\pi}$; and instead of $\frac{\tau}{\pi}$, σ , deduced from the original Equations, you'll have $R^{\lambda-\pi} S^{\frac{\lambda\phi}{\pi}-\pi} \times a + bx^{\sigma}$, that is, because $R^{\pi} S^{\phi} = x$, and $\frac{\lambda-\pi}{\pi} = \vartheta$, $x^{\vartheta} \times a + bx^{\sigma}$: which is that mentioned by our Author, and is the Ordinate of a Curve equal in Area to the Curve proposed; and having x for it's Absciss.

239. Again, if the Values of R , r , S and s be the same as formerly, and the Ordinate of a Curve be $\pi Sr + \phi Rj \times R^{\lambda-1} S^{\mu-1} \times aS^{-v} + bR^{\tau}$; and moreover $\frac{\mu+v}{\lambda} = \frac{v}{\tau} = \frac{\phi}{\pi}$, $\frac{\tau}{\pi} = \sigma$, $\frac{\lambda-\pi}{\pi} = \vartheta$, $R^{\pi} S^{\phi} = x$, the Curve will pass into another equal to it, whose Ordinate is $x^{\vartheta} \times a + bx^{\sigma}$.

For

For we prove, as formerly, that the Ordinate of the new Curve is

$R^{\lambda-\pi} S^{\mu-\phi} \times \overline{aS^v + bx^{\frac{\tau}{\pi}} S^{-\frac{\sigma\tau}{\pi}}}$, i. e. because $-\frac{v}{\tau} = \frac{\phi}{\pi}$, or $-\frac{\phi\tau}{\pi} = v$,
 $R^{\lambda-\pi} S^{\mu-\phi+\nu\omega} \times \overline{a + bx^{\frac{\tau}{\pi}}}$; which, by inserting $\frac{\lambda\phi}{\pi}$ for $\mu + \nu\omega$ and σ
 for $\frac{\tau}{\pi}$, from the original Equations, stands thus, $R^{\lambda-\pi} S^{\frac{\lambda\phi-\pi\phi}{\pi}} \times \overline{a + bx^{\sigma}}$,
 that is $x^{\phi} \times \overline{a + bx^{\sigma}}$ as formerly.

240. But if the Signs of all the Quantities be retained the same as in the printed Copies, then it may be demonstrated, as above, that the Ordinate of the Curve will be $R^{\lambda-\pi} S^{\mu-\phi} \times \overline{aS^v + bR^{\tau}}^{\omega}$ or $R^{\lambda-\pi} S^{\mu-\phi}$

$\times \overline{aS^v + bx^{\frac{\tau}{\pi}} S^{-\frac{\sigma\tau}{\pi}}}$, i. e. $R^{\lambda-\pi} S^{\mu-\phi} \times \overline{aS^v + bx^{\sigma} S^{-v}}$ or $R^{\lambda-\pi} S^{\frac{\lambda\phi-\pi\phi}{\pi}} \times \overline{aS^{2v} + bx^{\sigma}}$ = $x^{\phi} \times \overline{aS^{2v} + bx^{\sigma}}$: whereas it ought to be $x^{\phi} \times \overline{a + bx^{\sigma}}$.

241. Our Author takes notice of certain Cases, among many others, in which the former of the two Ordinates becomes more simple.

S E C T. IX.

Containing Notes on Art. 52—64.

PROP. 10. and it's Corollaries explained and illustrated.

242. **T**HE Design of this Proposition is to discover the most simple Figures, with which any proposed Curve may be geometrically compared: which is done by the Help of the foregoing Propositions. When any proposed Curve is capable of being compared with a rectilinear Figure, so as the Relation betwixt them may be exactly determined, then such a Curve is strictly *quadrable*: because we can find a Square equal to it by Prop. 14. B. 2^d. Elem. When this can't be done, the Curve, properly speaking, is not quadrable: yet a Curve may be such, that, though it's Area be not capable of a geometrical Comparison with a right-lined Figure, it may be compared with the Area of another Curve, or Curves more simple than it: by which means we may have a more clear and distinct Conception of it. Thus if a Curve be such, that it is capable of being compared geometrically with a Circle, one shall have a more distinct and satisfactory Notion of it upon that very account: because the Circle is a Figure so fami-

B b

liar

liar and well understood in Geometry, and so easily described. So likewise with respect to such Figures as may be compared with the Ellipse and Hyperbola, they will be better conceived of, upon that very account, than if they were not capable of such Comparison, and their Areas more easily exhibited. So that, considering how well the Conic Sections are known, any Curve that admits of a geometrical Comparison with them, may, in some Sense, be accounted *quadrable*. Accordingly our Author in the second of the two following Tables, has given us a Catalogue of the more simple Kind of Curves that admit of a geometrical Comparison with the Ellipse (including the Circle) and Hyperbola: even as in the first, we have a Catalogue of the more simple Curves, that are strictly quadrable.

C A S E 1.

243. This Case is plain: every Curve of this Sort being quadrable. See Art. 52. and 143 *.

* Of this
Explica-
tion.

C A S E 2.

244. If $ax^{b-1} \times e + fz^n + gz^{2n} + \mathcal{E}c.\}^{\lambda-1}$ be the Ordinate of a Curve, the Curve may admit of a Comparison with a rectilinear Figure or not.

If it do, you'll have it's Area exactly by Prop. 5. and that you may know when this will happen, in the Case of the more simple Curves, belonging to this general Class, and have the Canons at hand for determining these Areas, consult Art. 147—161 *.

* Of this
Explica-
tion.

245. But if the proposed Curve cannot be compared with a right-lin'd Figure, the next thing is to find out the most simple Curve or Curves, with which it may be compared, whether it be Conic Section or any others. To which purpose, you first make use of Cor. 2^d. Prop. 9. which see. By it, if you assume ν any Quantity you please, and put $x = z^{\frac{1}{\nu}}$, the Curve whose Ordinate is $ax^{b-1} \times e + fz^n + gz^{2n} + \mathcal{E}c.\}^{\lambda-1}$ may be changed into another of an equal Area, whose Ordinate is $\frac{\nu}{\nu-1} ax^{\frac{\nu b-1}{\nu-1}} \times e + fx^{\nu} + gx^{2\nu} + \mathcal{E}c.\}^{\lambda-1}$: and therefore if you put $\nu = 1$, the Curve proposed will be changed into one more simple whose Ordinate is $\frac{a}{\nu} x^{\frac{\nu b-1}{\nu-1}} \times e + fx + gx^2 + \mathcal{E}c.\}^{\lambda-1}$, where the Indexes of x under the Vinculum, go on in the natural Order of the Numbers.

Ex.

Ex. Let $\frac{4x^2}{2+3x^3} = 4x^{3-1} \times 2 + 3x^{3|0-1}$ be the Ordinate of a Curve: I perceive it cannot be compared with a rectilinear Figure, by Art. 139, or 149*. Wherefore, by comparing this Ordinate with the general Ordinate $ax^{\theta-1} \times e + fx^{\eta} + \mathcal{G}c.^{\lambda-1}$, I find $\theta=3$. $\eta=3$. $\lambda=0$. $a=4$. $e=2$. $f=3$: therefore the Curve proposed is converted into another equal to it, whose Ordinate is $\frac{4}{3} \times 2 + 3x|^{-1}$ or $\frac{4}{6+9x}$, whose Absciss is $x=z^3$: and which Ordinate belongs to the rectangular Hyperbola, as it is applied to the Assymptote: and therefore the Area of the Curve propos'd, whose Ordinate is $\frac{4x^2}{2+3x^3}$, is reducible to an hyperbolical Area.

* Of this
Explica-
tion.

The Comparifon is thus. Reduce the Ordinate $\frac{4}{6+9x}$ to $\frac{\frac{4}{3}}{\frac{2}{3}+x}$, where the numeral Coefficient of x is Unity: wherefore, having drawn the right Line ABN for one of the Assymptotes, (see Fig. 1. Tab. 2. p. 25.) in it take $Aa = \frac{2}{3}$, (linear Unity being suppos'd to be already determined) and $aB = x$: likewise having drawn aG perpendicular to AB, and equal to $\frac{2}{3}$: if with the Center A and Assymptote ABN you describe a rectangular Hyperbola GDS through the Point G; and then draw the perpendicular Ordinate BD, the Area $aBDG$ shall be equal to the Area of the Curve proposed. For from the Property of the Hyperbola $AB = \frac{2}{3} + x$: $Aa = \frac{2}{3}$: $aG = \frac{2}{3}$: BD, which therefore is equal to $\frac{\frac{4}{3}}{\frac{2}{3}+x}$ or $\frac{4}{6+9x}$: so that $aBDG$ is equal to the Area of the Curve proposed: for that is the Area belonging to the Ordinate BD and Absciss $aB = x$ *.

* Art. 190.
of this Ex-
plication.

N. B. It would have come to the same, if you had expressed the Ordinate $\frac{4}{6+9x}$ thus $\frac{4}{9} \times \frac{1}{\frac{2}{3}+x}$; only in that Case you must take $aG = \frac{2}{3}$, and then the Area of the Curve will be to the hyperbolical Area $aBDG$ as 4 to 9.

246. But if the given Ordinate $ax^{\theta-1} \times e + fx^{\eta} + gx^{2\eta} + \mathcal{G}c.^{\lambda-1}$ being reduced to the Ordinate $\frac{a}{\eta} x^{\frac{\theta}{\eta}-1} \times e + fx + gx^2 + \mathcal{G}c.^{\lambda-1}$, this last one is not yet so simple as may be, you must then diminish the Indexes $\frac{\theta}{\eta} - 1$ and $\lambda - 1$ by Unity always, by Prop. 7. until they be as low as possible, and so you shall have the most simple Figures, which can be discovered by this Method, with which the Curve proposed may be geometrically compared.

Ex. 1. Let the Ordinate $\frac{dx^{3\eta-1}}{e+fx^\eta}$ or $dz^{3\eta-1} \times e+fx^\eta)^{-1}$ be proposed. By putting $x=z^\eta$, I reduce it, by the Method already shewn, to this other Ordinate $\frac{d}{\eta} x^2 \times e+fx^{-1}$ belonging to a Curve of an equal Area. Then, by rejecting the Units from the Index of x without the Vinculum, by Prop. 7. Case 1, I reduce the Ordinate further, viz. thus: call the Area belonging to the Curve proposed, t , and consequently the Area belonging to the Ordinate $x^2 \times e+fx^{-1}$, $\frac{\eta t}{d}$, you'll find, by applying the general Theorem at Art. 203 *, viz. $A = \frac{x^\theta R^\lambda - \theta + \lambda \eta \times fB}{\theta e}$ that $\frac{dx^2 - 2\eta ft}{2ed}$ is the Area of a Curve whose Ordinate is $x \times e+fx)^{-1}$. Again, by repeating the Application of the same Theorem, you'll find $\frac{2edx - dfx^2 + 2\eta fft}{2ee}$ the Area belonging to the Ordinate $\frac{d}{e+fx}$, which is to the Hyperbola as before. Therefore if we call this hyperbolical Area, s , we will have $\frac{2edx - dfx^2 + 2\eta fft}{2e^2} = s$: whence, by proper Reduction, $t = (\frac{2e^2s - 2edx + dfx^2}{2\eta f^2}) = \frac{e^2s}{\eta f^2} - \frac{dex^\eta}{\eta f^2} + \frac{dx^{2\eta}}{2\eta f}$, and therefore, if the hyperbolical Area s be given, the Area of the Curve proposed is given likewise.

Of this
Explica-
tion.

Ex. 2. Let the Ordinate $3x^3\sqrt{1+4x^2+3x^4}$ be proposed: which, reduced to Form, is $3x^{4-1} \times 1+4x^2+3x^4)^{\frac{1}{2}-1}$. By Cor. 2. Prop. 9, if you put $x^2=x$, you'll have $\frac{3}{2}x\sqrt{1+4x+3x^2}$ for the Ordinate of another more simple but equal Curve.

This done, I next reject Unity from the Index of x without the Vinculum, twice. To do which, bring it to this Form $\frac{3}{2}x^{2-1} \times 1+4x+3x^2)^{\frac{1}{2}-1}$, and call the Area belonging to it, t ; that is the Area belonging to the Ordinate $x^{2-1} \times 1+4x+3x^2)^{\frac{1}{2}-1}$ is $\frac{2t}{3}$. Now suppose s and σ denote the Areas belonging to the Ordinates $\sqrt{1+4x+3x^2}$ and $x^{-1}\sqrt{1+4x+3x^2}$: then, here are three Ordinates $x^{0-1} \times 1+4x+3x^2)^{\frac{1}{2}-1}$, and $x^{1-1} \times 1+4x+3x^2)^{\frac{1}{2}-1}$ and $x^{2-1} \times 1+4x+3x^2)^{\frac{1}{2}-1}$, whose Areas are denoted by σ , s and $\frac{2t}{3}$ respectively. Therefore, by applying the Theor. in Art. 206 *, viz. $C = \frac{x^\theta R^\lambda - \theta eA - \theta + \lambda \eta \times fB}{\theta + 2\lambda \eta \times g}$, you'll have $\theta = 0$. $\eta = 1$. $\lambda = \frac{1}{2}$. $e = 1$.

• Of this
Explica-
tion.

$$f = 4$$

$f=4 \cdot g=3 \cdot A=\sigma \cdot B=s \cdot C=\frac{2f}{3} \cdot z=x$. Whence $\frac{2f}{3} = \frac{1+4x+3x^2}{9} - 6s$; for since $\theta=0$, σ goes out. Wherefore if you

multiply by $\frac{1}{2}$ you have $t = \frac{1+4x+3x^2}{6} - s$; the Area of the Curve proposed: so that when the Area s belonging to the Ordinate $\sqrt{1+4x+3x^2}$ is given, the Area t will likewise be given. Now the Ordinate $\sqrt{1+4x+3x^2}$ is an Ordinate to the Hyperbola: which is thus constructed. (See Fig. 3^d. Tab. 2^d. p. 25.) With the determinate Axis $Ka = \frac{2}{3}$ and Latus Rectum 2, describe the Hyperbola $aGDS$, having A for it's Center: take $aa = Aa$, $aB = x = z^2$, and draw the Ordinates aG , BD : then you'll have $BD = \sqrt{1+4x+3x^2}$ and $aBDG = s$, and consequently $\frac{1}{6}BD^3 - aBDG = t$ the Area of the Curve whose Ordinate was proposed, viz. $3z^3\sqrt{1+4z^2+3z^4}$: for, from the Property of the Hyperbola, putting L for the Latus Rectum, it is $Ka + aB \times aB : BD^2 :: Ka : L$, that is in the present Case (where we have $Ka = \frac{2}{3}$, $aB = \frac{1}{3} + x$ and $L = 2$) $\frac{1}{3} + x \times \frac{1}{3} + x : BD^2 :: \frac{2}{3} : 2$: whence $BD = \sqrt{1+4x+3x^2}$: and the initial Limit of the Area being aG , as may be collected from Sect. 5. concerning the Position and Limits of curvilinear Areas compared with Prop. 5. it must be $aBDG = s$: and therefore $\frac{1}{6}BD^3 - aBDG$ is the Area of the Curve proposed.

247. But although in the last Example, the Area of a trinomial Curve is compared geometrically with the Area of one Curve only, which is more simple than itself: yet it most frequently happens, that the Areas of two Curves must be supposed to be given, that you may have the Area of a trinomial Curve. See Art. 209*. Schol. 1.

* Of this
Explica-
tion.

Thus if $\frac{\sqrt{1+4x^2+3x^4}}{3}$ was the Ordinate of a Curve proposed, in order to find the most simple Curves with which it may be compared, you will find that it requires two Conic Sections to it's Quadrature. The Method of proceeding is thus. The proposed Ordinate $z^{0-1} \times \frac{1+4z^2+3z^4}{3}$ by putting $z^2 = x$, according to Cor. 2. Prop. 9, is changed into the Ordinate $\frac{1}{3}x^{-1} \times \sqrt{1+4x+3x^2}$: where $\sqrt{1+4x+3x^2}$ is the Ordinate of an Hyperbola as was shewn formerly: so that if we should call the Area of the Curve proposed, t ; and the Area of the Hyperbola, s , we would have $2t$ and s , for the Areas of two Curves

* Of this
Explica-
tion.

Curves whose Ordinates are $x^{-1} \times \sqrt{1+4x+3x^2}$ and $x^0 \times \sqrt{1+4x+3x^2}$: from one of which being given, the other cannot be found: as you will find by trial, by applying any of the Canons contain'd in Art. 206, 207, or 208 *. Wherefore, as our Author next directs, I transform the Ordinate $x^0 \times \sqrt{1+4x+3x^2}$ into this other $-\frac{2}{\xi^2} \times \sqrt{3+4\xi+\xi^2}$, which belongs to an equal Curve by Cor. 5.

Prop. 9, by putting $\frac{1}{x} = \xi$: where the radical Part $\sqrt{3+4\xi+\xi^2}$ is the Ordinate of an Hyperbola. Suppose you call the hyperbolic Area corresponding to the Ordinate $\sqrt{3+4\xi+\xi^2}$, σ : then by means of the two Areas s and σ you may find t the Area sought. For by Cor. 5. Prop. 9, the Ordinate $\sqrt{1+4x+3x^2}$ is transform'd into the Ordinate $-\frac{1}{\xi^2} \times \sqrt{3+4\xi+\xi^2}$, whose corresponding Area is s : wherefore there are three Ordinates, viz.

$$1^o. \xi^{-2-1} \times \sqrt{3+4\xi+\xi^2}^{-1-1} \quad 2^o. \xi^{-1-1} \times \sqrt{3+4\xi+\xi^2}^{-1-1} \quad 3^o. \xi^{1-1} \times \sqrt{3+4\xi+\xi^2}^{-1-1},$$

all belonging to the same Form of Curves; whose corresponding Areas are $-s$, $-2t$ and σ , two whereof being given the other is given likewise by Prop. 7. For by assuming the Areas $-s$ and $-2t$ belonging to the Ordinates $\xi^{-2-1} \times \sqrt{3+4\xi+\xi^2}^{-1-1}$ and $\xi^{-1-1} \times \sqrt{3+4\xi+\xi^2}^{-1-1}$, you'll find, by applying the Canon Art. 206 *, viz.

* Of this
Explica-
tion.

$C = \frac{2^0 R^{\lambda} - 0^0 A - 0^0 \lambda n \times f^B}{0^0 + 2\lambda n \times g}$, that $\xi^{-2} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1} = 6s - 4t$, is the Area of the Curve next in order, viz. that whose Ordinate is $\xi^{0-1} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1}$. Renew the Application of the same Canon to find the Relation of the Areas belonging to the three Ordinates $\xi^{-1-1} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1}$, $\xi^{0-1} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1}$ and $\xi^{1-1} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1}$: which Areas are $-2t$, $\xi^{-2} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1} = 6s - 4t$, and σ , and you will obtain this Equation $\sigma = \frac{\xi^{-1} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1} - 2\xi^{-2} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1} + 2t + 12s}{2}$, that is, by reducing, $t =$

$\sigma - 6s + 3\xi^{-2} + \frac{1}{2}\xi^{-1} - 1 - \frac{1}{2}\xi \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1}$, which, by substituting r for $\sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1}$, and v for $\xi^{-1} \times \sqrt{3+4\xi+\xi^2}^{\frac{1}{2}-1} = \sqrt{1+4x+3x^2}$, will give $t = \sigma - 6s + 3xv + \frac{1}{2}v - r - \frac{1}{2}\xi r$, the Area sought. And the Description of the Conic Sections may be easily gathered from what was said formerly.

248. Now,

248. Now, after you have found out the most simple Curves with which any proposed Curve may be compared, by the preceding Methods: you may sometimes find Curves yet more simple, provided the former ones can be so compared and joined together, by the Help of Prop. 3, as to constitute a new Curve, either of the Form mentioned in Cor. 9; or of that mentioned Cor. 10. Prop. 9: which will sometimes happen.

C A S E 3.

249. When any proposed Ordinate is of this Form $x^{n-1} \times a + bx^n + cx^{2n} + \&c. \times e + fx^n + gx^{2n} + \&c. \sqrt[n-1]{}$: and the Curve cannot be squared by Prop. 5, then you are to distinguish it's Ordinate into it's constituent Parts, which are to be considered as so many distinct Ordinates: every one of which must be handled, as the Ordinate in Case 2: and having by that means discovered the Areas belonging to these several Ordinates, join them all together by their proper Signs, and you'll have the Area required. See Art. 165*. Yet it may be proper here to observe, in order to prevent Mistakes, that such a Curve may be quadrable, although all the Parts of which the Ordinate is compos'd are not quadrable, as appears by comparing Art. 147 with Art. 162 and 163*: such is the Ordinate $\frac{2x-aa}{\sqrt{x^2-aa}}$.

* Of this Explication.

* Of this Explication.

C A S E 4.

250. This Case respects Curves of the Form mentioned in Prop. 4th and 6th: which are to be treated much after the same Manner as those mentioned in Case 2^d: according as our Author directs.

C A S E 5.

251. The Meaning of what is affirmed by our Author in this Case, together with the Reason of it, appears from what has been said formerly in Case 3^d, and Art. 164 and 165*: and will be further illustrated by what is said in the Cor. succeeding.

* Of this Explication.

C O R. 1.

252. It is evident from what has been said, that, if the Relation of the Absciss and Ordinate be expressed by an Equation in which the Ordinate y ascends not above the Square, such Curve may be compared with the most simple Figures with which it admits of a geometrical Comparison, whether rectilinear or curvilinear. Thus in the Example adduced

adduced by our Author: from the Equation $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$, is deduced $y = \frac{a^3 + z^3 \pm a\sqrt{a^4 + 2az^3 - z^4}}{a^2 + z^2}$: where the Value of y is made up of two Parts, which are to be considered as two distinct Ordinates, viz. $\frac{a^3 + z^3}{a^2 + z^2}$ and $\frac{\pm a\sqrt{a^4 + 2az^3 - z^4}}{a^2 + z^2}$: which must belong to Curves that may either be squared, or else compared with the most simple Curves, by the Methods already delivered.

253. It may be proper here to observe, that, when the Ordinate y has different Values (as in the above Example there are two different Values, according as the irrational Part is affected with the positive or negative Sign) there will be as many distinct Values of the Area, as there are Values of y : for in so many different Points will the Ordinate meet with the Curve: and so each particular Area must be considered as extending along that Part of the Curve, to which the corresponding Value of the Ordinate belongs: by observing of which, the several Areas may easily be distinguished from each other.

C O R. 2.

254. By means of this Corollary, any Curve whose Ordinate is defin'd by an affected Equation of that Sort which is mentioned in Cor. 7. Prop. 9. viz. such a Curve as may be made to pass into another equal to itself, whose Absciss may be determined from the Ordinate given, by an Equation not affected, may either be squared, or else compared with the most simple Curves it can be compared with.

For the general Equation to that Sort of Curves is

$$y^a \times e + fy^n z^d + gy^{2n} z^{2d} + \mathcal{E}c. = x^\beta \times k + ly^n z^d + my^{2n} z^{2d} + \mathcal{E}c.$$

which by putting $s = \frac{n-d}{n}$. $x = \frac{1}{s} z^s$ and $\lambda = \frac{n-d}{a\delta + \beta n}$. passes into another equal Curve, in which the Relation of the Absciss and Ordinate is defin'd by the Equation $\frac{1}{s} v^{a\lambda} \times \overline{e + fv^n + gv^{2n} + \mathcal{E}c.}^\lambda \times \overline{k + lv^n + mv^{2n} + \mathcal{E}c.}^{-\lambda} = x$; by which, from the Ordinate v given, the Absciss x is determined, and that by an Equation not affected. Therefore, by taking the Fluxions, according to Prop. 1, we have $\dot{x} = \frac{a\lambda}{s} v^{a\lambda-1} \dot{v} \times \overline{e + fv^n + \mathcal{E}c.}^\lambda \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda} + \lambda \times \overline{fv^{n-1} \dot{v} + 2ngv^{2n-1} \dot{v} + \mathcal{E}c.} \times \overline{e + fv^n + \mathcal{E}c.}^{\lambda-1} \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda} \times \frac{1}{s} v^{a\lambda} - \lambda \times \overline{lv^{n-1} \dot{v} + 2nmv^{2n-1} \dot{v} + \mathcal{E}c.} \times \overline{k + lv^n + \mathcal{E}c.}^{\lambda-1} \times \overline{e + fv^n + \mathcal{E}c.}^\lambda \times \frac{1}{s} v^{a\lambda}.$

And

And if you multiply both Sides of the Equation by v , it becomes
 $\dot{x}v = \dot{v} \times \frac{a\lambda}{s} v^{a\lambda} \times \overline{e + fv^n + \mathcal{E}c.}^\lambda \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda} + \dot{v} \times \frac{\lambda}{s} v^{a\lambda} \times$
 $\overline{\eta fv^n + 2\eta gv^{2n} + \mathcal{E}c.} \times \overline{e + fv^n + \mathcal{E}c.}^{\lambda-1} \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda} - \dot{v} \times$
 $\frac{\lambda}{s} v^{a\lambda} \times \overline{\eta lv^n + 2\eta mv^{2n} + \mathcal{E}c.} \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda-1} \times \overline{e + fv^n + \mathcal{E}c.}^\lambda$
 $= \dot{z}y$, because $\dot{x}v$ and $\dot{z}y$ are the Fluxions of equal Curves: conse-
 quently, by Prop. 10, a Curve having v for it's Absciss, and $\frac{a\lambda}{s} v^{a\lambda} \times$
 $\overline{e + fv^n + \mathcal{E}c.}^\lambda \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda} + \frac{\lambda}{s} v^{a\lambda} \times \overline{\eta fv^n + 2\eta gv^{2n} + \mathcal{E}c.} \times$
 $\overline{e + fv^n + \mathcal{E}c.}^{\lambda-1} \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda} - \frac{\lambda}{s} v^{a\lambda} \times \overline{\eta lv^n + 2\eta mv^{2n} + \mathcal{E}c.}$
 $\times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda-1} \times \overline{e + fv^n + \mathcal{E}c.}^\lambda$ for it's Ordinate, is equal
 to the Curve proposed, having z and y for it's Absciss and Ordinate.

Now the Ordinate of the Curve having v for it's Absciss, is made
 up of these Parts.

- 1^o. $\frac{a\lambda}{s} v^{a\lambda} \times \overline{e + fv^n + \mathcal{E}c.}^\lambda \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda}$.
- 2^o. $\frac{\lambda}{s} v^{a\lambda} \times \overline{\eta fv^n + 2\eta gv^{2n} + \mathcal{E}c.} \times \overline{e + fv^n + \mathcal{E}c.}^{\lambda-1} \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda}$.
- 3^o. $-\frac{\lambda}{s} v^{a\lambda} \times \overline{\eta lv^n + 2\eta mv^{2n} + \mathcal{E}c.} \times \overline{e + fv^n + \mathcal{E}c.}^\lambda \times \overline{k + lv^n + \mathcal{E}c.}^{-\lambda-1}$.

And the Curves having these several Parts of their Ordinates, and v
 for their common Absciss, may either be squared by Prop. fifth and
 sixth, or compared with the most simple Curves, they can be com-
 pared with by Cases third and fourth of this Prop. and consequently
 the Curve proposed is either squared; or compared with the most
 simple Curves, it admits of a geometrical Comparifon with.

255. Accordingly, as our Author affirms, every Curve whose Pro-
 perty is defin'd by any Equation consisting of three Terms only, how-
 ever affected, may either be squared by this Corollary; or compared
 with the most simple Curves possible.

Ex. Let $y^3 - axy + z^3 = 0$ be such an Equation: by reducing it
 to the Form in Cor. 7. Prop. 9, it may stand thus $y^3 \times 1 = z^3 \times$
 $-1 + ax^{-2}y$: which compared with the general Form, gives $\alpha = 3$
 $= \beta$, $\delta = -2$, $\eta = 1$. $e = 1$. $k = -1$. $l = a$. $s = (\frac{\eta - \delta}{\eta}) = 3$,
 $x = \frac{1}{3}z^3$, $\lambda = -1$: whence by a proper Substitution you'll obtain $-\frac{1}{3v^3} + \frac{a}{3v^2} = x$ for the Relation of the Absciss and Ordinate of an
 equal Curve: and by taking the Fluxions; $\frac{\dot{v}}{v^4} - \frac{2a\dot{v}}{3v^3} = \dot{x}$; and fur-

ther, by multiplying both Sides of the Equation by v , it is $\frac{\dot{v}}{v^3} - \frac{2a\dot{v}}{3v^3}$
 $= \dot{x}v = \dot{z}y$: wherefore the Area of the Curve, which has v for it's
 Abscifs, and $\frac{1}{v^3} - \frac{2a}{3v^2}$ for it's Ordinate, is equal to the Area of the
 Curve proposed: which therefore, by the Method of Quadratures, is —

*Art. 145. $\frac{1}{2v^2} + \frac{2a}{3v}$ *. Now to reduce this to an equivalent Expression in Terms
 of this Ex- of z and y : you have $\dot{x} : \dot{z} :: y : v$, by what has been shewn, and
 plication. again from the Equation $\frac{1}{3}z^3 = x$, by taking the Fluxions, you have
 $\dot{x} : \dot{z} :: z^2 : 1$: therefore $y : v :: z^2 : 1$; whence $v = \frac{y}{z^2}$. Substitute
 this Value of v in the Expression $-\frac{1}{2v^2} + \frac{2a}{3v}$, and you'll obtain $\frac{2az^2}{3y}$
 $-\frac{z^4}{2y^2}$ for the Value of the Area required in Terms of z and y .

Therefore, if you assume z or y of any determined Value, and sub-
 stitute that Value of z or y , in the Equation to the Curve, you'll, by
 the Resolution of the Equation, find the corresponding Value or Values
 of the other (for there may be as many Values of the unknown Quan-
 tity, retain'd in the Equation, as it has Dimensions) and so by sub-
 stituting these Values of z and y in the Expression of the Area, you'll
 obtain so many Values of the Area, as the Nature of the Curve requires.
 Thus in the foregoing Example, if you take the Abscifs $z = \frac{1}{2}a$, then
 by substituting this Value for z in the Equation to the Curve, viz.
 $z^3 - az y + y^3 = 0$, it becomes $\frac{1}{8}a^3 - \frac{1}{2}aay + y^3 = 0$; whence you'll
 obtain three Values of y corresponding to that Value of z , viz. 1°. y
 $= \frac{1}{2}a$: 2°. $y = \frac{\sqrt{5}-1}{4}a$. 3°. $y = -\frac{\sqrt{5}+1}{4}a$: the first two posi-
 tive; the other negative. Wherefore if you substitute these Values of
 y for it, and $\frac{1}{2}a$ for z in the Expression $\frac{2az^2}{3y} - \frac{z^4}{2y^2}$, you'll have 1°. $\frac{5}{24}aa$.
 2°. $\frac{27-11\sqrt{5}}{-96+48\sqrt{5}}a^2$. 3°. $-\frac{27+11\sqrt{5}}{96+48\sqrt{5}}aa$ for the three corresponding
 Values of the Area, according to the different Parts of the Curve, to
 which the different Values of y belong: which Areas are to be com-
 puted from the Beginning of the Abscifs *; the first two upon the one
 Side of the Abscifs; and the last, which is negative, upon the other
 Side of the Abscifs, because of the negative Value of the Ordinate.

*Art. 169.
 of this Ex-
 plication.

256. The Result will be the same, if we proceed as follows. After
 having found, as above, that $x = \frac{a}{3v^3} - \frac{1}{3v^3}$, instead of considering
 x as the Abscifs and v as it's corresponding Ordinate, you may, on the
 contrary, consider v as the Abscifs, and x as it's Ordinate: and then
 multiply-

multiplying both by $\dot{v}$, it is $x\dot{v} = \frac{a\dot{v}}{3v^2} - \frac{\dot{v}}{3v^3}$, which is the Fluxion of the curvilinear Space, that, together with the curvilinear Area, belonging to the Absciss x and Ordinate v , makes up the circumscribed Rectangle xv : wherefore find the Area of the Curve having v for it's Absciss and $\frac{a}{3v^2} - \frac{1}{3v^3}$ for it's Ordinate: which is $-\frac{a}{3v} + \frac{1}{6v^2}$: subtract it from $xv = \frac{a}{3v} - \frac{1}{3v^2}$, and there remains $\frac{2a}{3v} - \frac{1}{2v^2}$, for the Area required, to be converted into Terms of z and y as formerly.

C O R. 3.

257. What our Author lays down in this Corollary is evident, *viz.* that any Curve whose Ordinate is defin'd by any affected Equation whatsoever, that by the Application of Cor. 8. Prop. 9. can be made to pass into an affected quadratic Equation, may either be squared; or else compared with the most simple Figures, that it can be compared with. Which appears sufficiently from what has been said.

258. Schol. But if the Equation defining the Relation of the Absciss and Ordinate of any Curve proposed, be such, that none of the preceding Methods are sufficient for finding it's Area: then we must have recourse to the Method of infinite Series, laid down and explain'd by our Author in his Analysis by Equations of an infinite Number of Terms: by means of which, any Equation how high or affected soever it be, that defines the Relation of the Absciss and Ordinate of a Curve, may be reduced into an infinite converging Series, which exhibits the Value of the Ordinate, by an infinite Number of simple Terms, made up of the Absciss and known Quantities: each of which Terms is quadrable by Art. 145*: and therefore we thence find an approximate Value of the curvilinear Area. An Example of this we gave already Art. 146*: more of which, and the Method of Resolution, may be seen in the Treatise itself: which, because of the Affinity of the Subject, I have here annexed.

* Of this Explication.
* Of this Explication.

259. And thus far I have endeavoured to explain the Principles upon which the Quadrature of Curves, and the Doctrine of Fluxions in general, are founded; and to illustrate the foregoing Propositions contained in this Treatise, in which, the general Methods for finding the Areas of Curves are laid down. These the skilful and cautious Geometrician may have recourse to and consult, as often as Occasion requires: but in regard it would be a troublesome and tedious Affair to be obliged always to have recourse to those general Methods, when the Area of any Curve is required: therefore our Author has furnished

us with the two following Tables. The first of which exhibits immediately, the Areas of the more simple Kinds of Curves, that are capable of being squared geometrically. The other contains the more simple Kinds of Curves, that are capable of a geometrical Comparison with the Ellipse and Hyperbola, under the first of which the Circle is comprehended: in which the Relation of the curvilinear Areas, to the Areas of the Conic Sections, is set before our View, so as that it may be seen at once. They are both divided into certain Forms, Orders, or Classes, which contain certain Species's of Curves under them: which Tables are a sort of *Speculum*, exhibiting the Areas of infinite Numbers and Varieties of Curves.

EXPLICATION of T A B L E I.

260. This Table contains four different Forms or Orders of Curves: the first two Orders contain only one Species each; the other two Forms contain under them innumerable Species's of Curves, although the first four Species's, which are the most simple, be only inserted in the Table.

Form 1st. 261. The Curves of the first Form are quadrable: and the Area * Of this is that set down in the Table as appears by Art. 143 *. But if it be η = 0, appearing thus $dx^{-1} = y$, it belongs to the Hyperbola: for Explication. which see Art. 190, 191 *.

Explication. 262. All Curves belonging to the second Form are doubly quadrable, as appears by Art. 149 *: and the Area is to be found as our Form 2d. Author directs Art. 30 †: that is to say, you must bring the Expression of the Ordinate, viz. $\frac{dx^{\eta-1}}{e^2 + 2efxz^{\eta} + f^2 x^{2\eta}}$ to stand thus, $\frac{dx^{\eta-1}}{e + f x^{\eta}}$, which

† Of the is expressed thus $dx^{\eta-1} \times e + f x^{\eta}^{-2}$; or thus $dx^{\eta-1} \times f + e x^{\eta-1}^{-2}$ *. Quadrature of Curves.

* Art. 127. of this Explication. to the first Form of the Ordinate belongs the Area $\frac{dx^{\eta}}{\eta e^2 + \eta e f x^{\eta}}$; to the second

* Of this Explication. cond the Area $\frac{-d}{\eta e f + \eta f^2 x^{\eta}}$, as may be easily collected from Art. 147 *;

in which the third Example belongs to this Form. The first Area commences at the Beginning of the Absciss; the other lyes along the Absciss infinitely produced beyond the Ordinate. See Sect. 5. concerning the Position and Limits of Areas.

263. The other Species of Curves under this Form, whose Ordinates are denoted thus $\frac{dx^{p-1}}{e + f x^{\eta}}$, where p is any Integer, positive or negative, are not quadrable, as appears by Art. 147 and 149 *.

* Of this Explication.

264. The

264. The Curves belonging to Form third are quadrable by Art. Form 3d. 147 and 149 *: which may be thus generally expressed $dx^{p-1} \sqrt{e+fx^n}$ * Of this Explication. $\sqrt{e+fx^n} = y$: p being a positive Integer: but by them it appears that the Area, arising from the Ordinate retained in this Form, runs out into an infinite Series: and therefore it must be converted into the other Form, viz. $dx^{\frac{2p+n}{2}} - 1 \times \sqrt{f+ex^{-n}}$: and thence you'll find the Area by Prop. 5. according to Art. 147 *. The fourth Example in * Of this Explication. that Art. is the same with the Curve of Species second of this Order: which you may see.

265. Now that we may have a general Theorem for the Areas of all Curves belonging to this Form; and know how these Areas may be continued through all the different Species's in inf. let the general Ordinate $dx^{p-1} \sqrt{e+fx^n}$, turned into this Shape $dx^{\frac{2p+n}{2}} - 1 \times \sqrt{f+ex^{-n}}$, be compared with the Ordinate for binomial Curves Art. 147 *, and you'll find by substituting $\frac{2p+n}{2}$ for θ , $-n$ for η , * Of this Explication. $\frac{1}{2}$ for λ , $-\frac{2p+1}{2}$ for r , and $-p+1$ for s , that the Series for the

Area will be $dx^{\frac{2p+n}{2}} \times \sqrt{f+ex^{-n}}$ into $\frac{2}{2p+1 \times \eta f} - \frac{2 \times 2 \times p-1 \times e}{2p+1 \times 2p-1 \times \eta f^2} x^{-n}$
 $+ \frac{2 \times 2 \times 2 \times p-1 \times p-2 \times e^2}{2p+1 \times 2p-1 \times 2p-3 \times \eta f^3} x^{-2n} - \frac{2 \times 2 \times 2 \times 2 \times p-1 \times p-2 \times p-3 \times e^3}{2p+1 \times 2p-1 \times 2p-3 \times 2p-5 \times \eta f^4} x^{-3n} +$
 $\frac{2 \times 2 \times 2 \times 2 \times 2 \times p-1 \times p-2 \times p-3 \times p-4 \times e^4}{2p+1 \times 2p-1 \times 2p-3 \times 2p-5 \times 2p-7 \times \eta f^5} x^{-4n} - \&c.$ where you may observe,

1°. That the Number of Terms must be equal to p .
 2°. Supposing p to be 1. 2. 3. 4. 5, &c. successively, the numeral Coefficient of the Numerator of the last Term will be 2, - 4, + 16, - 96, + 868, &c. respectively: which are produced by multiplying by - 2, - 4, - 6, - 8, &c. continually, i. e. multiply the first, viz. 2 by - 2 that produces the 2^d, - 4; multiply this by - 4, it produces + 16; this by - 6, produces - 96, &c. as easily appears from the Progression of the Terms.

3°. p standing for any positive Integer whatsoever, it will easily appear, by considering the Progression of the Series, that the numeral Coefficients of the Numerators of the several Terms, beginning with the last and going backwards, are formed by dividing by the Terms of this Series, 2. 4. 6. 8, &c. and the Denominators of the Terms,

in

in the same Order, are formed by dividing by the Terms of this Series 3. 5. 7. 9, &c. Wherefore, the numeral Coefficients of the several Terms, beginning with the last and going backwards, if you respect the Signs, are formed by multiplying by the Terms of this Series — $\frac{1}{3}$, — $\frac{1}{4}$, — $\frac{1}{6}$, — $\frac{1}{8}$, &c.

Therefore, if the foregoing Expression of the Area be duly reduced, by bringing all the Terms under one common Denominator; and the Order of the Terms be inverted, the Relation of the numeral Coefficients of the several Terms will remain the same as before, the first Term now coming in place of the last formerly; the second in place of the last save one, &c. and since that which becomes the common Denominator, is the same with the Denominator of the last Term formerly, as in the Series above: hence it appears, that, after the Expressions of the Areas are reduced to the same Denominator, if p be 1. 2. 3. 4. &c. successively, the Denominator will be 3. 15. 105. 945, &c. respectively, formed by multiplying the Numbers 1. 3. 5. 7. 9, &c. into one another continually.

And so you have the Demonstration of what Sir Isaac Newton says at the End of the second Table, with respect to the Continuation of the Series of Curves belonging to this third Form. Only you may observe here, that the first Factor, viz. $dx^{\frac{2p-1}{2}} \times \sqrt{e+fx^n}^{\frac{1}{2}}$, being multiplied by the negative Power of x contain'd in the last Term of the Series for the Area above, viz. $-\frac{p-1}{p} \times \eta$, will always become $d \times e + f \times z^n^{\frac{1}{2}}$ or $dR^{\frac{1}{2}}$: to compensate which Multiplication, all the other Terms are divided by the same negative Power of x ; and so the Expression for the Area shall put on the same Form as in the Table.

Form 4th. 266. The Species's of Curves belonging to Form fourth are likewise infinite, and may be thus generally expressed $\frac{dx^{2p-1}}{\sqrt{e+fx^n}}$. The Area

arising from the Form of the Ordinate $dx^{2p-1} \times e + f \times z^n^{-\frac{1}{2}}$, runs out into an infinite Series, as appears from Art. 147 and 149*: but, when the

* Of this
Explica-
tion.

Ordinate is turned into this other Form, viz. $dx^{\frac{2p-1}{2}} \times \sqrt{e+fx^n}^{-\frac{1}{2}}$, you'll find, by these same Art. that the Expression for the Area terminates: and therefore all the Curves belonging to this Form in inf. are quadrable.

267. Let the Curve of the first Species be proposed, viz. $dx^{2p-1} \times \sqrt{e+fx^n}^{-\frac{1}{2}} = y$: I turn the Ordinate into this other Form, viz. $dx^{\frac{2p-1}{2}} \times \sqrt{e+fx^n}^{-\frac{1}{2}}$: then by comparing it with the general Form of Ordinate

nates of binomial Curves Art. 147 *, I find $\theta = \frac{1}{2}\eta$, $\eta = -\eta$, $\lambda = \frac{1}{2}$. * Of this Explication.
 $r = -\frac{1}{2}$, $s = 0$, $a = d$, $e = f$, $f = e$. Therefore by a due Substitution, the Area comes out $\frac{d}{-\eta} x^{\frac{1}{2}\eta} \times f + ex^{-\eta} \times \frac{1}{-\frac{1}{2}\eta} = \frac{2d}{\eta} \sqrt{e + fx^{\eta}}$; as in the Table.

And after the same manner, the Quadratures of all the other Species's of Curves, belonging to this Form, may be demonstrated, viz. by the direct and immediate Application of Art. 147 *.

268. But the Areas of all the Curves in this Species, and the preceding in *inf.* may be found, and the Series of Areas continued after the Area of the first is known, by the Help of the Theorem Art. 202 *, * Of this Explication.

viz. $B = \frac{z^{\theta} R^{\lambda} - \theta e A}{\theta + \lambda \eta \times f}$: thus, to find the Area of the second Species of

this Form, whose Ordinate is $dx^{2\eta-1} \times e + fx^{\eta} - \frac{1}{2}$, you'll have $\theta = \eta$.

$\lambda = \frac{1}{2}$. $A = \frac{2}{\eta} \sqrt{e + fx^{\eta}}$: therefore, by a due Substitution of Values,

it is $B = \frac{z^{\eta} \sqrt{e + fx^{\eta}} - \frac{2e}{f} \sqrt{e + fx^{\eta}}}{\frac{1}{2}\eta f} = \frac{2fx^{\eta} - 4e}{3\eta f^2} \sqrt{e + fx^{\eta}}$: which being

multiplied by d , gives $\frac{-4e + 2fx^{\eta}}{3\eta f^2} d\sqrt{e + fx^{\eta}}$ for the Area of the Curve constituting the second Species of this Order: and so for the others. But the Species's of Curves belonging to the Progression, which runs backward above the first Species, both in this and the preceding Form,

expressed thus $dx^{-p\eta-1} \sqrt{e + fx^{\eta}}$ and $\frac{dx^{-p\eta-1}}{\sqrt{e + fx^{\eta}}} = y$, cannot be found

by the Help of the Theor. Art. 203 *: See Schol. 2. Art. 209 †: * Of this Explication. † Of this Explication.

269. You may likewise, by Prop. 5, and Art. 147 *, find a general Area corresponding to the general Ordinate $dx^{p\eta-1} \times e + fx^{\eta} - \frac{1}{2}$, as was shewn in the Curves belonging to the third Order: which general

Expression will stand thus $dx^{\frac{2p\eta-\eta}{2}} \times f + ex^{-\eta} - \frac{1}{2}$ into

$$\frac{2}{2p-1 \times \eta f} - \frac{4 \times p-1 \times e}{2p-1 \times 2p-3 \times \eta f^2} x^{-\eta} + \frac{8 \times p-1 \times p-2 \times e^2}{2p-1 \times 2p-3 \times 2p-5 \times \eta f^3} x^{-2\eta} - \frac{16 \times p-1 \times p-2 \times p-3 \times e^3}{2p-1 \times 2p-3 \times 2p-5 \times 2p-7 \times \eta f^4} x^{-3\eta} + \text{Ec.}$$

Which Series will serve to find the Area of any Curve belonging to this Order, since the Progression of the Series is evident. And by considering

Considering it in the same Light, in which the general Series for the Areas of the Curves of the preceding Order, was considered above, you'll easily see the Demonstration of what Sir Isaac says, concerning the Continuation of the Series of Curves of this Order *in inf.* at the End of the second Tab. which it would be needless to insist upon particularly, after what has been shewn at length, with respect to this Matter, upon Order third.

270. To shew the Application of the preceding general Canon, put $p=4$, then you'll have $dx^{\frac{1}{2}} \times f^{+ez^{-n}}^{\frac{1}{2}}$ into $\frac{2}{7nf} - \frac{4 \times 3 \times e}{7 \times 5 \times nf^2} z^{-1}$ $+$ $\frac{8 \times 3 \times 2 \times e^2}{7 \times 5 \times 3 \times nf^3} z^{-2}$ $-$ $\frac{16 \times 3 \times 2 \times 1 \times e^3}{7 \times 5 \times 3 \times 1 \times nf^4} z^{-3}$, which, by reducing all to the Denominator of the last Term, and freeing the Expression of the negative Powers of z , becomes $d\sqrt{e} + fz^n \times \frac{30f^3 z^{3n} - 36ef^2 z^{2n} + 48e^2 fz^n - 96e^3}{105nf^4}$ the same as in the Tab.

271. If in the Ordinates of Curves belonging to the last two general Forms, the Index of the Radical was $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. in the third Form; or $-\frac{3}{2}$, $-\frac{5}{2}$, $-\frac{7}{2}$, &c. in the fourth, they may easily be reduced to the Curves in the Tab. by Case second of Prop. 7, viz. by diminishing or increasing that Index by Unity, till it become $\frac{1}{2}$ or $-\frac{1}{2}$, according to the Canons in Art. 211 and 212*.

* Of this
Explica-
tion.

EXPLICATION OF TABLE II.

272. When we enquire into the Area of any Curve proposed, that Area may be exhibited either *arithmetically*: which is done by Prop. fifth and sixth; or *geometrically* by finding and describing other more simple Curves; with which it may be compared. Now by the Help of Prop. ninth and tenth, our Author has constructed this Table or Catalogue of Curves, that are capable of being compared geometrically with the Ellipse and Hyperbola; so that their Areas may be exhibited by the Description of these Figures: and consequently given, when these Figures are given.

273. It is not my Design here to go through all the Orders of Curves contained in this Tab. and demonstrate the Quadratures of the several Species of Curves, belonging to each Order. All I shall do, is to shew by a few Examples, the Manner of demonstrating all the rest: then make some general Remarks upon the Construction and Use of this Table: and finally shew the Application of it in some Examples mostly taken from the Author himself.

274. The Curve of the first Species of Form first, is expressed thus $\frac{dx^{\eta-1}}{e+fx^{\eta}} = y$: which is not capable of a geometrical Comparison with a right-lined Figure, as appears by Art. 147 and 149 *. Wherefore, putting $z^{\eta} = x$, (by Cor. 4. Prop. 9.) it passes into another equal Curve, whose Ordinate is $\frac{1}{\eta} \times \frac{d}{e+fx}$: x being the Absciss. Therefore, if the Area of the Curve which hath $\frac{d}{e+fx}$ for it's Ordinate, be called s , the Area of the Curve proposed shall be $\frac{1}{\eta}s$, since η is a given Quantity. Now $\frac{d}{e+fx} = v$ is an Equation to the rectangular Hyperbola, as was shewn formerly *.

* Of this Explication.

* Art. 245. of this Explication.

The Construction of the Hyperbola, which serves equally for all the Species's of Curves under this general Form, is thus. See Fig. 1. belonging to this Table, p. 25. Draw the right Line ABN for one of the Asymptotes, make f linear Unity, take $Aa = e$, and $aB = x$: draw aG perpendicular to ABN, and equal to $\frac{d}{e}$, or a third Proportional to the three Lines e , f and d : through G , with the Center A and Asymptote ABN, describe the rectangular Hyperbola GDS, draw the Ordinate BD: then you have $v = BD = \frac{d}{e+fx}$: and $s = aGDB$: therefore it is $\frac{aGDB}{\eta} = t$ the Area required *.

* Art. 198. of this Explication.

275. The Quadratures of the other Species's of this Order, are demonstrated by the Help of the Canon in Art. 211 *: the Area of the first Species being once found.

* Of this Explication.

The Series of Curves belonging to this Order, may be carried on likewise *in inf.* by Division. Thus let the fourth Species be proposed

$\frac{dx^{4\eta-1}}{e+fx^{\eta}} = y$: divide $dx^{4\eta-1}$ by $fx^{\eta} + e$, and the Quote is $\frac{d}{f}x^{3\eta-1} -$

$\frac{de}{ff^2}x^{2\eta-1} + \frac{de^2}{f^3}x^{\eta-1} - \frac{\frac{de^3}{f^3}x^{\eta-1}}{e+fx^{\eta}}$, which Division is to be carried on, till you have as many Terms in the Quote, as the Number prefixt to η in the Index of x without the Vinculum, less one: to which annex the Remainder with it's proper Sign, and $e + fx^{\eta}$ under it for it's Denominator: and then all the Terms save the last, denote Ordinates of Curves belonging to Order first, Table first, which are thence quadrable; and the last Term is the Ordinate of a Curve, belonging to Species first of this Order: therefore all the Areas, joined with their pro-

D d

per

per Signs, make the Area required, viz. $\frac{d}{3\eta f} z^{3\eta} - \frac{de}{2\eta f^2} z^{2\eta} + \frac{de^2}{\eta f^3} z^{\eta} - \frac{e^3}{\eta f^3} s$. And so of others.

If we put $\frac{dz^{p\eta-1}}{e + fz^{\eta}} = y$, it is easy to see what is the Progression of the general Series for the Area, viz.

$$\frac{d}{p-1 \times \eta f} z^{p\eta-\eta} - \frac{de}{p-2 \times \eta f^2} z^{p\eta-2\eta} + \frac{de^2}{p-3 \times \eta f^3} z^{p\eta-3\eta} - \dots + \frac{e^{p-1}}{\eta f^{p-1}} s.$$

Form 3d. 276. The Quadrature at Species first, Order third, is thus demonstrated. Since the Property of the Curve is expressed thus $\frac{d}{z} \sqrt{e + fz^{\eta}}$

Art. 42. of the Quadrature of Curves. $= y$: by putting $\frac{1}{z} = x^2$ *, an equal Curve, whose Absciss is x , will have it's Ordinate $v = -\frac{2d}{\eta} x^{-2} \sqrt{f + ex^2}$. Wherefore, from the Assumption of the Area t , belonging to the Ordinate $-\frac{2d}{\eta} x^{-2} \sqrt{f + ex^2}$,

or of the Area $-\frac{\eta t}{2d}$ belonging to the Ordinate $x^{-2} \sqrt{f + ex^2}$, find (by Prop. 7.) the Area belonging to the Ordinate $\sqrt{f + ex^2}$: which

* Of this Explication. you do by the Theorem in Art. 202 *, viz. $B = \frac{x^{\theta} R^{\lambda} - \theta e A}{\theta + \lambda \eta \times f}$. For by

assuming the Area $-\frac{\eta t}{2d}$ agreeing to the Ordinate $x^{-1-1} \times \sqrt{f + ex^2}^{\frac{1}{2}-1}$, I find the Area belonging to the Ordinate $x^{-1-1} \times \sqrt{f + ex^2}^{\frac{1}{2}-1}$, by putting $\theta = -1$. $\eta = 2$. $\lambda = \frac{1}{2}$. $e = f$. $f = e$ and $A = -\frac{\eta t}{2d}$:

which accordingly is $\frac{x^{-1} \times \sqrt{f + ex^2}^{\frac{3}{2}} - \frac{\eta t}{2d}}{2e}$: call this Area s , then you

have this Equation $s = \frac{\sqrt{f + ex^2}^{\frac{3}{2}}}{2ex} - \frac{\eta t}{4de}$: by reducing of which, you

obtain $t = \frac{4de}{\eta f} \times \frac{\sqrt{f + ex^2}^{\frac{3}{2}}}{2ex} - s$ the Area of the Curve proposed: which therefore is given, when s is given.

Now s denotes the Area of a Conic Section, for it belongs to the Curve whose Ordinate is $\sqrt{f + ex^2}$, which, when f and e are both positive, is an Hyperbola, viz. that represented at Fig. 2. p. 25. but if f be negative, it belongs to the Hyperbola at Fig. 3. p. 25. lastly, if e be negative, it belongs to the Ellipse at Fig. 4. p. 25.

The

The Construction for the first Case is thus: with the Center A, the half transverse Axis $AP = \sqrt{f}$ (that is a mean Proportional betwixt Unity and f) and Parameter of the Axis $\frac{2\sqrt{f}}{e}$ (that is a fourth Proportional to the Lines e , 2 and AP) describe the Hyperbola PGD: draw the indeterminate Axis AN, in which take $AB = x = \frac{1}{x^{\frac{1}{2}}}$: to which draw the Ordinate BD: so shall it be $BD = v = \sqrt{f + ex^2}$, and $s = APDB$. For from the Property of the Hyperbola, calling the Latus Transversum t , and it's Parameter p , it is $BD + AP \times BD - AP: AB^2 :: t : p$ or $BD^2 - AP^2 : AB^2 :: t : p$, that is, by inserting the Values according to the Construction, $BD^2 - f : x^2 :: 2\sqrt{f} : \frac{2}{e}\sqrt{f}$: whence $BD = \sqrt{f + ex^2} = v$, as it ought to be; and so you'll have $s = APDB$.

Now in order to find what the Area of the Curve proposed is equal to, in the geometrical Description; we must find the Value of the Expression $\frac{f + ex^2}{2ex}$, or which is the same, $\frac{v^2}{2ex}$. For this end, draw the Tangent DT meeting AB in T: then from the Property of the Tangent, you'll have $AT = \frac{AB \times AP^2}{AO^2 - AP^2}$, that is, by substituting the Symbols, $AT = \frac{\frac{1}{x}}{ex}$: add $AB = x$ to each, and it is $BT = \frac{ex^2 + \frac{1}{x}}{ex}$, that is, because $\frac{1}{x} = f$, $BT = (\frac{ex^2 + f}{ex}) \frac{v^2}{ex}$: therefore, multiplying by $\frac{BD}{2} = \frac{v}{2}$, you obtain $\frac{BT \times BD}{2}$, i. e. $TDB = \frac{v^3}{2ex}$.

Hence the Area of the Curve proposed, whose Ordinate is $\frac{d}{x}\sqrt{e + fx^2}$, which was shewn to be $\frac{4de}{ef} \times \frac{v^3}{2ex} = \frac{4de}{ef}$ into $TDB - APDB$: which is set down in the Table thus $\frac{4de}{ef}$ into $APDB - TDB$, denoting that it is ambiguous, whether the second is to be subtracted from the first, or the first from the second: for sometimes TDB may exceed $APDB$, sometimes be less: but the Area of the Curve proposed is equal to that Difference: if $TDB - APDB$ be positive, the Area required being positive, lyes above the Ordinate; if $TDB - APDB$ be negative, the Area is situate upon the other Side of the Ordinate; and adjacent to the Absciss produced beyond the Ordinate.

277. If the Quantity f be negative in the Value of y , so that it be $y = \frac{d}{x}\sqrt{e - fx^2}$, then the Area of the Curve is found by the Descrip-

tion of the Hyperbola likewise, but according to a different Construction, viz. at Fig. 3. p. 25. For now the Curve proposed, will be equal to that whose Ordinate is $-\frac{2d}{\eta}x - \sqrt{-f + ex^2}$: and so the Conic Section must be such, as to have it's Ordinate $\sqrt{-f + ex^2}$. The Construction is thus: with the Center A, transverse Axe $Ka = 2\sqrt{\frac{f}{e}}$, and Parameter $2\sqrt{ef}$, describe the Hyperbola $aGDS$: take $AB = x$, draw the Ordinate BD and the Tangent DT : and the Area, which now by changing the Sign of f , becomes $-\frac{4de}{\eta f} \times \frac{v^3}{2ex} - s$ or $\frac{4de}{\eta f} \times s - \frac{v^3}{2ex}$, will be, according to this Construction, $\frac{4de}{\eta f}$ into $aGDB - TDB$ or $\frac{4de}{\eta f} \times -aGDT$: which shews the Area to be on the further Side of the Ordinate.

For from the Property of the Hyperbola, (using the Symbols t and p in the same Sense as formerly) $KB \times Ba : BD^2 :: t : p$; or $x^2 - \frac{1}{4}tt : v^2 :: t : p$; that is, by inserting the Values of t and p from the Construction, $x^2 - \frac{f}{e} : v^2 :: 2\sqrt{\frac{f}{e}} : 2\sqrt{ef}$: whence we have $v = \sqrt{-f + ex^2}$, as it ought to be. Again, from the Property of the Tangent, the Subtangent $BT = x - \frac{\frac{1}{4}tt}{x} = \frac{x^2 - \frac{1}{4}tt}{x} =$ (by inserting $\frac{v^2}{p}$ for $x^2 - \frac{1}{4}tt$,

as above) $\frac{v^2}{px}$ or $\frac{2\sqrt{\frac{f}{e}} \times v^2}{2\sqrt{fe} \times x} = \frac{v^2}{ex} = BT$: therefore, multiplying by $\frac{v}{2} = \frac{BD}{2}$, it is $\frac{v^3}{2ex} = \frac{BT \times BD}{2} = \Delta TDB$. Wherefore, since $s =$

$aGDB$, you'll have $\frac{4de}{\eta f} \times s - \frac{v^3}{2ex} = \frac{4de}{\eta f}$ into $aGDB - TDB$ or $\frac{4de}{\eta f}$ into $-aGDT$. Which negative Value shews that the Area of the Curve is adjacent to the Abciss produced beyond the Ordinate.

278. If e be negative, the Construction is by means of the Ellipse at Fig. 4. p. 25. for now it is $v = \sqrt{f - ex^2}$. The Construction is thus: upon the Center A, with the transverse Axis $Ka = 2\sqrt{\frac{f}{e}}$, and half conjugate Axis $AP = \sqrt{f}$, describe the Ellipse $aDPSK$: take $AB = x$, draw the Ordinate DB , then it will be $DB = (v =) \sqrt{f - ex^2}$, and $APDB = s$: consequently the Area, which now, by changing the Sign of e , becomes $-\frac{4de}{\eta f} \times \frac{v^3}{-2ex} - s$, or $\frac{4de}{\eta f} \times \frac{v^3}{2ex} + s$, will be $\frac{4de}{\eta f}$ into $TDB + APDB$; or else $\frac{4de}{\eta f}$ into $TDB - aGDB$, that is $\frac{4de}{\eta f} \times aGDT$.

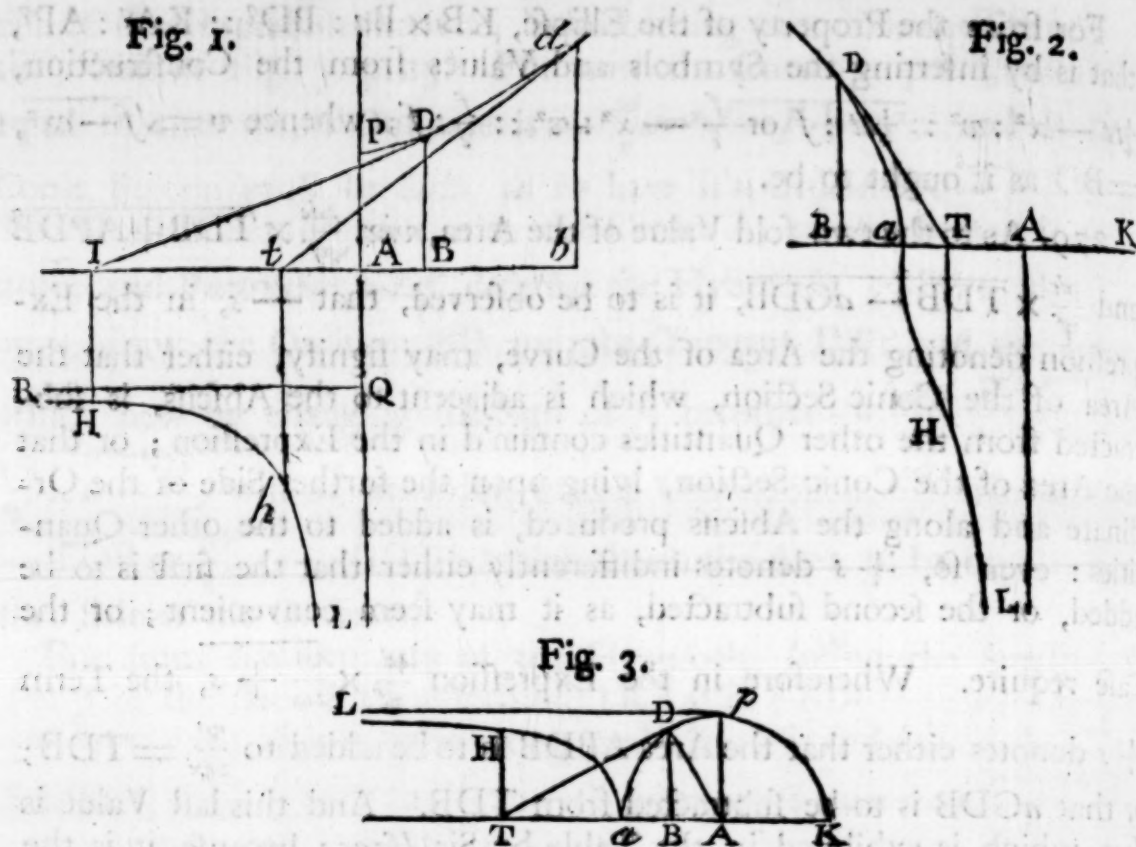
For

For from the Property of the Ellipse, $KB \times Ba : BD^2 :: KA^2 : AP^2$, that is by inserting the Symbols and Values from the Construction, $\frac{1}{4}tt - x^2 : v^2 :: \frac{1}{4}tt : f$ or $\frac{f}{e} - x^2 : v^2 :: \frac{f}{e} : f$; whence $v = \sqrt{f - ex^2}$, $= BD$ as it ought to be.

279. As to the two-fold Value of the Area, viz. $\frac{4de}{\eta f} \times \overline{TDB + APDB}$ and $\frac{4de}{\eta f} \times \overline{TDB - aGDB}$, it is to be observed, that $-s$, in the Expression denoting the Area of the Curve, may signify, either that the Area of the Conic Section, which is adjacent to the Absciss, is subtracted from the other Quantities contain'd in the Expression; or that the Area of the Conic Section, lying upon the further Side of the Ordinate and along the Absciss produced, is added to the other Quantities: even so, $+s$ denotes indifferently either that the first is to be added, or the second subtracted, as it may seem convenient; or the Case require. Wherefore in the Expression $\frac{4de}{\eta f} \times \frac{v^3}{2ex} + s$, the Term $+s$ denotes either that the Area APDB is to be added to $\frac{v^3}{2ex} = TDB$; or that aGDB is to be subtracted from TDB. And this last Value is that which is exhibited in the Table by Sir Isaac: because it is the proper Value in this Case. For if you express the Area of the Curve, whose Ordinate is $\frac{d}{z}\sqrt{-e + fz^n} = dx^{\frac{1}{n}-1}\sqrt{f - ex^n}$, analytically by a converging Series, according to Prop. 5, you'll find by the Method of determining the Limits of Areas (Sect. 5.) that the Area vanishes when it is $z^n = \frac{e}{f}$, that is when $x = \sqrt{\frac{f}{e}}$: because $z^n = \frac{1}{x^n}$: but according to the Construction $\sqrt{\frac{f}{e}} = Aa$; therefore the Area of the Curve vanishes, when AB ($= x$) becomes equal to Aa: which cannot happen to the Expression $\frac{4de}{\eta f} \times \frac{v^3}{2ex} + s$, but when $+s$ signifies that aGDB is to be subtracted from $\frac{v^3}{2ex} = TDB$.

And thus I have shewn at full length, what way the Conic Sections are to be constructed, with which the Curves belonging to Species first, Order third, are geometrically compared, in all possible Cases: and what the Value of the Area in each Case will be.

How-



However, to render this Affair as plain as possible, and assist the young Geometrician how to proceed in like Cases, I shall illustrate what hath been said in these last three Articles by an Example.

Let HL be a Curve having the Absciss $AT = x$, and Ordinate $TH = y = \frac{\sqrt{1+x^2}}{x}$ Fig. 1. or $\frac{\sqrt{1-x^2}}{x}$, Fig. 2. or $\frac{\sqrt{-1+x^2}}{x}$, Fig. 3. where $e = 1 = f = d$, $\eta = 2$, $x^2 = \frac{1}{x^2} = \frac{1}{x^2}$ or $x = \frac{1}{x}$. and $v = (\sqrt{f+ex^2}) = \sqrt{1+x^2}$, Fig. 1; or $\sqrt{-1+x^2}$, Fig. 2. or $\sqrt{1-x^2}$, Fig. 3. so that the Axes and Parameters of the Conic Sections being equal they become equilateral Hyperbolas in the first two Cases, and the Circle in the other Case: in all which the Semiaxes is 1.

Wherefore, when it is $y = \frac{\sqrt{1+x^2}}{x}$, upon the Center A with the semitransverse Axis $AP = 1$, and indeterminate Axis AT (Fig. 1.) describe the equilateral Hyperbola PDd , to which draw the Tangent TD through the Point T, and through D the Ordinate DB: then from the Property of the Tangent (as in Art. 276 *.) $x = (AT =$

* Of this
Explica-
tion.

$$\frac{AB \times AP^2}{BD^2 - AP^2} = \frac{AP^2}{AB} = \frac{1}{AB}, \text{ whence } AB = x \text{ and } BD = v = \sqrt{1+x^2}$$

=

$$= \sqrt{f + ex^2}; BT = \frac{1+x^2}{x} = \frac{f+ex^2}{x} = \frac{v^2}{2x} \text{ and } \Delta TDB = \frac{v^3}{2ex}; \text{ like-}$$

wife $APDB = s$: therefore the Area of the Curve, viz. $\frac{4de}{7f} \times \frac{v^3}{2ex} = s$, is in this Case twice $\Delta TDB - APDB$. Whence (by Sect. 5.) if the Point t be such, that $\Delta tdb - APdb = 0$, the curvilinear Area is nothing, and the Ordinate tb is the initial Limit, from whence the Area must be reckoned: so that if $TDB - APDB$ be positive, the curvilinear Area $THbt$ lyes upon the same Side of the Ordinate TH with the Point A the Beginning of the Absciss: but if $TDB - APDB$ be negative, it lyes upon the opposite Side of the Ordinate. Now although this initial Limit cannot be found in this Case, without drawing td so as to make $tdb = APDB$, which is equivalent to the Quadrature of the Hyperbola, yet nevertheless the curvilinear Area $THbt$ lying betwixt any two Ordinates TH , tb , is equal to the hyperbolical Area $TDdt$. For the Area corresponding to the Ordinate TH having the greatest Absciss AT , is $TDB - APDB$; and that corresponding to the Ordinate tb , having the lesser Absciss At , is $tdb - APdb$; therefore it will appear from the Consideration of what hath been said, that, whatever way the Areas lye, the Area $THbt$ inclosed betwixt the two Ordinates TH , tb , is $TDB - APDB - tdb + APdb = TDdt$, the Difference betwixt the other two: agreeably to what was shewn Art. 185-187*.

If through the opposite Vertex of the Hyperbola PDd , viz. Q , you draw QR perpendicular to PQ ; then PQ and QR are two Asymptotes to the Curve HbL .

* Of this
Explica-
tion.

But, 2°. If it be $y = \frac{\sqrt{1-x^2}}{x}$ (see Fig. 2. p. 206,) with the Center A , and semitransverse Axis $Aa = 1$, describe the equilateral Hyperbola aD : from the Point T draw the Line TD touching the Hyperbola in D ; and from D , the Ordinate DB : then it is $AB = x$, since $AT = z$, $Aa = 1$ and AB are proportional, from the Property of the Tangent. Moreover the Curve HL passes through the Point a : for

since $TH = y = \frac{\sqrt{1-x^2}}{x}$, put $\frac{\sqrt{1-x^2}}{x} = 0$, and thence you have $x = 1 = * Aa$.

Now it may be easily shewn here, as in the preceding Case, that $\Delta TDB = \frac{v^3}{2x} = \frac{v^3}{2ex}$; and $aDB = s$: therefore since

* Art. 171.
of this Ex-
plication.

the curvilinear Area is $\frac{4de}{7f} \times s = \frac{v^3}{2ex}$, as was shewn above, it becomes in

this Case $2 \times aDB - TDB = -2aDT$ negative: wherefore, since

aDT

$aDT = 0$ when $AT = Aa$, the curvilinear Area commences at a , being aTH , which lyes on that Side of the Ordinate, which is opposite to A . The indeterminate Axis of the Hyperbola, *viz.* AL is an Assymptote to the Curve AHL : and if AT be taken $= 0,816$, &c. of Aa , the Ordinate TH will pass through a Point of contrary Flexure.

Again, 3°. If it be $y = \frac{\sqrt{-1+x^2}}{x}$ (see Fig. 3.) with the Radius $Aa = 1$, describe the Semicircle $ADPK$; through T draw the Tangent TD , and through D the Ordinate DB : then since $AT = x$, $Aa = 1$ and AB are proportional, you have $AB = x$: and as formerly $\Delta TDB = \frac{v^3}{2ex}$; and now it is $APDB = s$ and $aDB = -s$: wherefore the curvilinear Area, as was shewn above, must either be $2 \times TDB + APDB$ or $2 \times TDB - ADB = 2aDT$: the last of which is equal to the curvilinear Area aTH : for since the Curve HL passes through the Point a , the Areas aTH and aDT both commence at the Point a : and the Fluxion of ATH is double the Fluxion of aDT , since $HT = (y = \frac{\sqrt{-1+x^2}}{x} = \sqrt{1-x^2} = v =)$ DB , and the Fluxion of AHT is to the

* Of this
Explica-
tion.

* Of this
Explica-
tion.

Fluxion of aDT , as $HT \times$ Fluxion of aT to $\frac{1}{2}DB \times$ Fluxion of aT (by Art. 48 *.) that is as 2 to 1: therefore (by Art. 40 *.) $aTH = 2aDT$. Draw AP perpendicular to Aa ; and the Tangent PL is an Assymptote to the Curve HL .

280. But by supposing a different Relation of the Abscisses x and x , *viz.* that $\frac{1}{x^n} = x$, as in the Table, you may obtain the Area of the Curve after a different Manner. For upon that Supposition, you'll find the Curve proposed to pass into another equal Curve having $-\frac{d}{n}x^{-n}\sqrt{fx+ex^2}$ it's Ordinate: so that, by Prop. 7, you may find the Relation of the Curve proposed, to the Curve whose Ordinate is $\sqrt{fx+ex^2}$: which you do by turning the Ordinate $-\frac{d}{n}x^{-n}\sqrt{fx+ex^2}$ into $-\frac{d}{n}x^{-\frac{n}{2}}\sqrt{f+ex}$, then increasing the Exponent of x , *viz.* $-\frac{n}{2}$ by 2: and so you may construct the Conic Section, with which the Curve may be compared. The other Species's of Curves of this Form are deduced from the first by Prop. 7.

Form 5th.

281. I shall next shew the Demonstration of the Quadrature of the Curve, constituting Species second of Form fifth, which requires two Conic Sections.

The

The Curve is expressed thus $\frac{dx^{2\eta-1}}{e + fz^\eta + gz^{2\eta}} = y$: and the Relation of the Absciss z to the Abscisses x and ξ , which are these of the two Conic Sections, is expressed thus $\sqrt{\frac{d}{e + fz^\eta + gz^{2\eta}}} = x$, and $fz^\eta + gz^{2\eta} = \xi$. Wherefore, I first seek for the Ordinate of a Curve, which having it's Absciss x , shall be equal to the Curve proposed, viz. thus. The Ordinate sought (by Prop. 9.) is equal to $\frac{yz}{x}$, that is, by suppo-

sing $z = 1$, $\frac{y}{x}$: or by putting in the Value of y , $\frac{dx^{2\eta-1}}{x \times e + fz^\eta + gz^{2\eta}}$:

wherefore from the Equation $\sqrt{\frac{d}{e + fz^\eta + gz^{2\eta}}} = x$, find the Relation

of the Fluxions, which is $\dot{x} = \frac{\eta fz^{\eta-1}x + 2\eta gz^{2\eta-1}x}{2e + 2fz^\eta + 2gz^{2\eta}}$: insert this in

place of $\dot{x}$, and the Ordinate sought will be $\frac{-2dx^\eta}{\eta fx - 2\eta gz^\eta}$: where, if you

put for z^η it's Value, $\sqrt{\frac{ff-4eg}{4gg}} + \frac{d}{gx^2} = \frac{f}{2g}$, from the preceding

Equation, and then reduce, you'll obtain $\frac{df}{2\eta g^2 \sqrt{\frac{d}{g} + \frac{ff-4eg}{4gg}x^2}} - \frac{d}{\eta gx^3}$

for the Value of the Ordinate of a Curve, which has x for it's Absciss, and is equal to the Curve proposed at first: which Ordinate is made up of two distinct Parts. The Area belonging to the first Part of the Ordinate, may be compared with the Area of the Conic Section whose

Ordinate is $v = \sqrt{\frac{d}{g} + \frac{ff-4eg}{4gg}x^2}$, by help of the Theorem in Art.

211 *, viz. $B = \frac{\theta + \lambda \eta \times A - z^0 R^\lambda}{\lambda \eta e}$: by applying of which, you'll obtain

$\frac{2fs - f \cdot \text{area}}{2\eta g}$ for the Area belonging to the Ordinate $\frac{df}{2\eta g^2 \sqrt{\frac{d}{g} + \frac{ff-4eg}{4gg}x^2}}$:

* Of this Explication.

which might also be deduced from the Quadrature of the Curve of the first Species of this Order.

The other Part of the Ordinate is $-\frac{d}{\eta gx}$: and in order to find the

Area belonging to this Part of the Ordinate, take $x = \sqrt{\frac{d}{e+k}}$ for the Relation of the Abscisses of two Conic Sections, as it is in the Table,

E e

Table,

Table, whose Areas are equal; the Ordinate of the first being $-\frac{d}{ngx}$: from whence find the Ordinate of the other. It will be, by Prop. 9,

$\frac{x \times -\frac{d}{ngx}}{\xi}$, or $-\frac{d}{ngx\xi}$, by putting $x = 1$: instead of x insert it's Value $\sqrt{\frac{d}{e+\xi}}$; and instead of ξ it's Value $\frac{2e+2\xi}{-\sqrt{\frac{d}{e+\xi}}}$, deduced from the

Equation $x = \sqrt{\frac{d}{e+\xi}}$, and so you'll obtain $\frac{d}{2ng} \times \frac{1}{e+\xi}$ for the Ordinate of a Curve, which having ξ for it's Abscifs, is equal to the Curve which has x for it's Abscifs and $-\frac{d}{ngx}$ for it's Ordinate.

* See Form 1st.

Now, according to the Table, $\gamma = \frac{1}{e+\xi}$: which denotes the Ordinate to the Affymptote in a rectangular Hyperbola *, and to which the correspondent Area is σ : that is the Area belonging to the Ordinate $\frac{d}{2ng} \times \frac{1}{e+\xi}$ is $\frac{d\sigma}{2ng}$: which therefore is the Area belonging to the Ordinate

* Of this Explication.

$-\frac{d}{ngx}$. See Art. 190—192 *.

Whence by putting the two Areas together, you'll have $\frac{d\sigma + 2fs - fxv}{2ng} = t$ the Area of the Curve proposed, expreffible by the Description of two Conic Sections.

282. The other Species's of Curves of this Order *in inf.* may be found by Division, the Areas of the two first being once found. Thus

for the third Species, where it is $\frac{dx^{3n-1}}{e+fx^n+gx^{2n}} = y$, divide dx^{3n-1} by

$gx^{2n} + fx^n + e$, and the Quotient is $\frac{dx^{n-1}}{e} - \frac{\frac{df}{e}x^{2n-1}}{e+fx^n+gx^{2n}} - \frac{\frac{dx^{n-1}}{e}}{e+fx^n+gx^{2n}}$,

which is the Value of the Ordinate of the Curve constituting the third Species of this Form: the first Part of which is the Ordinate of a Curve quadrable by Theorem first, Table first; the other two belong to the first and second Species of this Form fifth: whence the Area may be found.

Forms 7th and 11th.

283. I have said as much already upon Example 2^d. Case 2^d, Prop. 10, as may serve to illustrate the Quadratures of the Curves belonging to Form 7. Therefore I shall only take one Instance more: which shall be the Curve constituting the first Species of the last Form, whose

Property

Property is thus expressed $dz^{-1} \sqrt{\frac{e+fx^n}{g+bz^n}} = y$: and two Conic Sections are required to it's Quadrature. In order to which, I find the Ordinate of a Curve, which having x for it's Absciss, shall be equal to the Curve proposed, supposing $x = \sqrt{g+bz^n}$: the Ordinate sought is $\frac{y}{x}$ or $\frac{y}{\sqrt{g+bz^n}}$, putting $z=1$. Wherefore, if you put in the Value of y , viz. $dz^{-1} \sqrt{\frac{e+fx^n}{g+bz^n}}$, and the Value of x , viz. $\frac{bz^{n-1}}{2x}$, deduced from

the Equation $x = \sqrt{g+bz^n}$, the Ordinate sought will be $\frac{2dx \sqrt{e+fx^n}}{bz^n \sqrt{g+bz^n}}$:

which, by inserting $\frac{x^2-g}{b}$ for z^n and x for $\sqrt{g+bz^n}$, and bringing to Order, gives $\frac{2d}{n} \sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2 \times -g+x^2}^{-1}$ for the Ordinate of a Curve equal to the Curve at first proposed.

I next find the Ordinate of a Curve, whose Absciss, as before, being x , shall have it's Area equal to σ , which is the Area of the second Conic Section, whose Absciss being ξ , has it's Ordinate $r =$

$\sqrt{\frac{fg-eb}{g} + \frac{e}{g}\xi^2}$, the Relation of the Abscisses being $\frac{x^2-g}{b} = \frac{g}{\xi^2-b}$: which last is easily deduced from the Values of x and ξ in the Tab. And by proceeding after the same manner, as in the preceding Part

of this Art. you'll find that Ordinate to be $-gb \sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2 \times -g+x^2}^{-2}$. So that now you have three Ordinates, viz.

1°. $\sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2 \times -g+x^2}^0$. 2°. $\sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2 \times -g+x^2}^{-1}$.

3°. $\sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2 \times -g+x^2}^{-2}$, to which the three corresponding

Areas are s , $\frac{xt}{2d}$ and $-\frac{\sigma}{gb}$: so that from any two of them given, the third may be found by Prop. 8: by applying of which to the present Case,

you'll obtain $t = 2fs - bx \times -g+x^2}^{-1} \times \frac{eb-fg}{b} + \frac{f}{b}x^2}^{\frac{3}{2}} + 2e\sigma \times$

$\frac{2d}{neb-nfg}$; or by a due Substitution of xz^{-n} for $bx \times -g+x^2}^{-1}$,

and v^3 for $\frac{eb-fg}{b} + \frac{f}{b}x^2}^{\frac{3}{2}}$, $t = \frac{2dxv^3z^{-n} - 4dfz - 4de\sigma}{nfg-neb}$, the Area required.

284. And thus I have shewn in a Variety of Cases the Demonstration of the Quadratures of Curves belonging to several different

Forms in this Table: by observing of which carefully, all the others may be demonstrated in like manner: wherefore, I now proceed to make some general Observations upon this Affair, taken mostly from our Author.

285. 1°. The Series of Curves in each Form may be continued *inf.* towards both Hands, by the Help of Proposition third, fourth, seventh and eighth: that is, the Progression of Curves may be carried backwards by diminishing the Index of z continually by 1; as well as forwards according to the Order in the Table, by increasing the Index of z continually by 1. For which purpose you may consult the Theorems contained in Art. 202, 203, 206, 207, 208, 209*: the Application of which Theorems has been exemplified in several Instances, in the preceding Work. And the Series of Curves belonging to the first, second, fifth, sixth, ninth and tenth Forms may be carried forward by Division only, after the Manner shewn above in Order fifth*. But whereas these Theorems will not serve for finding the Curves belonging to the backward Progression in some Forms. See Art. 209*, therefore we shall speak of every Form particularly afterwards*.

* Of this
Explica-
tion.

* Art. 282.
of this Ex-
plication.

* Of this
Explica-
tion.

* Art. 318.
of this Ex-
plication.

286. 2°. Although the Signs of the given Quantities d, e, f, g, b and i are all affirmative in the Equations to the Curves, contained in this Table: yet one or more of these may be negative: and when that happens, such Quantity or Quantities must be made negative in the Values of the Absciss and Ordinate of the Conic Section or Sections, with which any such Curve is compared; and likewise in the Expression of the Area. This I illustrated in the Curves belonging to Species first Form third*. Where it appears, which I would have to be observed, that the changing the Signs of these Quantities, alters the Construction of the Conic Section or Sections with which the Curve is compared. Thus I shew'd that when the Equation belonging to the Curve, is $\frac{d}{z}\sqrt{e + fz^n} = y$; and consequently that to the Conic Section, $\sqrt{f + ex^2} = v$, such Conic Section, with which the given Curve is compared, is at Fig. 2. p. 25: if it be $\frac{d}{z}\sqrt{e - fz^n} = y$, and consequently $\sqrt{-f + ex^2} = v$; the Conic Section is at Fig. 3. p. 25: but if it be $\frac{d}{z}\sqrt{-e + fz^n} = y$, and $\sqrt{f - ex^2} = v$, in that Case, the Area of the Curve depends upon the Construction of the Ellipse at Fig. 4. p. 25. And the like happens in other Curves, upon like Suppositions.

* Art. 276.
-279. of
this Ex-
plication.

287. 3°. These Quantities $d, e, f, g, &c.$ may be any Quantities whatsoever, *i. e.* bear any possible Relations to one another; and be affected with any Signs; excepting only these Cases. 1°. That they be not such as to make the Sum of the Quantities under any of the radical Signs, negative: for that would make an impossible or imaginary Expression. And therefore also, 2°. That in the Curves belonging to Form sixth, when the Quantities e and g are affected with the same Sign, $4eg$ do not exceed ff : for then $p = \sqrt{ff - 4eg}$, would be impossible, and so render the Ordinate of the Conic Section, and consequently the Expression of the Area in the Table impossible.

288. 4°. In order to construct the Conic Sections, which belong to the several Orders, and Species's of these Orders; and every particular Case of any Curve, coming under any Species, according to any Supposition, either with respect to the Signs, or Proportions of the given Quantities $d, e, f, g, &c.$ You must consider the four Figures annexed to the Table, p. 25; and investigate the Equations expressing the Relations of the Abscisses and Ordinates in each Figure, from the primary Properties of these Conic Sections, and that, according as the Absciss x is either AB , aB , or αB , the Ordinate v being always BD : and then compare these Equations with the Equations to the Conic Sections contained in the Table. For which purpose I shall run through the several Figures.

For Fig. I.

289. Let it be $\alpha B = x$, $BD = v$, $A\alpha = a$ a given Line; $\alpha G = b$ a given Line likewise. Then, when the Point B is further distant from the Center than α is, we have from the Property of the rectangular Hyperbola $AB \times BD = A\alpha \times \alpha G$, that is in Symbols $a + x \times v = ab$, whence - - - - - 1°. $v = \frac{ab}{a+x}$.

290. But if B fall betwixt A and α , in which Case $\alpha B = x$ is negative, the Equation puts on this Form - - - 2°. $v = \frac{ab}{a-x}$.

291. And these two Forms, or Values of the Ordinate, will serve for all the Varieties of Curves belonging to Order first. The first Form, *viz.* $\frac{ab}{a+x} = v$, answers exactly to that in the Table $\frac{d}{e+fx} = v$: which is the proper Form of the Ordinate, when it is $\frac{d}{e+fx^n} = y$. If it be $\frac{d}{e-fx^n} = y$, and consequently $\frac{d}{e-fx} = v$; the second Form, *viz.* $\frac{ab}{a-x} = v$, answers to that Case. If it be $\frac{-d}{e+fx^n} = y$ that only denotes the Area

Area belonging to the first Form taken negatively: and so it will be evident that all the Varieties that can happen, by varying the Signs of the Quantities d, e, f , after any manner whatsoever, may be reduced to these two Forms expressed above.

292. As to the Comparison of the corresponding Parts of the two different Expressions of the hyperbolical Ordinate, it is made thus: since $\frac{d}{e+fx} = v = \frac{ab}{a+x}$ you'll have $a = e$, $1 = f$ and $ab = d$ or $b = \frac{d}{e}$. Therefore when the Conic Section is to be constructed, you use f for linear Unity, take $Aa = e$, $aG = \frac{d}{e}$ or $\frac{df}{e}$ that is a fourth Proportional to the three Lines e, d and f ; or Aa, d , and Unity: and therefore, if through the Point G you describe a rectangular Hyperbola, with the Center A and Affymptote ABN , that shall be it you want: so that taking $aB = x$, and drawing the Ordinate BD , you'll have

*Art. 190.
of this Ex-
plication.

$$BD = v = \frac{d}{e+fx}; \text{ and } aGDB = s^*.$$

And after the like manner, you make the Construction upon other Suppositions.

For Fig. 2.

293. Let it be $aB = x$, $BD = v$, $Aa = a$ a given Line; the transverse Axe t ; it's Parameter p . Then from the Property of the Figure $BD + AP \times BD - AP : AB^2 :: t : p$, that is in Symbols, $v^2 - \frac{1}{4}tt : a^2 + 2ax + x^2 :: t : p$. Whence

$$1^{\circ}. v = \sqrt{\frac{a^2t + \frac{1}{4}ptt}{p} + \frac{2at}{p}x + \frac{t}{p}x^2}.$$

294. Which Form of the Ordinate, when B falls betwixt A and a , i. e. when x becomes negative, falls into this other.

$$2^{\circ}. v = \sqrt{\frac{a^2t + \frac{1}{4}ptt}{p} - \frac{2at}{p}x + \frac{t}{p}x^2}.$$

295. Again, the other Things remaining the same as in the first Supposition: only now let $Aa = a$ vanish, or the Points A and a coincide, by which it becomes $AB = x$: in that Case, the Equation will appear in this Form.

$$3^{\circ}. v = \sqrt{\frac{1}{4}tt + \frac{t}{p}x^2}.$$

For Fig. 3.

296. Let it be $aB = x$, $BD = v$, $aa = a$, t and p as formerly: and let a fall betwixt a and B . Then from the Property of the Hyperbola,

perbola, $KB \times aB : BD^2 :: t : p$, that is $t + a + x \times a + x : v^2 :: t : p$.
Whence we deduce,

$$1^{\circ}. v = \sqrt{\frac{apt + a^2p}{t} + \frac{pt + 2ap}{t}x + \frac{p}{t}x^2}.$$

297. Which by supposing B to fall betwixt a and a , and by that means x to become negative, falls into this other.

$$2^{\circ}. v = \sqrt{\frac{apt + aap}{t} - \frac{pt + 2ap}{t}x + \frac{p}{t}x^2}.$$

298. If in the first Form, you make $aa = a$ to vanish, so that it be $aB = x$, then you'll have

$$3^{\circ}. v = \sqrt{px + \frac{p}{t}x^2}.$$

299. But further, if you suppose that the Point a falls beyond the Vertex a without the Hyperbola, aB being equal to x , as in the first Form: then it will turn into this

$$4^{\circ}. v = \sqrt{\frac{-apt + a^2p}{t} + \frac{pt - 2ap}{t}x + \frac{p}{t}x^2}.$$

Which includes under it three different Forms, according to the various Positions of a .

For, 1° . If a fall betwixt a and A , so that t be greater than $2a$, the Order of the Signs will be - - - - - + +.

But if a fall betwixt A and K , so that t be greater than a ; but less than $2a$, you'll have this Order of Signs - - - - - +.

Again, if a fall beyond K , so that a be greater than t , this Order of the Signs will arise, viz. - - - - - + - +.

In the other two Positions of a , viz. when it falls in either with A or K , the Radical is reduced to two Terms.

300. Thus if a fall in with the Center A , so that it be $2a = t$ and $AB = x$, then it is

$$5^{\circ}. v = \sqrt{-\frac{1}{4}pt + \frac{p}{t}x^2}.$$

301. Lastly, if a fall in with K , so that K be the Beginning of the Absciss; and $t = a$, then it is

$$6^{\circ}. v = \sqrt{-px + \frac{p}{t}x^2}.$$

So that in all, there are seven different Forms under which the Value of the Ordinate $BD = v$ appears: and one of these admits of a two-fold Construction, viz. Form 2^d and Case 3^d of Form 4^{th} .

For Fig. 4.

302. Put $aB = x$. $BD = v$. $aa = a$. t the transverse Axe and p it's Parameter: and let a fall betwixt a and B . Then from the Property

perty of the Ellipse $KB \times Ba : BD^2 :: t : p$, that is in Symbols $t - a - x$
 $\times a + x : v^2 :: t : p$, or $at - aa + t - 2ax - x^2 : v^2 :: t : p$.
 Whence we have

$$1^{\circ}. v = \sqrt{\frac{apt - aap}{t} + \frac{t - 2ap}{t}x - \frac{p}{t}x^2}$$

Which according to the different Values of $aa = a$, will appear differently: for if a be less than $\frac{1}{2}t$, the Order of the Signs will be

If a be greater than $\frac{1}{2}t$, the Order is - - + + - -.

303. But if a fall in with the Center so that $aB = x = AB$, and $a = \frac{1}{2}t$, then the Ordinate appears in this Form

$$2^{\circ}. v = \sqrt{\frac{1}{4}t - \frac{p}{t}x^2}$$

304. Again, if you suppose $a = aa$ to vanish, by which means $aB = x = aB$, then the Ordinate appears thus

$$3^{\circ}. v = \sqrt{px - \frac{p}{t}x^2}$$

305. Now, if instead of supposing the Point a the Beginning of the Absciss, to fall betwixt a and B , you should suppose B to fall in betwixt the Points a and a , so that $x = aB$ become negative, then the first Form of the Ordinate will turn into this other

$$4^{\circ}. v = \sqrt{\frac{apt - aap}{t} - \frac{t - 2ap}{t}x - \frac{p}{t}x^2}$$

Where you'll have different Orders of the Signs according as a is less or greater than $\frac{1}{2}t$. For if it be $a = aa$ less than $\frac{1}{2}t$, the Order of the Signs will be this

+ - - -

But if a be greater than $\frac{1}{2}t$, but so as that the Point a still fall within the Ellipse, and therefore that a be less than t , then the Order of the Signs will be

+ + - -

306. Again, supposing the same things: only that now the Point a fall in with the Center so that $a = (aa =) \frac{1}{2}t$, then the 4th Form appears thus

$$5^{\circ}. v = \sqrt{\frac{1}{4}pt - \frac{p}{t}x^2}$$

The very same with the second.

307. And by supposing the Point a to fall in with the Point K , so that $a = (aa =) t$ you'll obtain

$$6^{\circ}. v = \sqrt{px - \frac{p}{t}x^2}$$

The very same with the third.

308. But now the same things remaining as in Form fourth, only suppose $a = aa$ to be greater than t , so that the Beginning of the Absciss lye beyond K, then you'll have that Form always putting on this Order of Signs

— + —.

309. Finally, the same things remaining as mentioned in Form first: only now suppose the Point a to fall without the Ellipse, beyond the Vertex a , then the first Form will appear thus

$$7^{\circ}. v = \sqrt{-\frac{apt + a^2p}{t} + \frac{pt + 2ap}{t}x - \frac{p}{t}x^2}.$$

Which always gives this Order of Signs, viz. — + —.

310. By considering the several Forms under which the Ordinate of the Conic Section appears in the foregoing Articles, it will be found, that, whatever be the Signs, or Proportions of the given Quantities d, e, f, g , &c. in the Equations expressing the Nature of the Curves, contained in the several Orders succeeding the first (of which we spoke already: *) and whatever be the Value of v and r in the Table, which denote the Ordinates of the Conic Sections, by the Construction of which, these Curves are determined, you'll always find some of these Forms, that may be compared with the Value of v and r in the Table. For there are three general Forms, under which the Value of v appears in the Orders succeeding the first in the Table: which may be represented without the Signs thus. 1^o. $v = \sqrt{\alpha \cdot \beta x^2}$. 2^o. $v = \sqrt{\alpha x \cdot \beta x^2}$.

* Art. 289
— 292.
of this Ex-
plication.

3^o. $v = \sqrt{\alpha \cdot \beta x \cdot \gamma x^2}$. The first two admit of three Varieties of the Signs, each: the last admits of seven Varieties, all possible. If it be $v =$

$\sqrt{\alpha + \beta x^2}$, you have it Art. 295 *: if it be $v = \sqrt{\alpha - \beta x^2}$, you have

it Art. 303 and 306 †: if $v = \sqrt{-\alpha + \beta x^2}$, you see it in Art. 300 ‡.

The Varieties of the second are contained in Art. 298, 301, 304, 307 ||.

And the seven possible Cases of the third are contained in Art. 293, 294,

296, 297, 299, 302, 305, 308, 309 *. By which it appears, that there

is no Case can happen, with respect to the various Dispositions of the

Signs, in any possible Value of the Ordinate of the Conic Sections in

the Table, but you'll find a corresponding Form of the Ordinate in these

Articles; and in several Cases, different corresponding Forms; so that

in all possible Cases, you may make Constructions of Conic Sections;

with which, the several Species's of Curves in all the different Orders,

according to all possible Varieties in the Signs, and Proportions of the

Quantities, d, e, f , &c. may be geometrically compared. In order to

which Construction, you need only to compare the homologous or

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tion.

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correspondent Terms: in doing of which if any impossible Root or Value arise, by comparing with one Form, you must make the Comparison with another.

311. Thus if it were required to construct the Conic Section necessary to the Quadrature of the Curve belonging to Species first Order fifth expressed by this Equation $\frac{dx^{n-1}}{-e + fx^n + gx^{2n}} = y$: I find the Ordinate of the Conic Section in the Table, agreeing to this to be

* Art. 286.
of this Ex-
plication.
* Of this
Explica-
tion.

$\sqrt{\frac{d}{g} + \frac{f^2 + 4eg}{4g^2}x^2} = v$: therefore I compare this with the Form of the Ordinate contained in Art. 295 *: which stands thus $v = \sqrt{\frac{1}{4}tt + \frac{t}{p}x^2}$ and belongs to Fig. 2. p. 25: wherefore I compare the corresponding Terms or Parts of these two Expressions of the Value of v : which gives these two Equations.

$$1^\circ. \frac{1}{4}tt = \frac{d}{g}. \quad 2^\circ. \frac{t}{p} = \frac{f^2 + 4eg}{4g^2}.$$

By the first you find $t = 2\sqrt{\frac{d}{g}}$: insert this Value of t in the second, and it becomes $\frac{2}{p}\sqrt{\frac{d}{g}} = \frac{ff + 4eg}{4gg}$, whence you obtain $p = \frac{8g^2}{ff + 4eg}\sqrt{\frac{d}{g}}$: so that the transverse Axis and Parameter of the Hyperbola being known, the Hyperbola may be constructed: with which the proposed Curve may be geometrically compared.

312. Again, suppose the Curve was proposed, whose Property is thus expressed $dx^{n-1}\sqrt{e + fx^n + gx^{2n}} = y$: which belongs to Species second Order seventh, you find the Ordinate of the Conic Section, with which it is compared, in the Table to be $v = \sqrt{e + fx + gx^2}$.

Where, since all the Quantities e, f and g are positive, it may be compared with the Form contained in Art. 293 *: which stands thus

* Of this
Explica-
tion.

$v = \sqrt{\frac{a^2t + \frac{1}{4}ptt}{p} + \frac{2at}{p}x + \frac{t}{p}x^2}$, belonging to Fig. 2. p. 25: or, it

* Of this
Explica-
tion.

may be compared with the Form contained in Art. 296 *, viz. $v = \sqrt{\frac{apt + aap}{t} + \frac{pt + 2ap}{t}x + \frac{p}{t}x^2}$, which belongs to Fig. 3. p. 25. By the first you have these three Equations.

$$1^\circ. \frac{a^2t + \frac{1}{4}ptt}{p} = e. \quad 2^\circ. \frac{2at}{p} = f. \quad 3^\circ. \frac{t}{p} = g.$$

From which, supposing e, f, g given, you'll obtain the Values of t, p and a , viz.

$t =$

$$t = \sqrt{\frac{4eg - ff}{g}}. p = \frac{1}{g} \sqrt{\frac{4eg - ff}{g}}. a = \frac{f}{2g}.$$

By the second Comparison you have these three Equations, viz.

$$1^{\circ}. \frac{apt + a^2p}{t} = e. \quad 2^{\circ}. \frac{pt + 2ap}{t} = f. \quad 3^{\circ}. \frac{p}{t} = g.$$

From whence you obtain the Values of t , p and a , viz.

$$t = \sqrt{\frac{ff - 4eg}{g}}. p = \sqrt{ff - 4eg}. a = \frac{f - \sqrt{ff - 4eg}}{2g}.$$

So that the Constructions of the Hyperbolas are manifest: only it must be observed that if $4eg$ exceed ff , you must use the Construction, arising from the first Comparison, represented at Fig. 2. p. 25: but if f^2 exceed $4eg$, you must make the Construction that arises from the second Comparison, and is represented at Fig. 3. p. 25: the Reason of which is manifest, from the Consideration of the radical Expressions.

313. And if the Signs of one or two of the Quantities e, f, g be negative, you'll always find one or more Forms of Ordinates in the preceding Articles, with which the Ordinate of the Conic Section in the Table may be compared; and one or more Constructions made accordingly: by means of which the Area of any such Curve may be found. And so in other Cases.

314. 5°. It must likewise be observed that when the Signs of the numeral Indexes η and θ are negative in the Equation expressing the Nature of the Curve, they must be changed in the Values of the Areas. Thus in the Curves belonging to Species third Order fourth: which are thus expressed $\frac{d}{z^{2\eta+1}\sqrt{e+fz^\eta}} = y$; if η be made negative,

so that it stands thus $\frac{d}{z^{-2\eta+1}\sqrt{e+fz^{-\eta}}} = y$, by which both the Sign of η and $\theta = 2\eta$ are changed, you must change the Sign of η , both in the Equation expressing the Relation of the Abscisses, and in the Value of the Area: by which means you'll have $z^\eta = x$, and $t = -\frac{d}{\eta e} \times \frac{3s-2xv}{3s-2xv}$: that is, if the Value of the Area was formerly $\frac{d}{\eta e} \times \frac{3s-2xv}{3s-2xv}$,

it will now, by the Change of the Sign of η , become $\frac{d}{\eta e} \times \frac{2xv-3s}{2xv-3s}$: and so of others. You may likewise observe that, when this Change is made upon the Sign of η , the Species of the Curve may belong to a new Order. Thus in the present Case, when you have

$$\frac{d}{z^{-2\eta+1}\sqrt{e+fz^{-\eta}}} = y, \text{ or } dx^{2\eta-1} \times e + fz^{-\eta} - \frac{1}{2} = y, \text{ it is the same}$$

thing with $dx^{3n-1} \times \sqrt{fx^n + ex^{2n}} - \frac{1}{2} = y$, which falls in with Species third of Order eight, by supposing the Term e in the Table to be nothing, and substituting e here instead of g there. By observing which, you'll have $z^3 = x : \sqrt{fx + ex^2} = v$, and $t = \left(\frac{3dfx - 2dfxv}{-v\sqrt{f}} \right) \frac{d}{dx} \times 2xv - 3s$ as formerly. Thus likewise the Curve defin'd by the Equation $\frac{d}{dx} \sqrt{e + fx^n} = y$, which belongs to Species first Order third, by changing the Sign of n , may be turned into this Shape $\frac{d}{dx} \sqrt{f + ex^n} = y$: which belongs to a new Order of Curves, as our Author observes at Art. 64 *.

* Of the
Quadrature of
Curves.

315. 6°. When the defining Equation of any Curve is compounded of several Equations of different Orders; or of different Species's of the same Order, it's Area must be compounded of the corresponding Areas: taking care however that they may be rightly connected with their proper Signs: for we must not always add or subtract Ordinates to or from Ordinates; and at the same time add or subtract corresponding Areas to or from corresponding Areas: but sometimes the Sum of the Ordinates, and the Difference of the Areas are to be taken to constitute a new Ordinate and corresponding Area. Which must be done, when the constituent Areas are posited on the opposite Sides of the Ordinate. For understanding of which, see Sect. 5. on the Position and Limits of Areas: which Matter must be determined from the Nature and Circumstances of the Curve. And that the cautious Geometrician may not fall into any Inconveniency upon this Account, our Author tells us, he has prefixt their proper Signs to the Values of the Areas, though sometimes negative, as in the fifth and seventh Orders. And as to the Value of t , if it comes out affirmative, it denotes the Area of the Curve proposed, adjoining to its Absciss: and contrarywise, if it be negative, it represents the Area on the other Side of the Ordinate. And as to the Meaning of $+$ s and $-$ s , it has been explain'd formerly *: and will further appear by the subsequent Examples.

* Art 279.
of this Ex-
plication.

316. 7°. As to the initial Limits of the Areas, in every Case; they are to be determined, as directed in Sect. 5. of the Position and Limits of Areas. From whence also you gather that if any Part of the Area is posited below the Absciss, t will denote the Difference betwixt that and the Part above the Absciss.

317. 8°. When any Curve is proposed to find it's Area, you must endeavour to bring the Value of it's Ordinate into such a Form, as that you may be able to compare it with some Species of Curves belonging

belonging to some of the Orders or Forms contained in the Tables. If this cannot be done, you must transform the Equation, defining the Nature of the Curve, into other Equations of Curves, bearing any assumed Relations to it, after the Manner shewn in Art. 225, 226 *, till you find a Curve, whose Area may be known by the Tables. And when all Endeavours are used, and yet no such Curve can be found: our Author tells us, it may be certainly concluded, that the Curve proposed cannot be compared either with rectilinear Figures; or with the Conic Sections. It may be observed likewise, that, when the Dimensions of the Terms, in the Values of x , y and t shall ascend too high or descend too low, in comparing any proposed Curve with the Forms of Curves in the Table, you may reduce them to a just Degree, by dividing or multiplying so often by any given Quantity, which may be supposed to perform the Office of Unity, as shall be necessary to bring these Terms to the just Number of Dimensions: as will appear in the subsequent Examples.

* Of this
Explica-
tion.

318. 9°. With respect to the Continuation of the several Forms, both forwards and backwards, let the following things be observed. 1°. The Series's belonging to Forms first, second, fifth, sixth, ninth and tenth may be carried forwards according to the Order of Progression in the Table by Division; as was said in the first general Remark *. 2°. Those belonging to Forms third, fourth, seventh and eighth are continued forwards by the two Theorems in Art. 203 and 206 *: and the Series's of Form eleventh by the Theorem Step fifth, Art. 219 *. But with respect to the Continuation of the Series's in the several Forms, the contrary way, by going backwards from the first Species in each Order, this cannot be done in every one of them, by Propositions seventh and eighth. Wherefore, 3°. It may be done in Forms second, sixth, seventh and eleventh, viz. in Form second, by Art. 203 *: in the sixth and seventh, by Art. 208 †: and in Form eleventh, by Art. 219 ‡. In the other Forms we cannot pass from the first Species's in the Table to the Species's immediately before them, by going backwards, by means of Propositions seventh and eighth, as will appear by consulting what we have said in Art. 209 *. Notwithstanding, these other Forms may be continued backwards, in the following Manner. 4°. The Series's in Form first, fifth and eighth

* Art. 285.
of this Ex-
plication.

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tion.

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thus, let p denote any positive Integer, so that $\frac{dx^{-1}y^{n-1}}{e+fx^n} = y$ may denote any Species in the backward Progression of Form first: divide the Numerator and Denominator by x^n , and it stands thus $\frac{dx^{-p}y^{n-1}}{f+ex^{-n}} = y$, which

which belongs to some Species in the other Progression, by only changing η into $-\eta$. See Remark fifth *. The same way you proceed in Form fifth and eighth : by converting $\frac{dx^{-p\eta-1}}{e+fx^\eta+gz^{2\eta}}$, into $\frac{dx^{-p\eta-2\eta-1}}{g+fx^{-\eta}+ex^{-2\eta}}$

in Form fifth; and $\frac{dx^{-p\eta-1}}{\sqrt{e+fx^\eta+gz^{2\eta}}}$, into $\frac{dx^{-p\eta-\eta-1}}{\sqrt{g+fx^{-\eta}+ex^{-2\eta}}}$, in Form 8th.

The same might be done likewise in Forms second and sixth. 5°. As to Forms third and fourth, the Curves belonging to the backward Progression are quadrable, and constitute the Curves of Forms third and fourth in Table 1. See Schol. 2. Art. 209 *. Wherefore, all that remains is to shew after what manner the Series of Forms ninth and tenth are to be carried backwards. 6°. Now in order to find the

Area of the Curve whose Ordinate is $\frac{dx^{-1}\sqrt{e+fx^\eta}}{g+bz^\eta}$, let A and B de-

note the Areas of two Curves whose Ordinates are $\frac{z^{\eta-1}\sqrt{e+fx^\eta}}{g+bz^\eta}$, and

$\frac{z^{-1}\sqrt{e+fx^\eta}}{g+bz^\eta} = z^{-1}\sqrt{e+fx^\eta}$: the first of which belongs to Species

first of this ninth Form; and the other to Species first Form third, therefore A and B are given. Now if you take the indefinite Quantities p and q , and multiply these Ordinates and Areas by them, and proceed as in Art. 219 *, you'll have, by putting $R = e + fx^\eta$ and $S = g + bz^\eta$, $pz^\eta \times z^{-1}R^{\frac{1}{2}}S^{-1}$, and $q \times g + bz^\eta \times z^{-1}R^{\frac{1}{2}}S^{-1}$, Ordinates to the Curves whose Areas are pA , and qB ; and by taking the

Sum of the Ordinates and Areas, it is $qg + qb + p \times z^\eta \times z^{-1}R^{\frac{1}{2}}S^{-1}$, and $pA + qB$, for corresponding Ordinate and Area: whence, by putting $qb + p = 0$ or $p = -qb$; and reducing, you'll obtain

$\frac{B-bA}{g}$ the Area belonging to the Ordinate $\frac{z^{-1}\sqrt{e+fx^\eta}}{g+bz^\eta}$, or $\frac{dB - dA}{g}$

belonging to the Ordinate $\frac{dx^{-1}\sqrt{e+fx^\eta}}{g+bz^\eta}$: which was that sought.

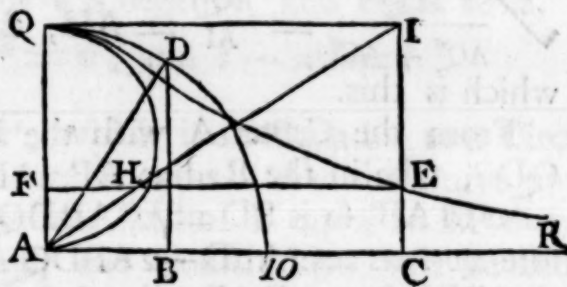
After the same manner, from the Areas of the Curves whose Ordinates are $\frac{z^{\eta-1}\sqrt{e+fx^\eta}}{g+bz^\eta}$ and $\frac{z^{-1}\sqrt{e+fx^\eta}}{\sqrt{e+fx^\eta}}$, belonging to Species's first Forms tenth and fourth, you may find the Area belonging to the Ordinate

nate $\frac{dz^{-1}}{g+bz^m\sqrt{e+fz^n}}$: which is that immediately preceding Species first of Form tenth: and so these Curves being found which immediately precede the first Species's in Forms ninth and tenth, the rest of the backward Progression *in inf.* may be found by Prop. 8, according to the Theor. Step fifth Art. 219 *. And thus I have shewn particularly, how the Series's of Curves belonging to each Form, may be continued both forwards and backwards *in inf.*

* Of this
Explica-
tion.

And now I proceed to shew the Use of the preceding Tables of Quadratures in some Examples.

319. Example 1. Let QER be a Conchoidal of such a kind, that the Semicircle QHA being described, and AC being erected perpendicular to the Diameter AQ: if the Parallelogram QACI be completed, the Diagonal AI be drawn, meeting the Semicircle in H; and the Perpendicular HE be let fall to IC, then the Point E will describe a Curve, whose Area ACEQ is sought.



Therefore making $AQ = a$, $AC = z$, $CE = y$: because of the continual Proportionals AI, AQ, AH, EC *, it will be EC or $y = \frac{a^3}{a^2 + z^2}$ * Elem. Prop. 4. B. 6.

Now that this may acquire the Form of the Equations in the Tables, make $\eta = 2$, and for z^2 in the Denominator write z^η , and $a^3 z^{\frac{1}{2}\eta - 1}$ for a^3 or $a^3 z^{1-1}$ in the Numerator, and there will arise $y = \frac{a^3 z^{\frac{1}{2}\eta - 1}}{a^2 + z^\eta}$, an Equation of the first Species of the second Order of Table second, which stands thus $\frac{dz^{\frac{1}{2}\eta - 1}}{e + fz^\eta} = y$: so that the Terms being compared it will be $d = a^3$, $e = a^2$, and $f = 1$; and further $\sqrt{\frac{a^3}{a^2 + z^2}} = x$; $\sqrt{a^3 - a^2 x^2} = v$ and $xv - 2s = t$.

Now that the Values found of x and v may be reduced to a just Number of Dimensions, chuse any given Quantity as a , by which, as Unity, a^3 may be multiplied once, in the Value of x ; and in the Value of v , a^3 may be divided once, and $a^2 x^2$ twice. And by this means you'll obtain $\sqrt{\frac{a^4}{a^2 + z^2}} = x$, $\sqrt{a^2 - x^2} = v$ and $xv - 2s = t$.

Where

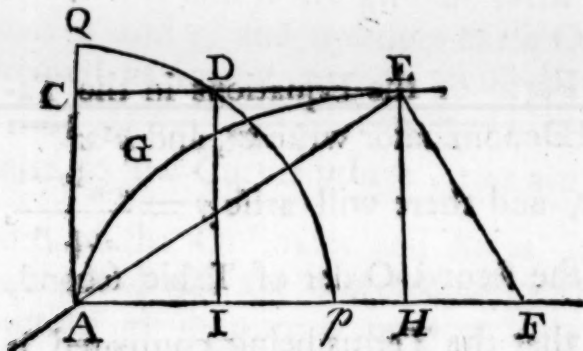
Where it is evident, that, since we have $\sqrt{a^2 - x^2} = v$ for the Ordinate of the Conic Section, with which the proposed Curve is compared, that Conic Section becomes a Circle of which $a = AQ$ is the Radius: for $\sqrt{a^2 - x^2} = v$ is the Equation to such a Circle. Which likewise appears by comparing this Value of v , with that contained

* Of this
Explica-
tion.

* Of this in Art. 303 *, viz. $\sqrt{\frac{1}{4}pt - \frac{p}{t}x^2} = v$: which gives $\frac{p}{t} = 1$ and $\frac{1}{4}pt = a^2$: that is the transverse Axe and it's Parameter are equal, and either of them equal to $2a$. Further, since it is $x = \sqrt{\frac{a^4}{a^2 + z^2}} =$

✓ $\frac{AQ^{2q}}{AQ^q + AC^q} = \frac{AQ^q}{AI} = AH$, the Construction will be manifest:
which is this.

From the Center A with the Radius AQ describe the Quadrant QDP, take in the Radius AP, $AB = x = AH$, draw BD perpendicular to AB, so is $BD = v$, $ABDQ = s$ and $ABD = \frac{1}{2} xv$: and therefore $xv - 2s = 2ABD - 2ABDQ = -2AQD$, or $xv - 2s = 2ABD + 2BDP^* = 2DAP$: the last of these Values is affirmative, and therefore denotes the Area ACEQ adjacent to the Absciss AC, and lying on the nearer Side of the Ordinate; the negative Value $-2AQD$ belongs to the Area RECR upon the further Side of the Ordinate, adjacent to the Absciss produced infinitely.



320. Ex. 2. Let AGE be a Curve, which is described by the angular Point E of the Norma AEF, while one of the Legs AE, being indefinite towards A, passes continually through that given Point; and the other EF of a given Length, slides upon the right Line AF given in Position.

Let fall EH perpendicular to AF, and complete the Parallelogram AHEC; and calling $AC = x$, $CE = y$ and $EF = z$: because HF,

* Prop 8.
B.6. Elem.

* Prop 8. HE, HA are continual Proportionals*, and $HF = \sqrt{EF^2 - EH^2} =$
 B.6. Elem. $\sqrt{a^2 - z^2}$, you'll have HA or $y = \frac{z^2}{\sqrt{a^2 - z^2}}$.

Now that the Area AGECE may be known, by comparing the Equation to the Curve with the Equation to some Species of Curves in the Table, suppose $z^2 = z^\eta$ or $2 = \eta$, then you'll have $\frac{z^{\frac{1}{2}\eta-1}}{\sqrt{a^2 - z^\eta}} = y$.

Where

Where, because z in the Numerator is of a fracted Dimension, and no Equation in the Table can be found, with which it may be immediately compared, I divide the Numerator and Denominator by $z^{\frac{1}{2}}$ or $\sqrt{z}$: by which means, it will be turned into this Shape

$\frac{z^{\frac{1}{2}-1}}{\sqrt{-1+az}z^{-\frac{1}{2}}}$ or $\frac{1}{z^{-\frac{1}{2}+1}\sqrt{-1+az}z^{-\frac{1}{2}}} = y$: which I find to be an Equation of the second Species of Order fourth Table second, which stand thus $\frac{d}{z^{\frac{1}{2}+1}\sqrt{e+fz^{\frac{1}{2}}}} = y$. By comparing of which Values, you have $d=1$, $e=-1$, and $f=a^2$: and because η is negative, and equal to 2, it is $z^2 = (\frac{1}{z^{-\eta}}) x^2$; $\sqrt{a^2 - x^2} = v$; and $s - xv = t$.

*Art. 314.
of this Ex-
plication.

Now since it is $v = \sqrt{a^2 - x^2}$, which is an Equation to a Circle having $a = EF$ for it's Radius, hence it appears that the Curve proposed is compared with a Circle whose Radius is equal to EF : and since moreover it is $x = z = AC$, the Construction will be thus. Upon the given Point A as a Center with the Radius $AQ = EF$, describe the Quadrant QDP , cutting CE in the Point D : complete the Parallelogram $ACDI$, then is $AC = z = x$, $CD = v$, $ACDP = s$, $ACDI = xv$: and the Area sought $AGEC = (s - xv =) ACDP - ACDI = IDP$.

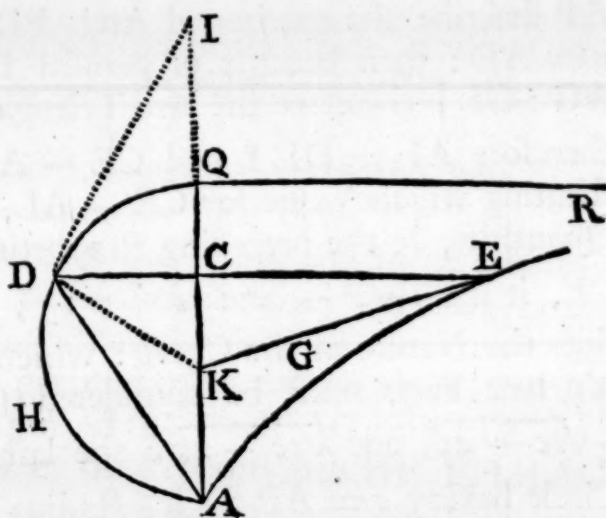
321. Ex. 3. Let AGE be the Cissoïd, belonging to the Circle ADQ , described with the Diameter AQ . Let DCE be drawn perpendicular to the Diameter, and meeting the Curves in D and E . And calling $AC = z$, $CE = y$ and $AQ = a$: because of CD, CA, CE continual Proportionals, from the Nature of the Cissoïd,

it will be CE or $y = \frac{z^2}{\sqrt{az - zz}}$, that is by dividing by z , $y =$

$\frac{z}{\sqrt{az - 1}} = \frac{1}{z^{-2} + 1\sqrt{-1+az}z^{-1}}$: where because it is $\eta = -1$ or z^{η}

$= z^{-1}$, it will be an Equation of the third Species of Order fourth Table second, which stands thus $\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}} = y$. So that by com-

paring



to Species first, Order third of Table second, which stands thus

$\frac{d}{z}\sqrt{e+fx^n}=y$: so that, by comparing the Parts, you'll have $d=b$,

$e=c^2$, $f=-1$, $\frac{1}{z}=\sqrt{\frac{1}{z^n}}=x$, $\sqrt{-1+c^2x^2}=v$. In order to

complete the Dimensions, by some given Quantity considered as Unity;

which let be c , there will arise $\frac{c^2}{z}=x$ and $\sqrt{-c^2+x^2}=v$: by com-

paring which with Art. 300 *, you'll find it to belong to an Hyper-

bola, having it's transverse Axe $2c=2AP$ and it's Latus Rectum the

* Of this
Explica-
tion.

same: and the tabular Area, viz. $\frac{4de}{nf} \times \frac{v^3}{2ex} - s$, becomes $-2bc^2 \times$

$-s + \frac{v^3}{2c^2x} = 2bc^2s - \frac{bv^3}{x}$; or, by completing the Dimensions $\frac{2bs}{c} -$

$$\frac{bv^3}{cx} = \frac{2b}{c} \times s - \frac{v^3}{2x} = t.$$

By considering of which things, the Construction is thus. With the Center A and Vertex Principalis P, describe the rectangular Hyperbola PK:

take $AM=x$ *, a third Proportional to AC and AP, for $x=\frac{c^2}{z}$: draw the

* Art. 300.
of this Ex-
plication.

Ordinate MK which is equal to v : join KC, which will touch the Hyper-

bola in the Point K from the Property of the Tangent. Then $PMKP=s$,

and $\Delta CMK=\frac{v^3}{2x}$ (for which see Art. 276 *.) Whence $-CKPC=s-\frac{v^3}{2x}$, and

* Of this
Explica-
tion.

$\frac{2GA}{AP} \times -CKPC = \frac{2b}{c} \times s - \frac{v^3}{2x} = \text{Area PDEP}$: which comes out

negative; because that Area lyes upon the further Side of the Ordi-

nate *. Therefore it is $AP : 2AG :: CKPC : \text{DPED}$ the Area

* Art. 316.
of this Ex-
plication.

required.

323. Ex. 5. Let the Norma GFE so revolve about the Pole G, as

that it's angular Point F may continually slide upon the right Line

AF, given in Position: then conceive the Curve PE to be described

by any Point E, in the Leg EF, while the other Leg FG always passes

through the given Point G. Now that the Area of this Curve may

be found, let fall GA and EH perpendicular to the right Line AF,

and completing the Parallelogram AHEC, call $AC=z$, $CE=y$,

$AG=b$ and $EF=c$; and because of the Proportionals $HF:EH::AG:$

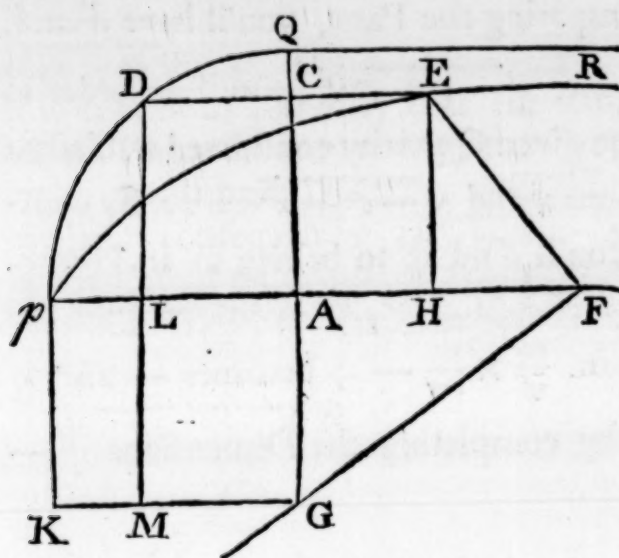
AF , i. e. $\sqrt{c^2-z^2}:z::b:AF$, you'll have $AF=\frac{bz}{\sqrt{cc-zz}}$: therefore

CE or $y=(AF-HF)=\frac{bz}{\sqrt{cc-zz}}-\sqrt{cc-zz}$. But whereas

$\sqrt{cc-zz}$ is the Ordinate of a Circle, described with the Semi-diameter c

G g 2 = EF

$= EF = AP$: about the Center A, let such a Circle PDQ be



described, which CE produced meets in D; then it will be $DE = (DC + CE =)$

$\frac{bz}{\sqrt{cc - zz}}$: by the Help of which

Equation, the Area PDEP on the nearer Side of the Ordinate, or DERQ on the further Side of the Ordinate, remains to be determined. Suppose therefore $\eta = 2$, and it

will be $\frac{bz^{\eta-1}}{\sqrt{cc - z^{\eta}}}$, an Equation of the 1st Species Order 4th Tab.

first, which stands thus $\frac{dz^{\eta-1}}{\sqrt{e + fz^{\eta}}} = y$: so that the Terms being compared,

it will be $d = b$, $e = cc$, and $f = -1$: so that $t = (\frac{2d}{\eta f} R =) -b\sqrt{cc - zz}$ the Value of the Area sought: which being negative, designs the Area lying beyond the Line DE.

In order therefore to know it's initial Limit upon that further Side, put $-b\sqrt{cc - zz} = 0$; thence arises $z = c = AQ$: wherefore, drawing the Ordinate QR, you have that for the Limit, and consequently the Area DQRED $= b\sqrt{cc - zz} =$ Rectangle contain'd under AG and CD: so that if you complete the Parallelogram PAGK, and draw DLM perpendicular to AP, you'll have the Area DQRED = Parallelogram AM.

If you would know the Quantity of the Area PDE, posited at the Absciss AC, and co-extended with it, without inquiring into the Limit QR, subtract the Value of t at the Beginning of the Absciss, *i. e.* when $z = 0$, which therefore is $-bc$, from it's Value, when $z = AC$, *viz.* $-b\sqrt{cc - zz}$; and the Difference $bc - b\sqrt{cc - zz} = AG \times AP - AG \times AL$ or the Rectangle PKML, is the Value of the Area PDE sought.

324. Ex. 6. Let AE and AF be two given right Lines at right Angles to each other: and let the right Line CD = AE, slide along AF with it's Extremity D, and at the same time, turn about D as a Pole or Center, in such manner that, if from it's Extremity C you draw

draw CB perpendicular to AF, it shall always be $AB : BD :: AF : AE$, the Point C touches a Curve, whose Nature and Area are required.

Call $AB = z$, $BC = y$, $AE = a$, $AF = b$: then because $AF = b : AE = a :: AB = z : BD$, hence we have $BD = \frac{az}{b}$: and be-

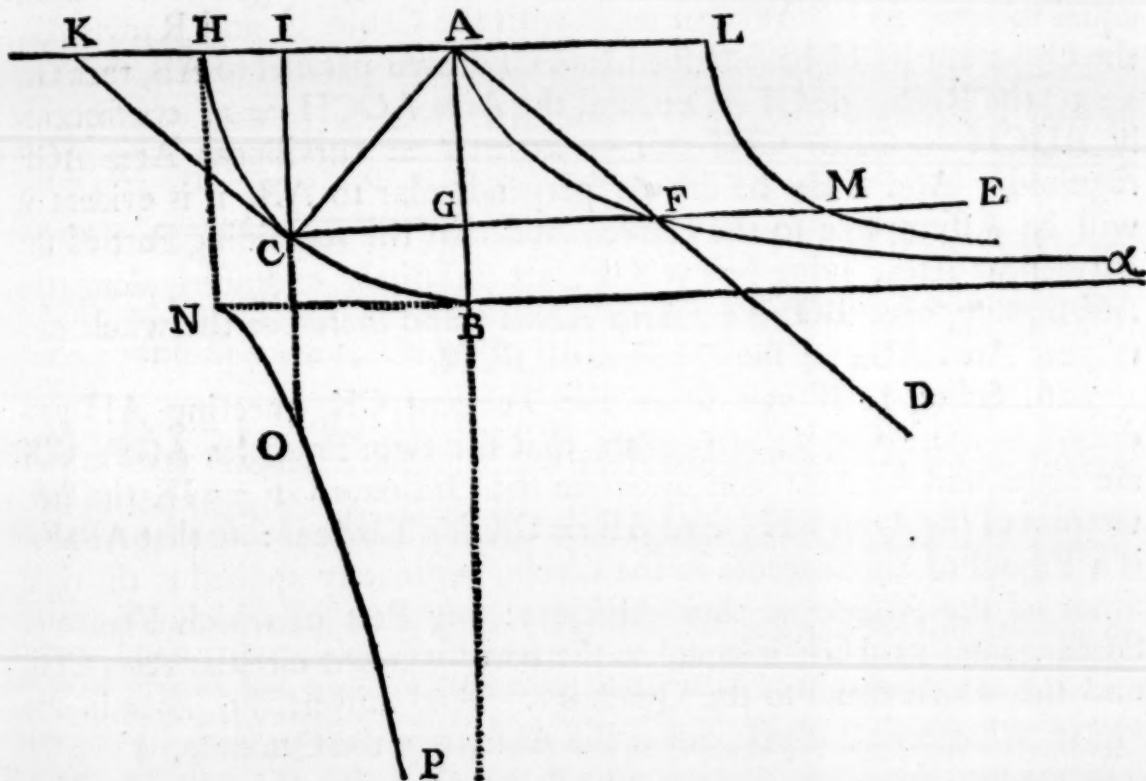
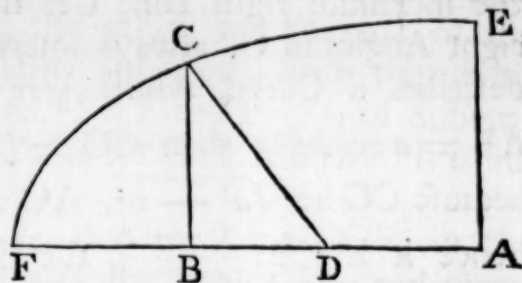
cause of the right-angled Triangle BCD, $BC = \sqrt{CD^2 - BD^2}$,

that is $y = \sqrt{a^2 - \frac{a^2 x^2}{b^2}}$: By comparing this with Art. 306 *, where
 you have $v = \sqrt{\frac{1}{4}pt - \frac{p}{t}x^2}$, it appears to belong to an Ellipse : and

* Of this
 Explica-
 tion.

by comparing the homologous Terms, you have $\frac{1}{4}pt = aa$ and $\frac{p}{t} = \frac{a^2}{b^2}$; whence arises $t = 2b = 2AF$, and $p = \frac{4a^2}{2b} = \frac{4AE^2}{2AF}$, that is the transverse Axis is $2AF$ and it's Parameter a third Proportional to $2AF$ and $2AE$; so that it's conjugate Axis is $2AE$, and A the Center.

325. Ex. 7. Let AB be a given right Line, CAD a rectangular Norma, revolving upon the Point A as a Pole, of which one Leg AC


$$\equiv AB$$

$= AB$; the other AD indefinite: and while the Norma revolves, let the indefinite right Line CE drawn through C and cutting AB at right Angles in G , always intersect AD in the Point F : the Point F describes a Curve, whose Area is sought. Call $AG = z$, $GF = y$, $AB = a = AC$: then $CG = (\sqrt{AC^2 - AG^2}) = \sqrt{aa - z^2}$: and because $CG = \sqrt{a^2 - z^2}$, $AG = z$, and $GF = y$ are proportional, hence it is $y = \frac{z^2}{\sqrt{a^2 - z^2}}$, the Equation to the Curve whose Area is

* Of this
Explica-
tion.

sought. By comparing which with Ex. 2. Art. 320 *, it appears to be the same Curve: and therefore it's Area is found, as is there shewn, by comparing it with Species second, Form fourth, Table second. But that you may see how one and the same Curve may belong to different Orders and Species's in the Table, you may compare it with Spe-

cies third, Form eight, Table second, viz. $\frac{dx^{3\eta-1}}{\sqrt{e+fx^\eta+gz^{2\eta}}} = y$, by putting $\eta = 1 = d$ and $f = 0$: and moreover $e = aa$, $g = -1$, $z = x$, $\sqrt{a^2 - x^2} = v$, and $s - xv = t$, as formerly. Now since it is $x = z$, and $v = \sqrt{a^2 - x^2} = \sqrt{a^2 - z^2}$, hence it appears that the Conic Section with which the Curve is compared is a Circle, whereof the Radius is $a = AB$, and the Absciss x , beginning at the Center, equal to $z = AG$. Wherefore, if with the Center A and Radius AB , the Quadrant BCH be described, and CI drawn parallel to AB , then $GC = v$, the Rectangle $GI = xv$, and the Area $AGCH = s$: consequently $AGCH - GI = CHI = s - xv = t =$ curvilinear Area AGF required. And if Ba be drawn perpendicular to AB , it is evident it will be Asymptote to the Curve; and that the remaining Part of the curvilinear Area, lying below GF , are infinitely extended along the Asymptote, viz. $BGFa =$ Area $ABCI$: and therefore the whole curvilinear Area $ABa =$ the Quadrant $ABCH$.

326. Schol. 1. If you draw the Tangent CK , meeting AH produced in the Point K , it appears that the two Triangles AGF , CIK are equal and similar: and therefore the Ordinate $GF = IK$ the Subtangent of the Arch CH ; and $AF = CK$ it's Tangent: so that $ABaFA$ is a Figure of Subtangents in the Circle, ordinately applied to the right Sines of the Arches as their Abscisses: any Part of which Figure of Subtangents, as AGF is equal to the correspondent circular Area CHI , and the whole equal to the Quadrant. And universally, since in any Curve whatsoever, if you call z the Absciss, y the Ordinate, and s the

* Art. 42.
of this Ex-
plication.

Subtangent, you have $y : z :: y : s$ *, or $ys = zy$, the first of which

is

is the Fluxion of the Figure of Subtangents applied to the Ordinates or right Sines as their Abscisses; and the other, the Fluxion of the curvilinear Area *: † hence the Fluents are equal, since they both begin to be generated at the same time *, i. e. the Figure of Subtangents applied to the Ordinates of any Curve for their Abscisses, is equal to the Area of the Curve.

* Art. 35.
of this Ex-
plication.
* Art. 38.
of this Ex-
plication.

327. Schol. 2. Moreover, it appears, that the Line CGF, in the preceding Figure, is equal to the Secant of the Arch CH: and therefore AHB_aFA is equal to a Figure of Secants ordinately applied to the right Sines as their Abscisses: which Figure of Secants, therefore, adjacent to the whole Radius, is equal to the Semicircle. It's Description may be thus: produce HA to L, so that AL = AH, and take every where upon GF produced, FM = CG, then the Curve LM, which M touches, belongs to a Figure of Secants, such as has been mentioned, having AG = IC for the Absciss, and GM = AK for it's Ordinate. And so the Area AGML = (AGCHA + CHI =) twice Sector ACH: and the whole Area AB_aML infinitely extended along the Affymptote B_a, is equal to twice the Quadrant AHB: and when the Arch HC = 45°, the Ordinate GM bisects the curvilinear Area, making each half equal to the Quadrant. Accordingly GM =

$$(GF + GC = \frac{z^2}{\sqrt{a^2 - z^2}} + \sqrt{a^2 - z^2} =) \frac{a^2}{\sqrt{a^2 - z^2}} \text{ or } \frac{a^2}{z\sqrt{-1 + aaz^{-2}}}$$

which belongs to Species 1. Order 4. Table 2, viz. $\frac{d}{z\sqrt{e + fz^n}} = y$;

as appears by putting $d = a^2$, $e = -1$, $f = a^2$, $n = -2$: so that it is $x^2 = (\frac{1}{z^2}) z^2$, or $x = z = AG$, $v = (\sqrt{a^2 - x^2} = \sqrt{aa - z^2}$

=) CG, and $\frac{4d}{nf} \times \frac{1}{2} xv \div s = 2 \times s - \frac{1}{2} xv = 2ACH = AGML$ required. The same may be found by comparing the given Ordinate

$\frac{a^2}{\sqrt{a^2 - z^2}}$ with Species 1. Order 8. as in Art. 325 *.

* Of this
Explica-
tion.

328. Schol. 3. But if the Secants be ordinately applied to the versed Sines of their Arches, that is, if the rectangular Ordinate IO be always equal to the Secant AK, then AI, AH, AO are proportional: therefore if you complete the Square AN, it is evident that the Curve

† When I say here that ys and zy are the Fluxions of the Figure of Subtangents and Area of the Curve, I hope the Reader won't mistake me, or alledge that I have departed from the true Notion of Fluxions laid down in the Beginning of this Work: as if I meant that these Expressions were in strict Propriety Fluxions or Velocities of flowing: it is enough that they are Representations or the Exponents of the Fluxions and proportional to them, which is all that the Reasoning requires. This I thought proper to observe in this Place once for all: that so the Reader may not mistake me in any Place, where he meets with the like way of speaking.

NOP.

let AH in this Figure, be equal to AH in the preceding: then the Quadrant HR being described is a fourth Part of a meridian Circle: draw QIS, cutting AH in I, and HR in S, then it is evident that AI is the Tangent of the Half of RS. Now suppose the Line QIS to revolve upon Q as a Pole, while IO in the preceding Figure moves parallel to itself, so that AI and HI in both Figures be always of the same Length: and let QIS move into the new Place Qis, the two Triangles QSs, QIi, in their nascent State, are similar: whence $Ss : Ii :: QS : Qi$ or QI ; but, if you draw ST perpendicular to AH at the Point T, from the similar Triangles AIQ, IST, it is $QS : QI :: AT : AI$, therefore $Ss : Ii :: AT : AI$; but $Ss : Ii ::$ Fluxion of HS : Fluxion of HI *, that is, calling $HS = z$ and $HI = x$, $\dot{z} : \dot{x} :: (AT : AI ::) \text{Cof. } HS : T. \frac{1}{2} \text{ comp. } HS$. Again, in the preceding Figure, $F.IN : F.IOHN :: HN : IO :: AI : AH :: T. \frac{1}{2} \text{ comp. } z : R$; but $F.IN : F.IOHN :: \frac{F.IN}{HN} = \dot{x} : \frac{F.IOHN}{HN}$, therefore $\dot{x} : \frac{F.IOHN}{HN} :: T. \frac{1}{2} \text{ comp. } z : R$. But it was just now demonstrated that $\dot{z} : \dot{x} :: \text{Cof. } z : T. \frac{1}{2} \text{ comp. } z$; therefore by Equality $\dot{z} : \frac{F.IOHN}{HN} :: \text{Cof. } z : R :: R : \text{Sec. } z$. But supposing p to express the Length of the meridional Arch HS or z , as protracted in the nautical Chart, according to *Mercator's* Projection, from the known Property of that Projection, it is $\dot{z} : \dot{p} :: R : \text{Sec. } z$; wherefore, by taking equal Ratios, it is $z : \frac{F.IOHN}{HN} :: \dot{z} : \dot{p}$; whence $\frac{F.IOHN}{HN} = p$, and therefore $\frac{IONH}{HN} = p$ * the protracted meridional Arch HS. Wherefore the hyperbolical Area IONH is equal to the protracted meridional Arch multiplied into the Radius. Now from the known Property of the Hyperbola, the Area IONH is the Logarithm or Measure of the Ratio of AH to AI, that is, calling the Rad. AH, 1, the Logarithm of $\frac{1}{1-x} = \frac{R}{T. \frac{1}{2} \text{ comp. } z}$: and whereas the hyperbolical Logarithm of any Ratio, is to the tabular Logarithm of that Ratio, taken from *Briggs's* or the common Tables as 2.302585, &c. to 1: therefore the Logarithm of $\frac{R}{T. \frac{1}{2} \text{ comp. } \text{lat.}}$ taken from the common Tables, being multiplied by 2.302585, &c. will give the hyperbolical Area IONH, or the Length of the protracted meridional Arch, the Radius being Unity: so that the Rule is this; subtract logarithmical Tangent of $\frac{1}{2}$ comp. lat. from logarithmical Rad. and multiply the Difference by 2.302585, &c. Moreover, if you put A and a for a greater and lesser Arch of the Meridian, reckoning from the Equator, and l for an hyperbolical Logarithm: then

H h it

* Art. 28.
of this Ex-
plication.

* Art. 40.
of this Ex-
plication.

it appears from what has been said, that the protracted Length of the meridional Arch $A - a$ is $l \cdot \frac{R}{T. \frac{1}{2} \text{ comp. } A} - l \cdot \frac{R}{T. \frac{1}{2} \text{ comp. } a} = l \cdot T. \frac{1}{2} \text{ comp. } a - l \cdot T. \frac{1}{2} \text{ comp. } A$, that is, subtract the Log. of the Tangent of the half Complement of the greater Arch from the Log. of the Tangent of the half Complement of the least Arch, and the Difference is the protracted Length of the Arch $A - a$: meaning by Logarithm the hyperbolical Logarithm. Wherefore when you take the logarithmical or artificial Tangents from the common trigonometrical Tables, the Difference mentioned above must be multiplied by 2.302585 &c. to have the protracted meridional Arch lying betwixt the two Latitudes A and a , the Rad. of the Globe being 1. But if you would have this meridional Arch expressed in the common way by Miles or Minutes, so that each Minute of Latitude be called 1, then the Expression already found must be multiplied by 3437.74677, &c. because so many times is one Minute of a great Circle contained in the Radius. Or, by one Operation, multiply the Difference of the artificial Tangents of the half Complements of the Latitudes taken from the common Tables, by 7915.70446789, &c. $= 2.30258509$, &c. $\times 3437.74677078$, &c. e. g. Let it be required to find the Length of the meridional Arch in *Mercator's* Chart, lying betwixt the Tropic in Lat. $23^\circ 30'$, and the polar Circle in $66^\circ 30'$: which being Complements to each other, therefore the Halves of their Complements are $33^\circ 15'$, and $11^\circ 45'$, whose artificial Tangents are 9.8166580 and 9.3180640, whose Difference is 0.498594, which multiplied by 7915.704, &c. gives 3946.722, &c. Miles or Minutes of a great Circle, contained in that meridional Arch protracted.

330. Schol. 4. Because this Affair is under Consideration, and the Figures suited to the Purpose, I shall shew another Method of investigating and demonstrating this Property of the meridian Line in *Mercator's* Projection. In the last Figure draw IC perpendicular to AH , meeting the Circumference of the Circle in C : at C and S draw the Tangents CK , SV , meeting AH produced in K and V . Then, using the Symbols x , z and p as formerly, and supposing the same things as before, from the Property of the Tangent, you have $AV : AH : AT ::$ and $AK : AH : AI ::$, whence $AV : AK :: (AI : AT :: QI : QS :: Ii : Ss ::) x : z$, that is $AV \times z = AK \times x$: but the first is the Fluxion of the Figure of Secants applied to their Arches as Abscisses; the other is the Fluxion of a Figure of Secants applied to the versed Sines as their Abscisses; the corresponding or synchronal Abscisses being HS and HI : therefore the Areas of these Figures are equal*, that

* Art. 38.
of this Ex-
plication.

is the hyperbolical Area HION (in the preceding Figure) is equal to the Area of the Figure of Secants ordinately applied to the Arches, having HS for the Absciss, in this last Figure. But the Area last mentioned is to a Rectangle having it's Base equal to HS or x , and Altitude, the Radius, as the protracted Length of the Arch HS, to it's natural Length (for the Fluxions of these Figures are as the Secant of HS to Radius) therefore $HION : x \times R :: p : x$; that is $HION \times x = x \times R \times p$, or $\frac{HION}{R} = p$: so that when $R = r$, $HION = p$ the protracted meridional Arch HS: and therefore the rest follows as before.

And here it may be observed that a Method is suggested of finding a geometrical Curve having it's Area equal to that of a mechanical Curve: since it has been demonstrated that the hyperbolical Area HION is equal to the Area of the Curve which has an Arch of a Circle, viz. HS for it's Absciss, and the Secant AV for it's Ordinate: of which more immediately.

P R O B L E M.

331. *To find the Areas of mechanical Curves.*

A Curve, whose Nature or Property may be expressed by an algebraical Equation of any finite Order or Degree, which defines the Relation of the Absciss and Ordinate to each other, is commonly called an *algebraical* or *geometrical* Curve: so that, according to the Number of Dimensions of the Equation by which such Relation is defined, Curves are distinguished into different Orders: when the Equation is of two Dimensions, the Curve is of the first Order, the Circle and Conic Sections are the only Curves of this Order: when the Equation is of three Dimensions, the Curves are of the second Order, such are the cubical Parabola, the Cissoid of *Diocles*; and others, an Enumeration of which our learned Author hath made in a separate Treatise by itself, in which he reduces them all to seventy-two Kinds. And after the like manner, Curves of the third, fourth and higher Orders are such, the Relation of whose Ordinates to their Abscisses, is defined by Equations of four, five or more Dimensions respectively. It amounts to the same to say that Curves of the first, second, third, &c. Orders are such as may be cut by a right Line in two, three, four, &c. Points respectively. And a Curve of the first Order is esteemed a Line of the second Order; a Curve of the second Order, a Line of the third, &c. the right Line being the first Order of Lines; which can be intersected by another, only in one Point. Accordingly a Curve which

H h 2

may

may be intersected by a right Line in an infinite Number of Points, is said to be of an infinitesimal Order: these Curves, or such, the Relation of whose Absciss to the Ordinate cannot be expressed by an algebraical Equation of any *finite* Order or Dimensions, are termed *mechanical* or *transcendental*, such as the common Cycloid, Quadratrix, Spiral and infinite others¹.

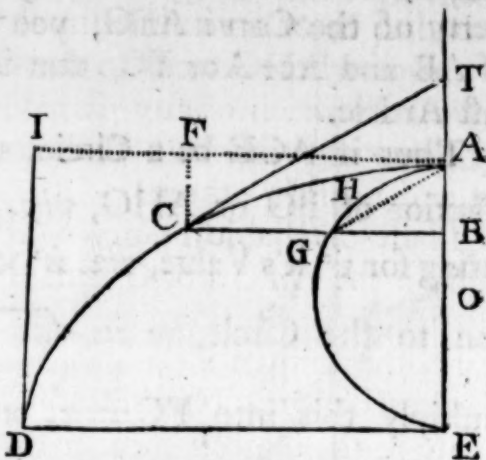
These mechanical Curves (at least the Order of them) are supposed to have the Absciss or Ordinate equal to some known Curve, as a Circle or Conic Section; or even one of some higher Order; or else to bear some known Relation to such Curve-line; or curvilinear Area applied to Unity. Now although the Relation of the Absciss and Ordinate in these mechanical or transcendental Curves, cannot be expressed by any algebraical Equation of finite Dimensions: yet the Relation of the Fluxions of the Absciss and Ordinate, may be expressed by an algebraical Equation; by means of which we may find a geometrical Curve which has the same Fluxion: and thence the Area of the mechanical Curve may be found, by the Quadrature of Curves: of which take the following Examples.

332. Ex. 1. Let the common Cycloid or Trochoid ABCD be proposed, generated by the Rolling of the Circle AHGE upon the Base

¹ *Des Cartes* calls those Curves, in which the Relation of the Absciss and Ordinate is expressed by an Equation of two Dimensions, Curves of the first Order or Kind, *viz.* the Circle, Ellipse, Parabola and Hyperbola: but under Curves of the second Order, he reckons both such, the Relation of whose Absciss and Ordinate is expressed by an Equation of three Dimensions; and likewise those, in which it is expressed by an Equation of four Dimensions. And under Curves of the third Kind or Order, he reckons all those in which the Relation of the Absciss and Ordinate is defined either by an Equation of four or of five Dimensions, and so on. See his *Geometry*, Book 2. But the Division above, which is Sir *Isaac Newton's*, is the most simple and natural. See his *Enumeratio Linearum tertii Ordinis* at the Beginning. And whereas there have been some Differences among learned Men, with respect to what Curves ought to be called geometrical; and as to their Preference in geometrical Constructions of Problems. See our Author's Opinion of this Matter in the Appendix to his *Arith. Univ. de Constructione Lineari*. Where he shews that the Rule by which to determine the Preference of one Curve to another in Geometry, is the Easiness of the Description; and not the greater Simplicity of the Equation by which it is defined: which last is a Consideration entirely algebraical, otherwise the Parabola ought to be preferred to the Circle in geometrical Constructions: which no one will affirm.

The general Division of Curves by our Author is into *geometrically rational*, and *geometrically irrational*. "*Curvas geometricè rationales appello, quarum puncta omnia per longitudines æquationibus definitas, i. e. per longitudinum rationes complicatas, determinari possunt; cæteræque (ut Spirales, Quadratrices, Trochoides) geometricè irrationales. Nam longitudines quæ sunt vel non sunt ut Numerus ad Numerum (quemadmodum in decimo Elementorum) sunt arithmeticè rationales vel irrationales.*" *Phil. Nat. Prin. Lib. 1. Cor. ad Lem. 28.* By which it appears that these Curves which he calls geometrically rational, are the same with such as we (after others) have called *geometrical* or *algebraical*: and the geometrically irrational, the same, we have termed *mechanical* or *transcendental*: which include those called *exponential* by Mr. *Leibnitz*, *viz.* such whose Equations involve some Power of the Absciss or Ordinate, that has a variable Quantity for it's Exponent as x^y , x^x &c.

ED, of which the Diameter AE is the Axis of the Cycloid: from any Point of which as B, having drawn the rectangular Ordinate BGC, meeting the Circumference of the Circle in G, and the Curve in C: complete the Parallelograms ABCF, AEDI; draw the Chord AG; and suppose CT a Tangent to the Curve at C. The Area ABC is required.



Call $AB = z$, $BC = y$, $BG = v$, $AE = d$: then from the Nature of the Curve (as shall likewise be shewn afterwards) CT is parallel to AG, so that the two Triangles BCT and BGA are similar, whence $\dot{z} : \dot{y} ::$ (TB : BC* ::) $z : v$, that is $\dot{y} = \frac{v\dot{z}}{z} =$ (by inserting $\sqrt{dz - zz}$ in-

* Art. 4th of this Explication.

stead of y , from the Property of the Circle) $\frac{\dot{z}\sqrt{dz - zz}}{z}$: whence you have the Relation of the Fluxions of the Absciss and Ordinate. Therefore, while AB flows with the Fluxion $\dot{z}$, BC or AF flows with the Fluxion $\frac{\dot{z}\sqrt{dz - zz}}{z}$: consequently, if you multiply this last Fluxion

into $CF = z$, you'll have $\dot{z}\sqrt{dz - zz}$, for the Fluxion of the external Area AFC, which, with ABC, completes the Parallelogram ABCF. Wherefore (by Prop. 9.) the Area AFC is equal to the Area of a geometrical Curve having z for it's Absciss, and $\sqrt{dz - zz}$ for it's Ordinate: which is the circular Area ABG described by the Ordinate $BG = \sqrt{dz - zz}$ and Absciss $AB = z$: so that the complemental Area AFC is equal to the circular Area AHGB; and the whole Area AIDCA equal to the Semicircle AGEA: therefore the Area $ABC = ABCF - AHGB$; and the Semicycloid $AED = AEDI - AHGE$, *i. e.* (because the Base ED is equal to the Semicircumference AHGE, and therefore the circumscribed Rectangle $AD = AE \times ED = AE \times AHGE = 4AHGEA$) the Semicycloid $AED = 3$ Semicircles AHGEA; and so the whole Cycloid is equal to three times the generating Circle, and ACDEGHA equal to that Circle.

333. Ex. 2. After the same manner, if ACDE was a Figure of Arches of any geometrical Curve, as the Circle, Ellipse or any other known Curve: so that the Ordinate BC applied to the versed Sine AB, were equal to the Arch AHG: you may find the complemental Area AFC, and thence the Area AHGB as formerly. For, since $BC =$
AHG,

* Art. 40.
of this Ex-
plication.

AHG, their Fluxions are equal*: therefore, from the known Property of the Curve ABG, you may find the Relation of the Fluxions of AB and AHG or BC, that is AF; and then you proceed as in the last Article.

* Art. 42.
of this Ex-
plication.

Thus if AGE be a Circle, using the same Symbols as before, the Fluxion of BG or AHG, viz. $\dot{y} = \sqrt{\dot{z}^2 + \dot{v}^2}$ *, that is, (by substituting for $\dot{v}$ it's Value, viz. $\dot{z} \times \frac{d^2 - 4dz + 4z^2}{4dz - 4z^2}$, deduced from the Equation to the Circle, $v = \sqrt{dz - z^2}$, and reducing) $\dot{y} = \frac{\frac{1}{2}dz}{\sqrt{dz - z^2}}$:

multiply this into $FC = z$, and you have $\dot{y}z = \frac{\frac{1}{2}dz}{\sqrt{dz - z^2}}$: wherefore a geometrical Curve, whose Absciss is $z = AB$, and Ordinate $\frac{\frac{1}{2}dz}{\sqrt{dz - z^2}}$, is equal to the complemental Area AFC*: which may be reduced to Species second, Order eight, Table second, expressed thus

* Prop. 9. $\frac{dz^{2\eta-1}}{\sqrt{e+fx^{\eta}+gx^{2\eta}}} = y$, by putting $\eta = 1$, $d = \frac{1}{2}d$, $e = 0$, $f = d$, $g = -1$, $x = (z^n =) z = AB$, $\sqrt{e+fx+gx^2} = v = \sqrt{dz - z^2} = BG$, and $t = 2s - xv = 2ABGHA - 2\Delta ABG = 2AGHA$: which therefore is equal to the complemental Area AFC; and so the whole Area AID is equal to twice the Semicircle. Whence, by subtracting twice the Segment AGHA from the given Rectangle AC, there remains the Area ABC required: and since $ED = AHGE$, and consequently the Rectangle $AD = 4AHGEA$, and $AIDCA = 2AHGEA$, therefore the Figure of Arches AEDCA is just equal to the Circle whose Diameter is AE: so that the Rectangle AD is bisected by the Curve-line ACD, and the Area AED by the Semicircumference AHGE.

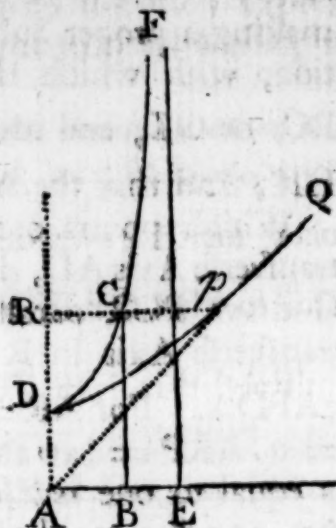
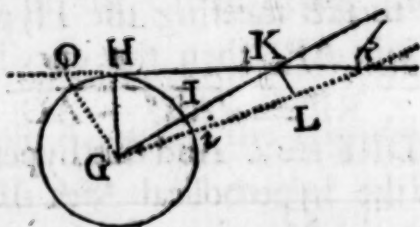
The same things may be found by comparing the Ordinate, after it is reduced to this Form $y = \frac{\frac{1}{2}dz}{\sqrt{-1 + dz}}$, with Case second, Species

second, Order fourth, Table second, by putting $\eta = -1$, &c. And thus you might find the Areas of Figures of Arches in the Ellipse and Hyperbola, and compare them with geometrical Curves: only in this Case, it will be proper to compute the Beginning of the Absciss from the Centers of the Conic Sections, rather than from the Extremities of the Diameters: as we have done in the Case of the Circle.

334. Ex. 3. Let GHI be a Circle, which the right Line HK touches in the Point H, and suppose the right Line GK to revolve about the Center G as a Pole, and always intersect the Tangent HK in the Point K,

K, and the Circumference of the Circle in I: and let FCD be a Curve

of such a Nature that it's Absciss A B be equal to the Arch HI, and it's rectangular Ordinate BC equal to the Secant GK, so as to constitute such a Figure of Secants as was mentioned formerly: the Area ABCD is required.



Join GH, and let GIK move into the new Place Gik; draw KL perpendicular to Gk: call HI or AB = z , and GK or BC = y : then z is to y in the prime Ratio of the nascent Augments Ii and Lk *, that is, (because the Triangles GIi, GKL, and LKk, GHK are in that Case similar) $z : y :: Ii : KL + KL : Lk$ or $z : y :: GI : GK + GH : HK$, that is, if you call the Radius r ; $z : y :: r^2 : y\sqrt{y^2 - rr}$, or $zy\sqrt{y^2 - rr} = r^2$, which defines the Relation of the Fluxions of AB and BC. But whereas the Value of y is mixt, having y in it's Composition, otherwise than happened in the preceding Examples, I can't find the complemental Area as in them; but I take the Value of z , viz. $z = \frac{r^2}{y\sqrt{y^2 - rr}}$, and multiply it into BC = y ,

*Art. 3. 22 of the Quadrature of Curves.

which produces $\frac{r^2}{\sqrt{y^2 - rr}} = zy$ the Fluxion of the curvilinear Area ABCD: and therefore that curvilinear Area is equal to the Area of a geometrical Curve having $y = BC$ or GK for it's Absciss, and $\frac{rr}{\sqrt{y^2 - rr}}$ for it's Ordinate*, that is, if you draw GO perpendicular to GK, meeting the Tangent KH produced, in O, the Line HO, which is the Cotangent of HI.

*Prop. 9.

Now this Ordinate $\frac{rr}{\sqrt{y^2 - rr}}$, being made to stand thus $\frac{r^2}{x\sqrt{1 - \frac{r^2}{y^2}}}$

belongs to Species 1. Order 4. Table 2, expressed thus $\frac{d}{x\sqrt{e + fz^n}} = y$,

by putting $d = r^2$, $e = 1$, $f = -r^2$, $z = y$, $n = -2$: in which Species,

Species, because $\frac{1}{x^2} = x^2$, $\sqrt{f + ex^2} = v$, and $\frac{4d}{xf} \times \frac{1}{2} xv = s$: by making a proper Substitution of Values you'll find that the Conic Section, with which the Curve is compared, has it's Absciss $x = y = BC$, or GK , and it's Ordinate $v = \sqrt{-r^2 + x^2} = \sqrt{BC^2 - AD^2} = HK$; and that the Area of the Curve sought is $2 \times \frac{1}{2} xv = s$.

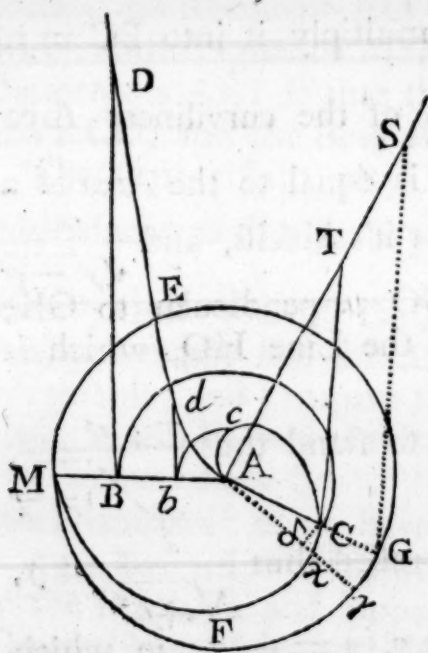
Which points out this Construction: with the Center A and half transverse Axis AD, describe the rectangular Hyperbola DPQ, through C draw PCR parallel to AB meeting the Hyperbola in P, and the transverse Axis in R; join AP, then the curvilinear Area $AECD = 2APDA$. For $AR = x$, $RP = \sqrt{AR^2 - AD^2} = HK = \sqrt{x^2 - r^2} = v$. $ARP = \frac{1}{2} xv$ and $DRP = s$. And this hyperbolical Sector APDA is equal to one half of the hyperbolical Area IONH in the Figure Art. 325*, according to the Meaning of Art. 329†, when the Arch HS in the Figure Art. 329‡, and the Arch HI in this Figure are equal, the Radius's of both Circles being supposed equal: as might be easily demonstrated.

* Of this Explication.

† Of this Explication.

‡ Of this Explication.

335. The Areas of Spirals may be determined in the following manner. Let FC be a Spiral of any Kind described from the Point A as a Pole; BC an Arch of a Circle described from the same Point A as a Center, moving uniformly along the indefinite right Line AB given in Position, as a Base or Absciss: and let the Arch BC, while it moves, always touch the Spiral with it's Extremity C: then it is evident that the spiral Space terminated at BC, is described with the same Fluxion as a Curve having AB for the Absciss, and a right Line equal to BC for the rectangular Ordinate; and therefore the spiral Area and curvilinear Area adjacent to the same Part of the Absciss, must be equal*. Thus, if at the Point B of the Base AB, you draw the right Line BD perpendicular to AB, and equal to the Arch BC, so that while BC moves along AB, BD may move together with it, and be always equal to it, so as to describe the Curve dED, the Fluxions of the spiral and curvilinear Areas are equal: therefore if from the Point b you draw the Perpendicular $bd = bc$, the Areas $bBDd$



* Art. 40. of this Explication.

bBDd

$bBDd$ and $bBCcb$ are equal *. Wherefore calling $AB = z$, BC or $BD = y$, the Relation of z and y are defined by an Equation expressing the Nature of the Spiral; and thence the Area may be found, as formerly, from the Quadrature of Curves. * Art. 40. of this Explication.

Thus let $y^m = a^p z^n$ be an Equation defining the Relation of z and y ; or AB and BC or BD ; which may denote an infinite Number of different Kinds of Spirals: thence we have $y = a^{\frac{p}{m}} z^{\frac{n}{m}}$ for the Value of the Ordinate BD of the Curve dED : therefore the Area is $\frac{m}{n+m}$

$a^{\frac{p}{m}} z^{\frac{n+m}{m}}$ * denoting the curvilinear Area ABD or spiral Space $ABCA$, * By Form 1. Tab. 1. beginning at the Pole A , when the Exponent $\frac{n+m}{m}$ is positive; and adjacent to the Absciss AB and bounded by BD and DA , or BC and CcA *: but if $\frac{n+m}{m}$ be negative, the Area lyes upon the other Side

of BD or BC †: and in every Case, the Area adjacent to any Part of the Absciss Bb , viz. $BbcCB$ is found, by subtracting the Area belonging to the lesser Absciss from that belonging to the greater *. * Art. 168. of this Explication. † Ibid.

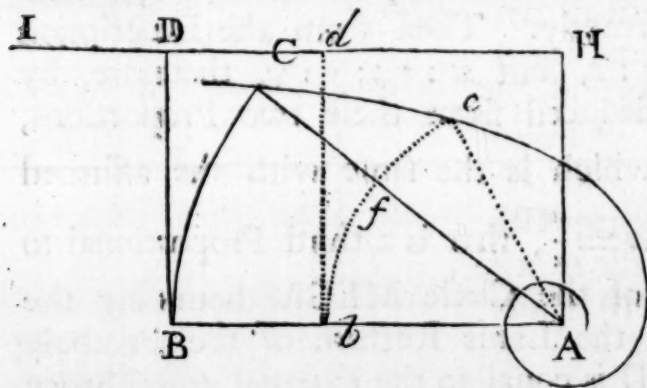
Ex. 1. Let the Equation be $y = a^{-1} z^2$, where we have $m = 1$, $p = -1$ and $n = 2$: therefore the Area is $\frac{1}{3} \frac{z^3}{a} =$ (by substituting y for $\frac{z^2}{a}$)

$\frac{1}{3} zy = ABCA = ABDA$. In which Case it appears that the Curve AED is the common Parabola having it's principal Vertex in A , where AB touches it, and BD a Diameter; and a the Latus Rectum to the Axis: for it is $a \times BD = AB^2$. And the Spiral is that of *Archimedes*. For suppose $ACFM$ to be his first Spiral, AM the Line of the first Rotation; draw ACG cutting the Circle $MEGM$, which bounds the Spiral, in the Point G : call $AM = r = AG$, $MEGM = c$, $MEG = v$, and retain z and y as formerly. Then from the Description of *Archimedes's* Spiral $z : r :: v : c$, but $z : r :: y : v$, therefore, by equating the two Values of v , deduced from these two Proportions, you have $\frac{cz}{r} = \frac{ry}{z}$, or $\frac{cz^2}{r^2} = y$: which is the same with the assumed Equation $\frac{z^2}{a} = y$, by supposing $a = \frac{r^2}{c}$, that is a third Proportional to the Circumference and Radius of the Circle $MEGM$ bounding the first Spiral: which therefore is the Latus Rectum of the Parabola AED , whose external Area ABD is equal to the external spiral Space $ABCA$.

Moreover, if you suppose the right Line AD drawn, the parabolical Segment $AEDA$ is equal to the internal spiral Area $AcCA$: for

the whole circular Sector $ABCA = \frac{AB \times AC}{2} = \frac{AB \times BD}{2} = \triangle ABD$: but the external spiral Space $ABCcA = ABDEA$, therefore the internal Space $AcCA = AEDA$. From which, as likewise from $\frac{1}{3}zy$ denoting the spiral Space $ABCcA$, as was demonstrated, it follows that $ABCcA = \frac{2}{3}$ Sector $ABCA$, and $AcCA = \frac{1}{3}$ of the same: that the whole internal spiral Area $AcCMA = \frac{1}{3}$ Circle $MEGM$, and the external Part lying without that, and within the Circumference of the Circle $MEGM$ is $\frac{2}{3}$ of the same Circle. And by a like way of reasoning it may be shewn, that if the right Line AM be supposed to be produced indefinitely, and by revolving to describe Spirals of the second, third, fourth, &c. Revolutions the Proportions of the several spiral Spaces, belonging to the several Revolutions, may be shewn to be what *Archimedes* has demonstrated in his *Treatise de Lineis Spiralibus*. Moreover, the spiral Space $AcCA$ is to the whole spiral Space $AcCMA$, as the circular Sector $ABCA$, to the whole Circle $MEGM$, that is in a Ratio compounded of the Sector $ABCA$ to the Sector $AMEGA$, and the Sector $AMEGA$ to the whole Circle $MEGM$; or of AC^2 to AG^2 and MEG to $MEGM$, or AC to AG , that is the Ratio of AC^{cub} to AM^{cub} , as *Pappus* has demonstrated[†].

Ex. 2. Let the Nature of the Spiral be expressed by this Equation $\frac{z^3}{a} = y^2$, or $\frac{z^{\frac{3}{2}}}{\sqrt{a}} = y$: whence the spiral Area $AcCBA = \frac{2}{3} \frac{z^{\frac{3}{2}}}{\sqrt{a}} = \frac{2}{3}zy = ABDEA$: where the Curve AED to whose Area the spiral Space is equal, is a Parabola of the second Kind: and the external spiral Space $AcCBA$ is $\frac{4}{5}$ of the circular Sector $ABCA$; and therefore the internal $AcCA$ is $\frac{1}{5}$ thereof: and the whole $AcCMA$ is $\frac{1}{5}$ of the whole Circle $MEGM$.



Ex. 3. Let AcC be a Spiral described from the Pole A , of such a Nature that if the right Line AB (called the first Radius) be drawn, and from the Center A , with any Distance AB , the Arch BC of a given Length, be described, it's Extremity C may always touch the Spiral CcA . In

this Case calling $AB = z$ and $BC = y$ as before; the general Equation $y^m = a^p z^q$, becomes $y = az^0$ or $y = a$, and therefore the Area is

[†] Pappi Collect. Mathem. Prop. 22. Lib. 4.

$$az = yz.$$

$ax = yz$. Wherefore if the right Line HA be drawn perpendicular to AB, so that AH be equal to BC; and through H, HI parallel to AB; HI shall be an Assymptote to the Spiral: and if through B, b , you draw BD, bd , perpendicular to AB, meeting HI in B, b , the spiral Space BCcb = Rectangle BDdb. Moreover let AC intersect bc in the Point f , then, if from BCcb = $Bb \times bc$, you take away BCfb = $Bb \times \frac{bc + bf}{2}$, there remains Cfc = $Bb \times \frac{fc}{2}$, to which add Afc = $Ab \times \frac{fc}{2}$, and you have the spiral Sector ACc = $\frac{AB \times fc}{2}$: therefore when the Abscisses AB, Ab are given, the spiral Sector ACc is found. This Spiral is called the reciprocal Spiral, with respect to the common Spiral of Archimedes: because, as in that of Archimedes, the Radius BC is proportional to the Angle BAC, which it makes with the first Radius, called the Beginning of the Circulation; so in this other, BC is reciprocally proportional to the Angle BAC, as easily appears.

Ex. 4. But if the Spiral CcA be of such a Nature, that the Arch BC be reciprocally proportional to AB, or $BC : bc :: Ab : AB$: then the Equation to it is $y = ax^{-1}$, which plainly appears to be an Equation to the Hyperbola, reckoning the Absciss x from the Center of the Hyperbola upon one of the Assymptotes, and the Ordinate drawn from the Hyperbola upon the Assymptote, parallel to the other Assymptote. Therefore, suppose the Point b such, that $bc = Ab$; complete the Square Ad; and with the Center A and principal Vertex d , describe an equilateral Hyperbola, to which AB, AH are the Assymptotes: draw the Ordinates BD, bd , then it is evident, the spiral Area BCcb is equal to the hyperbolical Area BDdb, the right Line dDI in the former Example, becoming an Hyperbola in this. This Spiral continually approaches to AB as an Assymptote. Moreover, since it is $AB : Ab :: bc : BC$, the Sectors ABC, Abc are always equal, hence it appears that the spiral Sector ACc = BCcb. Whence it follows that BCcb and ACc are as the Logarithm of the Ratio of AC to Ac.

And these Examples shall suffice to shew after what manner mechanical Curves may be reduced to geometrical Figures, and thereby their Areas determined: which was what we proposed to do in this Problem. I shall next shew after what manner the Areas of Curves may be determined, when they are supposed to be generated by the Revolution of a Radius round some immoveable Point as a Pole or Center.

P R O B L E M.

336. To find the Areas of Curves supposed to be generated by the Revolution of a Radius about a fixt Point.

Although in this Treatise of the Quadrature of Curves, the illustrious Author considers curvilinear Areas as generated by the Motion of a rectangular Ordinate along a Base or Absciss: which is certainly the most natural and convenient Conception of the Production of geometrical Figures; and likewise the most Part of mechanical Ones; in order to determine their Areas by his Method of Fluxions: yet the Method of Fluxions, as applied to the Quadrature of Curves, is not limited to this manner of conceiving the Production of curvilinear Areas; but may be applied to other manners of conceiving their Production or Generation, by Motion variously modified: particularly when they are produced, or supposed to be produced, by the Revolution of a right Line about an immoveable Point. In which Case, even as in the other, to which our Author confines himself, there is some certain Line supposed to flow with an uniform or constant Fluxion; and some Area likewise to flow with a constant Fluxion, which arises from the Multiplication of the Fluxion of the uniformly flowing Line into a given Line: then, the Relation of the Fluxion of the curvilinear Area, described by the Revolution of a Radius, to the Fluxion of the uniformly flowing Area, is investigated, and thence by means of the Equation to the Curve, the Relation of the Fluents is found as in the other Method. But whereas this way of conceiving the Description of Areas, is of very little use, unless in the Case of Spirals, I shall just shew in an Example or two, the Application of it to this sort of Curves.

* Of this
Explica-
tion.

Ex. 1. Let ACM be the Spiral of *Archimedes*, (see the Fig. at Art. 335 *. p. 240.) the Lines AM, ACG as before: so that the Radius ACG revolving about the Center A with an uniform or equable Motion, may with the Point G describe the Circumference of the Circle MEGM, and with the Point C the spiral Line AcCFM. In this Case, it is evident, that both the Circumference and Area of the circular Sector AMEGM flow with a constant Fluxion: and moreover that the Fluxion of the spiral Sector AcCA is to the Fluxion of the circular Sector AMEGM as AC^2 to AG^2 ; for this is the prime Ratio of the nascent Augments of these Sectors. Wherefore, call AG or $AM = a$, $MEGM = c$, $MEG = x$, $AC = y$, the circular Sector = S , the spiral

spiral Sector = s , and you'll have $\dot{S} : \dot{s} :: a^2 : y^2$, or $\dot{s} = \frac{y^2}{a^2} \dot{S}$. But since the Fluxion of MEG or z is $\dot{z}$; or of $1 \times z$ is $1 \times \dot{z}$, thence the Fluxion of the circular Sector is $\frac{az}{2}$ * : substitute this for $\dot{S}$ in the preceding Equation $\dot{s} = \frac{y^2}{a^2} \dot{S}$, and you'll have $\dot{s} = \frac{y^2 \dot{z}}{2a}$, which is the general Formula for the Fluxion of all Spirals. But from the Property of the Spiral, it is $y : a :: z : c$, or $y = \frac{az}{c}$; by substituting of which Value of y in place of it, you have $\dot{s} = \frac{az^2 \dot{z}}{2cc}$: wherefore it is $s = \frac{az^3}{6cc}$ *, * Art. 42. of this Explication, and Form 1. Tab. 1.

the Value of the spiral Sector $AcCA$: and when $z = c$, it is $\frac{ac}{6} = \frac{AM \times MEGM}{6} = \frac{1}{3}$ of the Circle $MEGM = AcCFMA$, as before. Since it is $s = \frac{az^3}{6cc}$, if you substitute for z it's Value, viz. $\frac{y}{a}$, the spiral Sector $AcCA$ will be otherwise expressed thus $\frac{y^3}{6a^2}$: which includes the same things that were demonstrated above.

Ex. 2. Let MFC (see the same Fig. p. 240.) be supposed a Spiral of such a sort, that if from it's Pole A with the given Distance AM, the Circle MGEM be described, and any Radius ACG be drawn, the Sum of the Squares of AC and MG be equal to a given Area, e. g. to the Square of AM. Then putting $AM = a$, $MG = z$, and $AC = y$, it is $y^2 + z^2 = a^2$, or $y^2 = a^2 - z^2$: but the Fluxion of the spiral Sector is $\frac{yy\dot{z}}{2a}$, as was shewn in the preceding Example, in which fluxionary Expression insert $a^2 - z^2$ for y^2 , and it becomes $\frac{az\dot{z}}{2} - \frac{z^3\dot{z}}{2a}$; the Fluent of which, viz. $\frac{az}{2} - \frac{z^3}{6a}$ is the spiral Area sought, viz. $AMFCA$: which Expression consists of two Parts; the first $\frac{az}{2}$ denotes the circular Sector $AMGA$; the other $\frac{z^3}{6a}$, the external spiral Space $MFCGM$. And when $z = a$ or $MG = AM$, the Point C falls into A; and the whole internal spiral Space is $\frac{1}{2}a^2 - \frac{1}{6}a^2 = \frac{1}{3}a^2 = \frac{1}{3}AM^2$.

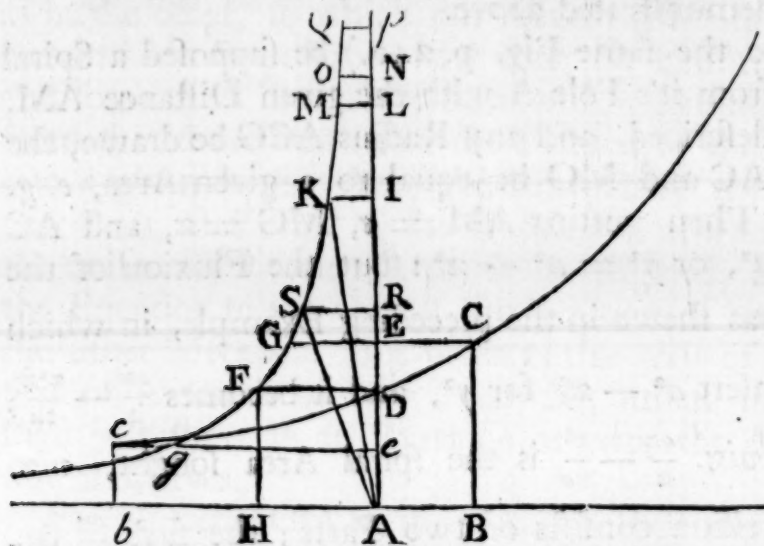
And thus you may feign any Relations of z and y , you please, by which an infinite Variety of Spirals will arise, whose Areas may be determined after the like manner as in these two Examples; or else may be compared with the Areas of Conic Sections, or other the most simple geometrical Curves, they can be compared with.

S E C T.

S E C T. X.

Of the Reduction of the Hyperbola and Ellipse; together with all curvilinear Areas, and Fluents, which depend upon these Conic Sections, to the Measures of Ratios, and Angles; or Logarithms and circular Arches.

337. **D**EF. Let the right Line BC move along the right Line AB in a given Angle, so that while AB increases or decreases with an equable or uniform Velocity, the Ordinate BC increases or decreases with a Velocity always proportional to itself: then AB is called the *Logarithm* of the Ratio of BC to AD; AD being the same with BC when the Points B and A coincide: and the Curve DC which the Point C describes is called the *logarithmical* or *logistical* Curve: because, if the Absciss AB be divided into very small



and equal Parts, and Ordinates drawn thro' all the Points of Division, they will constitute a Series of continual geometrical Proportionals, falling in betwixt BC and AD; so that if the Ratio of BC to AD be considered as compounded or made up of the equal Ratios of all the

adjoining Ordinates beginning with BC and ending with AD, the Absciss AB will express the Number of the equal compounding Ratios, which make up the Ratio of BC to AD: which is the true and original Notion of the Word *Logarithm*. The right Line AB is likewise called the *Measure* of the Ratio of BC to AD. According to which Meaning of the Word Ratio, the Measure of the Duplicate, Triplicate, Quadruplicate, &c. of any Ratio is double, triple, quadruple, &c. and the Measure of the Subduplicate, Subtriplicate, &c. is $\frac{1}{2}$, $\frac{1}{3}$, &c. of the Measure of the simple Ratio: the Measure of the Ratio of Equality is 0; because the Duplicate-triplicate, &c. or Sub-

duplicate-

duplicate-subtriplicate, &c. of a Ratio of Equality is still the same with itself, and therefore according to the former Notion of Ratio, it's Measure must be 0: accordingly when BC coincides with AD, the Absciss AB vanishes. And when the Measure of the Ratio of greater to less is considered as positive; upon the contrary the Measure of the Ratio of less to greater, is negative: for when BC by moving towards AD, and getting upon the opposite Side of it, comes into the Position *bc*, the Absciss AB by diminishing first becomes nothing, from being positive; and afterwards becomes negative, represented by *Ab*, which is negative with respect to AB, and denotes the Measure of the Ratio of *bc* to AD, or it's Logarithm.

338. Let the Parallelogram ADFH be completed; and with the Affymptotes AD, AH, and principal Vertex F, describe the Hyperbola FG: through C draw CEG parallel to AB, cutting AD in E, and the Hyperbola in G, the hyperbolic Area DEGF is proportional to the right Line AB. For, from the known Property of the Hyperbola, EG is reciprocally proportional to AE, that is to the Velocity or Fluxion of $AE = BC$, since BC was supposed to increase, or decrease with a Velocity proportional to itself *: in which Case the Fluxion of the hyperbolic Area DEGF is constant and invariable (as easily follows from Art. 37 *.) but so is the Fluxion of AB also; therefore the Fluents being as the Time in which they are described, these Fluents, viz. the right Line AB and hyperbolic Area DEGF are always proportional. Whence it follows that the Area DEGF is the Logarithm, or Measure, of the Ratio of AE to AD, or DF to EG. And in this Case likewise, as in the preceding, the Measure of the Ratio of Equality is 0, for DEGF vanishes, when the Point E falls in with D; and the Logarithm or Measure of the Ratio of less to greater is negative, for *DeG*F, the Measure of the Ratio of *Ae* to AD, is negative, viz. when the Point E falls in *e* betwixt A and D.

*Art. 337.
of this Ex-
plication.
* Of this
Explica-
tion.

339. If through any two Points K and M of the Hyperbola, the Ordinates KI, ML be drawn, parallel to the Affymptote AH, the Area IKML, will be the Logarithm or Measure of the Ratio of AL to AI or IK to LM: for if you take $AE : AD :: AL : AI$, and suppose the Fluxions of AE, AL, as formerly to be proportional to these Lines, the Lines DE, IL; and consequently the Areas DFGE, IKML, are generated in equal Times; and therefore, since the Fluxions with which the Augments DFGE, IKML are described, are likewise constantly the same * these Augments are equal: but DFGE is the Logarithm or Measure of the Ratio of AE to AD, therefore IKML is likewise the Measure of that Ratio, i. e. of AL to AI or IK to LM.

*Art. 338.
above.

340. Hence

340. Hence follows the fundamental Property upon which logarithmical Arithmetic is founded, *viz.* that the Logarithm of any Ratio made up or compounded of other Ratios is equal to the Sum of the Logarithms of the compounding Ratios: and the Logarithm of a Ratio, which arises by subducting ^{*} one Ratio from another is equal to the Difference of their Logarithms. For let the Ratio of AR to AD be compounded of the Ratios of AP to AN and AL to AI; through R, P, N draw the Ordinates RS, PQ, NO. Take the Point E such that it may be $AP : AN :: AR : AE$; then must it be $AL : AI :: AE : AD$, otherwise the Ratio of AR to AD would not be compounded of the Ratios of AP to AN, and AL to AI. Wherefore it appears, by the last Article, that the Sum of the two Areas NOPQ, IKML is equal to the Area DFSR: the first of which is the Sum of the Logarithms of the compounding Ratios, and the last is the Logarithm of the compounded Ratio, and the rest easily follows. Whence it follows that the Sum of the Logarithms of two Numbers is equal to the Logarithm of their Product, and the Difference of the Logarithms of two Numbers is the Logarithm of their Quotient, &c. For what is ordinarily called the Logarithm of any Number, is, properly speaking, the Logarithm of the Ratio of the Number to Unity.

341. If BC, and consequently AD, be perpendicular to AB, the Hyperbola FG is rectangular: in which Case, if you call $AD = a$, a constant Quantity, $DE = x$, and $EG = y$, two variable Quantities, then, from the Property of the Hyperbola $a + x \times y = a^2$, or $y = \frac{a^2}{a+x}$: but the Fluxion of DFGE, that is of the Logarithm, or Measure of the Ratio of AE to AD, or $a + x$ to a , is $\dot{x}y = \frac{a^2 \dot{x}}{a+x}$, so that if $AD = (a =) 1$, the Logarithm or Measure of the Ratio of $1 + x$ to 1 is $\frac{\dot{x}}{1+x} =$ (by Division) $\dot{x} - \dot{x}x + \dot{x}x^2 - \dot{x}x^3 + \dot{x}x^4 - \text{\&c. in inf.}$ all the Terms of which being squared, you'll have $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \text{\&c. in inf.}$ for the Value of the hyperbolical Area DFGE; and consequently of the Logarithm, or

^{*} By subducting of one Ratio from another, I mean the Reverse of that which is called compounding of Ratios, agreeable to the Way in which we consider the Doctrine of Ratios in this Place. Thus, if the Ratio of A to B were compounded or made up of the Ratio of C to D, and of E to F; which is when $A : B :: C \times E : D \times F$; so contrarily the Ratio of E to F subducted from the Ratio of A to B, leaves the Ratio of C to D, *viz.* when $\frac{A}{E} : \frac{B}{F} :: C : D$. This last Operation might be called Resolution of Ratios, being the opposite of Composition of Ratios. But it is all one how it be termed, if the thing itself be understood.

Measure of the Ratio of $1 + x$ to 1 : which is commonly called it's hyperbolical Logarithm, to distinguish it from the common or tabular Logarithm of that Ratio, commonly called simply the Logarithm of $1 + x$. By the same way of reasoning, the Area $DFge$ will be found to be $-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \text{Ec. in inf.}$ by supposing $De = -x^*$: which therefore is the Logarithm of the Ratio of $1 - x$ to 1 : or, which is the same, $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{Ec. in inf.}$ is the Logarithm of the Ratio of AD to Ae , or 1 to $1 - x$. Consequently if it be $De = DE$, the Logarithm of the Ratio of AE to Ae or $1 + x$ to $1 - x$, viz. $EGge$, is equal to the Sum of $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{Ec.}$ and $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{Ec.}$ that is $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7, \text{Ec. in inf.} = 2 \times x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \text{Ec. in inf.}$ which Series converges twice as fast as the former ones: and affords one of the most convenient and expeditious Methods of constructing and computing a Table of Logarithms. If the Difference betwixt the two Areas $DFGE$ and $DFge$ be sought, subtract the lesser $DFGE = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{Ec.}$ from the greater $DFge = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{Ec.}$ and there remains $x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \frac{1}{4}x^8 + \text{Ec.} = DFge - DFGE$.

* Art. 190, 191. of this Explication.

* Art. 340. of this Explication.

* Art. 191. of this Explication.

Now if you suppose $DG = Dg = 0.1$, that is $AG = 1.1$ and $Ag = 0.9$: by substituting 0.1 for x in the Series $2 \times x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \text{Ec.}$ you'll find the Area $EGge = 0.200670695462, \text{Ec.}^*$: which is therefore the hyperbolical Logarithm, or Measure of the Ratio of 1.1 to 0.9 or 11 to 9 , in this particular System. And by substituting 0.1 for x in the other Series $x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \frac{1}{4}x^8 + \text{Ec.}$ you'll obtain $0.0100503358, \text{Ec.}$ for the Value of $DFge - DFGE$: wherefore by adding this Number to and subtracting it from the former, the Half of the Sum, viz. $0.1053605156 = DFge$; and the Half of the Difference, viz. $0.0953101798, \text{Ec.} = DFGE$; the first of which is the Logarithm of the Ratio of 1.1 to 1 ; and the other the Logarithm of the Ratio of 1 to 0.9 . After the same manner by supposing $DG = 0.2 = Dg$, you'll find the Logarithm of the Ratio of 1.2 to 0.8 , or 12 to 8 , or 3 to 2 to be $0.4054651081, \text{Ec.}$ of the Ratio 1.2 to 1 to be $0.2231435513, \text{Ec.}$ and of 1 to 0.8 to be $0.1823215567, \text{Ec.}$ And thus by supposing $DE = De = 0.01, 0.02, \text{Ec. } 0.001, 0.002, \text{Ec.}$ you may investigate the Logarithms of other Ratios, in the same

* The Tables of Logarithms, which are in common use, are called *Briggs's* Logarithms, to distinguish them from that Form of Logarithms first invented by Lord *Neper*: which last are the same with those which I have mentioned under the Name of hyperbolical Logarithms. The Difference betwixt the two Forms will appear in what follows.

manner: by the Addition of which Logarithms, their Subtraction, Multiplication and Division, all other Logarithms belonging to this System may be found most conveniently and expeditiously: particularly since $\frac{2}{0.8} \times \frac{2}{0.8} = \frac{4}{0.64} = 2$, and $\frac{2}{1} \times \frac{1.2}{0.8} = \frac{2.4}{0.8} = 3$, and $\frac{2}{1} \times \frac{2}{0.8} = \frac{4}{0.8} = 5$, and $\frac{2}{1} \times \frac{5}{1} = \frac{10}{1} = 10$: hence you'll find by Addition of the Logarithms formerly found, agreeably to what was said in the preceding Article, the Logarithms of the Numbers 2, 3, 5, 10; or rather of their Ratios to Unity, to be 0.6931471805, &c. 1.0986122886, &c. 1.6094379124, &c. and 2.30258509299, &c. And since the Logarithm of 10, or of the decuple Ratio in the common or *Brigg's* Tables is 1, hence you see what Proportion the Logarithm or Measure of the decuple Ratio in the hyperbolical Logarithms, bears to the Logarithm of that same Ratio in the common System of Logarithms, viz. that of 2.30258509299, &c. to 1: and the same is the Proportion that the hyperbolical Logarithm of any other Ratio bears to the tabular Logarithm of that same Ratio: whence the hyperbolical Logarithms being found, in the Manner just now shewn, the tabular Logarithms may thence be found, and contrarily: more of which by and by.

342. Schol. As there may be an infinite Variety of different Hyperbolas, hence it appears that there may be an infinite Variety of different Systems or Scales of Logarithms, or Measures of Ratios: and what Proportion the Logarithms or Measures of the same Ratio, in the different Systems, bear to each other, will appear by what follows.

P R O P.

343. If from any two Points K and S of an Hyperbola FSK (see the foregoing Fig. p. 246.) the Semi-diameters KA, SA be drawn, and the two Ordinates KI, SR, as before, the hyperbolical Sector KAS is equal to the Area KIRS; and consequently, is the Logarithm or Measure of the Ratio of AI to AR, or RS to IK, in that particular System.

For the two Triangles ARS, AIK are equal, since the Angles at R and I are equal, and the Sides about these equal Angles reciprocally proportional: wherefore, if from the Area ASKIA, these Triangles be subtracted separately, there remains KIRS = KAS. Wherefore also KAS is the Logarithm or Measure of the Ratio of AI to AR, or RS to IK *.

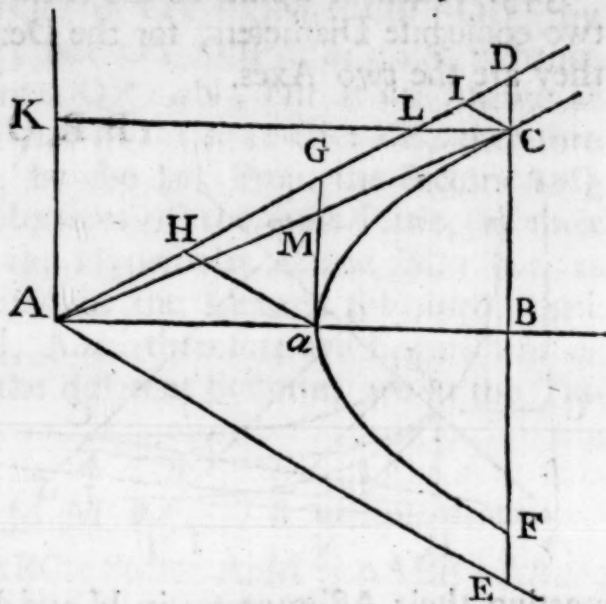
*Art. 338.
of this Ex-
plication.

P R O P.

344. Let *aC* be an Hyperbola describ'd with the Centre A, semi-transverse Aa, and Assymptotes AD, AE: through any Point of the Hyperbola

Hyperbola C draw the Ordinate CBF to the transverse Axis, cutting it in B, the Curve in C and F, and the Affymptotes in D and E; draw likewise CLK an Ordinate to the conjugate AK, meeting the Affymptote AD in L; draw

also the Semi-diameter AC: let aMG be a Tangent at a meeting AC, AD in M and G: and through a and C draw aH , CI parallel to the Affymptote AE, and meeting the other Affymptote AD in H and I: then I say the Area $aHIC$, or Sector ACa , is the Measure of the Ratio of FD to aG or aG to CD; or $KC + KL$ to Aa or Aa to $KC - KL$; or $BD + BC$ to $\sqrt{BD^2 - BC^2}$ or aG



+ aM to $\sqrt{aG^2 - aM^2}$; or $KC + KL$ to $\sqrt{KC^2 - KL^2}$. For from parallel Lines, $aH : CI :: aG : CD$. Again, from the Property of the Hyperbola, $aG : CD :: CE = DF : aG$. Likewise, from similar Triangles AaH , LCI , it is $aH : CI :: Aa : LC$; and from the Property of the Hyperbola $Aa : LC :: KC + KL : Aa$. Since then all these Ratios are equal to the Ratio of aH to CI , which the Area $aHIC$ or Sector ACa is the Measure of: hence it appears that $aHIC$ or ACa is the Measure of any of these Ratios, which have been mentioned, in the System belonging to the given Hyperbola. And as to the other Ratios mentioned in the Prop. viz. these of $BD + BC$ to $\sqrt{BD^2 - BC^2}$ &c. since it is $BD + BC$, aG , and $BD - BC$, proportional; hence $BD + BC = FD : aG :: \sqrt{BD + BC} : \sqrt{BD - BC} :: BD + BC : \sqrt{BD^2 - BC^2}$: hence it appears that the Ratio of $BD + BC$ to $\sqrt{BD^2 - BC^2}$ is the same with the Ratio of FD to aG . And by the same way it may be shewn that the Ratio of $KC + KL$ to $\sqrt{KC^2 - KL^2}$ is the same with that of $KC + KL$ to Aa . Moreover it is $BD : BC :: aG : aM$, therefore it follows that $BD + BC : \sqrt{BD^2 - BC^2} :: aG + aM : \sqrt{aG^2 - aM^2}$. Whence it follows, that $aHIC$ or AaC is likewise the Measure of the Ratio of $BD + BC$ to

Again, in comparing the two Hyperbolas aK , aI together, you'll have, from the Property of the Hyperbola, $OT^2 - OM^2 : OP^2 - OQ^2 :: (ac^2 : aN^2 ::) OT^2 : OP^2$; and by Alternation, Inversion, and Division, $OT^2 : OM^2 :: OP^2 : OQ^2$; or $OT : OM :: OP : OQ$; by Division, $OT : OP :: MT : QP$; but $OT : OP :: ac : aN$, whence $MT : QP :: ac : aN$, or $MT : ac :: QP : aN$; but it was shewn at the End of the last Paragraph, that $MT : ac :: CS : BE$, therefore $QP : aN :: CS : BE$: wherefore, by the last Prop. the Sectors AaQ and ABC are Logarithms or Measures of the same Ratio, in their respective Systems, belonging to the Hyperbolas aI and BC : but, as has been said, and is demonstrated in the Lemma subjoined, these Sectors are as the Triangles AaN , Aac : therefore the Logarithms or Measures of the same Ratio, in the different Systems, are as the Triangles AaN , Aac . Q. E. D.

L E M M A.

347. It was said that Sector $ABC : \text{Sector } AaM :: \triangle ABE : \triangle Aac$; and that Sector $AaM : \text{Sector } AaQ :: \triangle Aac : \triangle AaN$; and consequently, by Equality, that Sector $ABC : \text{Sector } AaQ :: \triangle ABE : \triangle AaN$. Which things appear thus.

Because the transverse and conjugate Axes of the two Hyperbolas BC and aK are in the same Ratio, therefore, from the Property of the Ordinates, $AR^2 - AB^2 : AO^2 :: Aa^2 :: (RC^2 : OM^2 ::) AR^2 : AO^2$; or by Division, $AB^2 : Aa^2 :: AR^2 : AO^2 :: AC^2 : AM^2$; but $F.ABC : F.AaM :: AC^2 : AM^2$, since this is the Ratio of their nascent Augments, and $AC^2 : AM^2 :: AB^2 : Aa^2 :: \triangle ABE : \triangle Aac$; therefore $F.ABC : F.AaM :: \triangle ABE : \triangle Aac$, which is a given or constant Ratio, hence it follows that Sector $ABC : \text{Sector } AaM^* :: \triangle ABE : \triangle Aac^*$ * Art. 40. of this Explication.

Again it is demonstrated, as in the Prop. itself, that $OM : OQ :: ac : aN :: \triangle Aac : \triangle AaN$, which is a given Ratio; and that being the Ratio of the Fluxions of the hyperbolic Areas, or half Segments aOM , aOQ , hence these Areas themselves are in the same Ratio: but the Triangles AOM , AOQ are in that Ratio, consequently the hyperbolic Sectors AaM , AaQ , which are their Differences, are in the same Ratio, viz. of the Triangles Aac , AaN . By comparing of which with the Proportion at the End of the last Paragraph, viz. Sector $ABC : \text{Sector } AaM :: \triangle ABE : \triangle Aac$, it is, by Equality, Sector $ABC : \text{Sector } AaQ :: \triangle ABE : \triangle Aac$. Q. E. D.

348. Def. The Triangle AaN is called by the learned and ingenious Mr. Cotes, the *Modulus* of the System, belonging to the Hyperbola.

bola aI ; and the like in other Hyperbolas. It is the same thing to take the Parallelogram $AHaG$, inscribed betwixt the Hyperbola, and it's Assymptotes AF , AG ; for $AN = 2AH$: yea, and although Aa be not the Axis, but any other Diameter, the Triangle AaN and Parallelogram $AHaG$ are always equal to one another, and to the Parallelogram $AHaG$ inscribed at the Vertex Principalis: therefore any of them will be the Modulus. Whence it follows that the Logarithm or Measure of any Ratio, in one System, or Hyperbola, is to the Logarithm or Measure of the same Ratio in another System; as the Modulus of the first, to the Modulus of the other.

* Of this
Explica-
tion.

349. Wherefore, by the Help of what was demonstrated Art. 341*, the Modulus of the common System of Logarithms, is easily found: for supposing AaI to be an equilateral Hyperbola, in which $AH = 1$ or $Aa = \sqrt{2} = aN$, the Modulus is 1, and the Measure of the decuple Ratio, as was there shewn, is 2.302585092994, &c. and the Logarithm of the Number 10, *i. e.* of the decuple Ratio in *Brigg's* Logarithms, is 1: therefore a third Porportional to 2.30258, &c. and 1, that is $\frac{1}{2.30258509, \&c.} = 0.434294481903, \&c.$ is the Modulus of the common System of Logarithms.

* Of this
Explica-
tion.

350. From what has been said, it likewise appears, that the hyperbolical Logarithm of any Ratio being found by Art. 341*, in which the Modulus is 1, that Logarithm multiplied by the Modulus of any other System, will give the Logarithm of that same Ratio, in that other System. Moreover, that, if the tabular Logarithm of any given Ratio be called l ; and M denote the Modulus of any other System, and in the Measure of the given Ratio in that System, then it will be $M = \frac{m \times 0.43429, \&c.}{l}$: and $m = \frac{M \times l}{0.43429, \&c.}$ or $M \times l \times 2.30258509, \&c.$ since the Measure of any given Ratio is always as the Modulus of the System to which it belongs.

* Art. 348.
of this Ex-
plication.

* Of this
Explica-
tion.

+ Of this
Explica-
tion.

351. Def. The Ratio, of which the Modulus in any System, is the Measure, is called, by the same ingenious Author, the *Ratio Modularis*. And it is evident, that this Ratio is the same in every System: for the Measure of any given Ratio is as the Modulus of the System*. Therefore, if in the forementioned equilateral Hyperbola, you find the Ratio whose Measure is 1, that shall be the Ratio Modularis. In order to which (see Fig. Art. 337*. p. 246.) if you put $AD = 1$, and $DE = x = De$, then by Art. 341+ you have the Logarithm or Measure of the Ratio of AE to $Ae = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7, \&c.$ which Series expresses the Value of the Area EGe . Where-

Wherefore, if you put that Series equal to Unity, and by the Method of Reversion of Series (taught by our Author ¹) find the Value of $x = DE$ or De , by a converging Series, and add that Value of x to $AD = 1$, and subtract it from AD ; the first will give AE the Antecedent, and the second Ae the Consequent of the Ratio Modularis, it's Measure being 1, which is here the Modulus. You might find the same Ratio by putting $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4, \&c. = 1$; or $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4, \&c. = 1$; and the Value of x , resulting from the first added to $AD = 1$, will give the Antecedent of the Ratio Modularis to AD or 1, the Consequent; and the Value of x resulting from the other, subtracted from 1, gives the Consequent of that Ratio, to the Antecedent 1: as appears from Art. 341 ^{*}, and what hath been said: but the former Series converging twice as fast as these two last, is to be preferred. By which means it will be found, that the Ratio Modularis is that of 2.71828181828459, $\&c.$ to 1, or 1 to 0.367879441171, $\&c.$ the first of which Numbers has 0.43429448, $\&c.$ it's Log. in the common Tables.

^{*} Of this Explanation.

352. And here by the by you see the Solution of this Problem, From the Logarithm of a Number or Ratio given, to find the Number or Ratio: which is the Converse of the Prob. treated of Art. 341 ^{*}.

^{*} Of this Explanation.

353. According to the Notation made use of by Mr. Cotes, let $M \left| \frac{A}{C} \right|$ denote the Measure of the Ratio of A to C, in the System whose Modulus is M; A being the Antecedent, and C the Consequent: then if n denote any Number, integral or fractional, you shall have $M \left| \frac{A}{C} \right|$

$= nM \left| \frac{A^{\frac{1}{n}}}{C^{\frac{1}{n}}} \right|$, and $M \left| \frac{A}{C} \right| = \frac{M \left| A^n \right|}{n \left| C^n \right|}$. For $M \left| \frac{A}{C} \right| : nM \left| \frac{A}{C} \right| :: 1 : n$ ^{*}; and

^{*} Art. 348. of this Explanation.

$nM \left| \frac{A}{C} \right| : nM \left| \frac{A^{\frac{1}{n}}}{C^{\frac{1}{n}}} \right| :: n : 1$ ^{*}. Therefore *ex æquo* $M \left| \frac{A}{C} \right| : nM \left| \frac{A^{\frac{1}{n}}}{C^{\frac{1}{n}}} \right| :: 1 : 1$.

^{*} Art. 337. of this Explanation.

And the same is the Reasoning in the other Case.

354. Hence it follows (see Fig. Art. 344 ^{*}. p. 251.) that the Area $aHICa$, or Sector $AaC = \frac{1}{2} \Delta AaG \left| \frac{EC}{CD} \right|$ or $\frac{1}{2} \Delta AaG \left| \frac{BD+BC}{BD-BC} \right|$: likewise

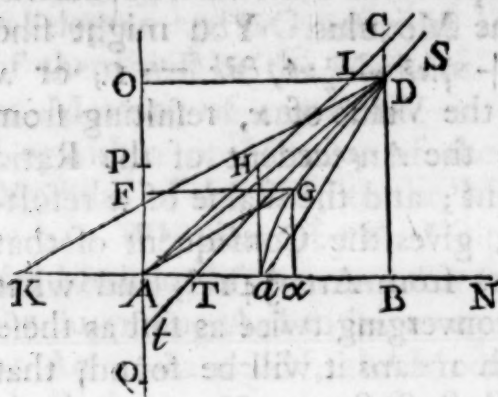
^{*} Of this Explanation.

$aHICa$, or $AaC = \frac{1}{2} \Delta AaG \left| \frac{KC+KL}{KC-KL} \right|$. For $EC : aG : CD ::$, and also $KC + KL : Aa : KC - KL ::$, from the Property of the Hyperbola: hence the thing is evident by comparing the last Article with Art. 348 ^{*}.

^{*} Of this Explanation.

¹ See Analysis by means of Equations infinite in Number of Terms, Sect. Invention of the Base from the Area given, likewise our Author's Letter to Mr. Oldenburg, 24 Octob. 1676.

355. Schol. What is principally to be remarked here is, that by means of a logarithmical Canon, any hyperbolic Space may be found. For let $aGDS$ be any conical Hyperbola, described with the Center A , transverse and conjugate Axes Ka , PQ ; and Affymptote AC . Draw the Ordinates DB , DO , meeting the Axes in B and O , and the Affymptote in C and I : likewise from any other Point G the Ordinates Ga , GF , meeting the transverse and conjugate Axes in the Points a and F : at a draw the Tangent aH , meeting the Affymptote in H : join AG , AD , KD ; and let DTt be a Tangent at D , meeting the transverse Axis in T , and Conjugate in t .



Then it appears from what has been said, that the hyperbolic Spaces $ADGa$, $aGDB$, $aGDB$, $aGDa$, $aGDT$, $aGDK$, $AODGa$, $FODG$, &c. may be found by logarithmical Canon. For the Sector $ADGa$ is the Measure of the Ratio of aH to DC , as it is variously expressed Art. 344*, in the System whose Modulus is the Triangle AaH , and therefore if the Logarithm of that Ratio be taken from the logarithmical Canon, the Modulus of the logarithmical Canon is the Logarithm of the Ratio of aH to DC ; as the Triangle AaH is to the Sector $ADGa$ *, which therefore is found. The same way is the Sector AGa found. And therefore, since all the other hyperbolic Areas are made up of these and known right-lined Figures, they are found.

* Of this
Explica-
tion.

* Art. 350.
of this Ex-
plication.

356. Wherefore, if the Lines belonging to the hyperbolic Areas, be expressed by Numbers, these Areas are most conveniently found by a common Table of Logarithms: which is a System of the Measures of Ratios expressed by Numbers (for there may be a great Variety of Systems, in which these Measures may be expressed by any other Kind of Quantities). Thus in this Figure, suppose $Ka = 1$. $aH = 0.4$. $aB = 0.5$: so that the Triangle $AaH = 0.1$, the Modulus of the System. And because $Aa = 0.5$: $aH = 0.4$:: $AB = 1$: BC , therefore $BC = 0.8$: again, since $aH^2 = BC^2 - BD^2$, therefore $BD = (\sqrt{BC^2 - aH^2} = \sqrt{0.64 - 0.16} =) 0.6928203$, &c. and therefore $DC = (BC - BD = 0.8 - 0.6928203 =) 0.1071796$, &c. Wherefore the Sector $AaGD = 0.1 \times 2.30258509 \times \text{Log. Ratio of } 0.4 \text{ to } 0.1071796$. Now the Logarithm of any Ratio taken from a Table of Logarithms, is found by subtracting the Logarithm of the Consequent from the Logarithm of the Antecedent: for if N and n denote

denote any two Numbers, then $N : n :: N : 1 + 1 : n$; but the Logarithm of N to 1 , is that which is called the Logarithm of N ; and the Logarithm of 1 to n is the opposite of the Logarithm of n to 1 , that is $-\text{Log. } n$: so that the Logarithm of N to n , is $\text{Log. } N - \text{Log. } n$ *. Therefore Sector $AaGD = 0.1 \times 2.30258509 \times \text{Log. } 0.4$ *Art. 340.
 $= 0.1071796 = 0.2302585$, &c. $\times 1.6020600 = 1.0301121$ of this Ex.
 $= 0.2302585 \times 0.5719479 = 0.131696$, &c. = Sector $AaGD$ nearly. plication.
 Subtract this from $\triangle ABD = \frac{AB \times BD}{2} = 0.346410$, &c. remains
 $0.214714 = aGDB$. Subtract the same Sector from $\triangle ADa =$
 $\frac{Aa \times DB}{2} = 0.1732050$, remains $0.041509 =$ Segment $aGDa$.
 Again, to the Sector $AaGD$ add that $\triangle ADa = \triangle ADK$, the Sum is
 $0.304901 = aGDK$. Then for the Space $aGDT$, from the Pro-
 perty of the Tangent, you have $AB = 1 : Aa = 0.5 :: Aa : AT =$
 0.25 . Therefore $\triangle ADT = \frac{AT \times BD}{2} = 0.0866020$; which subtract-
 ing from the Sector $AaGD = 0.131696$, remains $0.045094 = aGDT$.
 Further to the Sector $AaGD$ add $\triangle AOD = \frac{AB \times BD}{2} = 0.346410$,
 the Sum is $0.478106 = \text{Area } AODGa$. And it is abundantly evident,
 how, by the same Method, the Sector aGA may be found, and from
 thence the Areas aGa and $AFGa$: which being subtracted from $aGDB$
 and $AODGa$, you shall thereby find $aGDB$ and $FGDO$. Which were
 the things I proposed to shew.

357. Hence it appears that the Areas of all those Curves, which
 are capable of being compared with the Hyperbola, may be found,
 by the Measures of Ratios, or by the Help of a Table of Logarithms:
 and therefore also, all such Fluxions of any kind, whose Fluxions are
 analogous to the Fluxions of these Curves. More of which afterwards.

P R O P.

358. Every elliptical Area may be reduced to a circular Area: and
 to be found by means of the trigonometrical Canon, and the known
 Proportion of the Radius of a Circle to the Circumference.

Let aGS be any given Ellipse, described with the Center A , trans-
 verse and conjugate Axes Ka , PH : let AD be any Radius of the El-
 lipse and $aGDA$ a Sector adjacent to the transverse Axe, and DPA a
 Sector adjacent to the shorter Axe: with the Radius Aa , describe the
 quadrantal Arch aQR , meeting AP produced, in R : through D draw
 the Ordinate DB , meeting the transverse Axis in B , and produce it
 till it cut the Circle in Q , join AQ . I say the elliptical Sector $aGDA$:

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circular

circular Sector $aQA :: BD : BQ :: AP : Aa$; and therefore Sector $aGDA = \frac{1}{2}AP \times aQ$. And Sector $DPA = \frac{1}{2}AP \times QR$. For from the Property of the Ordinates $BD : BQ :: AP : AR$; therefore $aGDB : aQB :: AP : AR$, since their Fluxions are in that given Ratio: but $\Delta ABD : \Delta ABQ :: BD : BQ :: AP : AR$; therefore $aGDA : aQA :: AP : AR :: BD : BQ$; wherefore it is $aGDA = \frac{aQA \times AP}{AR} = \frac{1}{2}AP \times aQ$.

And thence it follows that Sector $DPA = \frac{1}{2}AP \times QR$.

PROP.

259. If with the Radius AP (see the preceding Figure) you describe the quadrantal Arch Pqr , meeting the longer Axe in r , and the Ordinate Db , belonging to the shorter Axe, in the Point q , the Sector $DPA = \frac{1}{2}Aa \times Pq$, and the Sector $aGDA = \frac{1}{2}Aa \times qr$. It is demonstrated the same way as the preceding.

360. Since it is $bD : bq :: (Aa : Ar ::) BQ : BD$, and by Division $bD = AB : qD :: DB : QD$, hence it appears that the Points A, q, Q lie in the same right Line: so that the circular Sectors RQA and PqA ; aQA and rqA ; and Arches RQ and Pq , aQ and rq , in this and the preceding Proposition, are similar. Moreover, if the Tangent aC be drawn at the Point a , and AqQ be produced till it meet it in C , the Arch aQ is such an Arch as has Aa for it's Radius, aC for it's Tangent and AC for it's Secant: which are proportional to AB, BQ and AQ or Aa . And it is evident that the like Construction may be made for the Circle Pqr ; and that the Arches RQ and Pq , are such that Ab, bq and Aq or AP are as the Radius, Tangent and Secant of these Arches.

361. Hence it appears that as the hyperbolical Sectors are Measures of Ratios, so the elliptical Sectors are Measures of Angles: thus the elliptical Sector $aGDA$ is the Measure of the Angle aAQ : for since $aGDA = \frac{1}{2}AP \times aQ$, it appears that $aGDA$ increases and diminishes

minishes in the same Ratio the Arch aQ doth, that is in the same Ratio the Angle aAQ doth. But as the Measure of any given Ratio is not sufficiently determined, unless you know to what System it belongs: which may be known from the Modulus (as was explained at full Length) so neither can the Measure of a given Angle be known, unless you know to what System of Measures of Angles it belongs. Therefore it is necessary to consider the Measure of some certain determinate Angle, as a Modulus, to which the Measures of all other Angles in the System, may be referred and compared, and thereby determined. Now there may be conceived various Kinds of Measures of Angles: Arches of Circles, the angular Point being at the Center, make one System of Measures: circular Sectors make another.

The Measures in the first Case being Lines, in the other plane Surfaces; but both increase and decrease in the same Ratio with the Angles*. There may be different Modulus's: but the most convenient in the Systems of Arches, upon several Accounts, particularly because of the *Analogy* betwixt the Measures of Ratios and Angles, and the *natural* Passage from the one to the other, is the Radius. Which being established for the Modulus in this Kind, the Half of the Square of the Radius will be the Modulus in the Systems of circular Sectors: for the Radius bears the same Ratio to an Arch, which subtends any given Angle, as half the Square of the Radius bears to the circular Sector, which has the same Arch for it's Base; and is the Measure of the same given Angle in the System of circular Sectors: which is the Property and Notion of a Modulus.

* Eucl.
Elem. lib.
6. Prop.
33.

362. Hence it follows that in a System of elliptical Sectors, considered as the Measures of Angles, the Modulus is half the Rectangle contained under the semitransverse and semiconjugate Axes, *i. e.* (in the foregoing Fig.) if you suppose the right Line Pa joined, the Triangle AaP : which is the same with the Modulus of the System of Measures of Ratios, belonging to an Hyperbola, having the same Axes with the Ellipse. For $\frac{AP \times Aa}{2} : \frac{AR \times Aa}{2} :: aGDA : aQA$; which two last are the Measures of the same Angle aAQ in the respective Systems: and therefore since $\frac{AR \times Aa}{2}$ is the Modulus of the one System, $\frac{AP \times Aa}{2}$ will be the Modulus of the other System.

363. From what has been said it appears, that, in a given Ellipse aP , whose transverse and conjugate Axes are known, if moreover the Ratio of Aa to aC or to AC , which are the Tangent and Secant of the Angle aAQ , in the Circle whose Radius is Aa , be given, the Arch

L 1 2

aQ ,

aQ , and elliptical Sector $aGDA$ may be found. Now the Ratio of Aa to aC , or, which is the same, of AB to BQ is given, when AB is given, for $BQ = \sqrt{Aa^2 - AB^2}$: so that the Sector $aGDA$ is the Measure of an Angle whereof the Radius, Tangent and Secant¹ are Aa , aC and AC ; or AB , BQ and Aa , the Modulus being $\frac{AP \times Aa}{2}$. And it is evident from Ab given, by the like means the other elliptical Sector PDA is found, by supposing a Tangent to the Circle Pqr , drawn at the Point P , and produced till it meet AqC . For it is the Measure of an Angle, whose Radius, Tangent, and Secant are as Ab , bq and AP , the Modulus being the same as before, viz. $\frac{AP \times Aa}{2}$.

364. Hence it appears, that, when the Lengths of the given Lines are expressed by Numbers, the Number of Degrees, Minutes, &c. contained in the Arches aQ and Pq may be found, with great ease, by means of the trigonometrical Canon: and consequently the elliptical Sectors will be found, from the known Ratio of the Radius of a Circle to it's Circumference, whereby the Length of an Arch of a known Number of Degrees, Minutes, &c. with respect to the Length of the Radius is determin'd: thus, in the preceding Fig. p. 258^{*}. $aGDA = \frac{1}{2}AP \times aQ$ ^{*}, and $PDA = \frac{1}{2}AP \times RQ$ ^{*}. And therefore, in this Case, the Areas of any other elliptical Spaces may be found. That is, if you join KD , draw the Tangent DT meeting the Axis in T , and the Ordinate Ga , you may find the Areas of the elliptical Spaces $aGDB$, $aDGa$, $aGDT$, $aGDK$, $AbDGa$, PbD , $ABDP$, $aGDB$, &c. after the same Manner as was shewn formerly, in the Case of the Hyperbola: since all these elliptical Spaces are made up of elliptical Sectors and known right-lined Figures, by Addition or Subtraction.

Of this
ica-
rt. 258.
of this Ex-
plication.

Ex. Let us suppose $Ka = 1$, $HP = 0.8$, $AB = 0.25$. Then $AB = 0.25$: $AQ = 0.5$:: Rad. : Sec. $\angle aAQ$; whence Sec. $\angle aAQ = 2 \times$ Rad. wherefore add the Log. of 2 to the logarithmical Rad. i. e. 0.3010300 to 10.0000000, and the Sum 10.3010300 will be found in the Table of artificial Secants, opposite to an Angle of 60° : therefore the Arch aQ is an Arch of 60° , i. e. $\frac{1}{6}$ of the whole Circumference. But the Circumference of a Circle is to the Rad. as 6.283185307179, &c. to 1: therefore that Number divided by 6, quotes 1.047197551196, &c. the Length of $\frac{1}{6}$ of the Periphery of a Circle having it's Radius 1: and since it is $Aa = \frac{1}{2}$, if you divide the former Number by 2, you'll

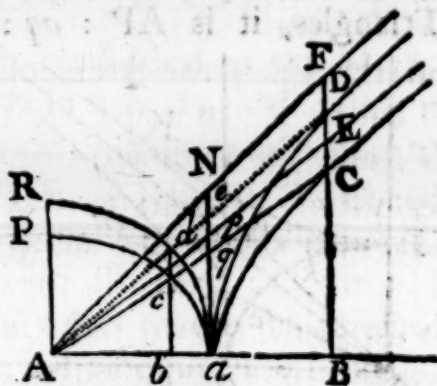
¹ When in this Place, or any other, I mention the Radius of an Angle, in conjunction with the Tangent and Secant, I mean, as is evident, the Radius, Tangent and Secant of a circular Arch which measures that Angle.

have 0.523598775598 the Length of aQ : which multiplied by $\frac{1}{2}AP = 0.2$, gives 0.1047197551196 = $aGDA$ nearly.

Again $aK^2 = 1 : HP^2 = 0.64 :: KB \times Ba = 0.1875 : BD^2$; therefore $BD = \sqrt{0.64 \times 0.1875} = 0.346410161513$, &c. and therefore the Triangle $ABD = \frac{AB \times BD}{2} = 0.043301270189$, which subtracted from $aGDA = 0.104719755119$, leaves $0.06141848493 = aGDB$ nearly. And we need not insist particularly, in shewing how all the other elliptical Areas mentioned, are deduced after the same Manner: after what has been done already, in the like Case of the hyperbolic Areas.

365. And from what has been said, with respect to the Ellipse, it appears, that the Areas of all such Curves as are capable of being compared geometrically with the Ellipse: and universally, all Fluents, whose Fluxions are analogous to the Fluxions of such curvilinear Areas, or are expressed by the same algebraical Quantities, may be found, and exhibited, by the Measures of Angles, *i. e.* by circular Arches, or Sectors, in a linear Description: or, by means of the trigonometrical Canon, (when the Values of the Quantities are given in Numbers) in an arithmetical Expression.

366. And now, it will be proper to shew the Analogy there is betwixt the Measures of Ratios, and the Measures of Angles, as they belong to the Hyperbola and Ellipse. In order to which let aC and aP be an Hyperbola and Ellipse, described with the Center A , and the same semitransverse and semi-conjugate Axes, Aa , AP ; and let AE be an Assymptote to the Hyperbola aC :



draw the right Line AaC , and from C , c , the Lines CB , cb , perpendicular to the Axis Aa , at the Points B , b : at the Vertex a , draw the Tangent apq , cutting AC , AE in q and p . I say the hyperbolic Sector $AaC = \frac{Aa \times AP}{2} \left| \frac{AP + aq}{\sqrt{AP^2 - aq^2}} \right|$: and the elliptical Sector $Aac = \frac{Aa \times AP}{2}$

$\left| \frac{AP \cdot aq}{\sqrt{AP^2 + aq^2}} \right|$: by the last of which two Notations, I mean the Measure of an Angle, whose Radius, Tangent, and Secant are AP , aq , and $\sqrt{AP^2 + aq^2}$, to the Modulus $\frac{Aa \times AP}{2}$: where it must be observed that of the two Quantities as $AP \cdot aq$ the first always designs the Radius and

and the second the Tangent. For if with the Center A, and Semiaxis Aa, the equilateral Hyperbola aD, and quadrantal Arch aR be described; the Asymptote ANF to the Hyperbola aD be drawn: and if BC be produced till it meet the Hyperbola aD in D, and the Asymptotes AE, AF, in E and F; and bc be produced till it meet the Circle in the Point d; and moreover aqp, produc'd, if need be, till it meet the Asymptote AF in N: then, because it is $bc : bd :: (AP = ap : AR = aN :: BE : BF ::) BC : BD$ *, therefore the Points A, d, D, lye in the same right Line. Let that right Line AdD be joined, and intersect aN, in the Point o.

* Art. 346.
of this Ex-
plication.

* Art. 344.
of this Ex-
plication.

* Art. 363.
of this Ex-
plication.

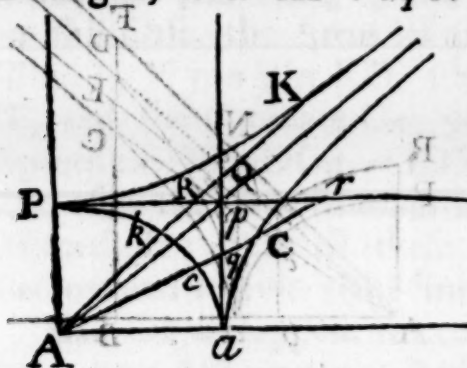
Now the hyperbolic Sector $AaC = \frac{Aa \times AP}{2} \left| \frac{AP + aq}{\sqrt{AP^2 - aq^2}} \right|$ * for AP

$= ap$. Again the elliptical Sector $Aac = \frac{Aa \times AP}{2} \left| \frac{Aa \cdot ao}{Ao} \right| = \frac{Aa \times AP}{2}$

$\left| \frac{aN \cdot ao}{\sqrt{aN^2 + ao^2}} \right| =$ (because $aN : ao :: BF : BD :: BE : BC :: ap = AP : aq$)

$\frac{Aa \times AP}{2} \left| \frac{AP : aq}{\sqrt{AP^2 + aq^2}} \right|$, as was to be shewn.

367. Cor. I. Let Pp be joined and produced indefinitely towards r, and AcC produc'd, if need be, meet it in r: then, from similar Triangles, it is $AP : aq :: Pr : Aa$; therefore it follows that the



same hyperbolic Sector $AaC = \frac{Aa \times AP}{2}$

$\left| \frac{Pr + Aa}{\sqrt{Pr^2 - Aa^2}} \right|$: and the elliptical Sector $Aac =$

$\frac{Aa \times AP}{2} \left| \frac{Pr \cdot Aa}{\sqrt{Pr^2 + Aa^2}} \right|$. And when the Points

q and r by approaching to the Point p, coincide with it, the hyperbolic Sector AaC becomes infinite, and so doth the

Measure of the Ratio; for the Consequent of the Ratio vanishes. But the elliptical Sector Aac becomes the Measure of an Angle of 45° , because AP and aq or Pr and Aa, that is the Radius and Tangent became equal. And if q and r appear upon the opposite Side of p, viz.

in Q and R, the hyperbolic Sector AaC becomes $\frac{Aa \times AP}{2} \left| \frac{AP + aQ}{\sqrt{AP^2 - aQ^2}} \right|$

or $\frac{Aa \times AP}{2} \left| \frac{PR + Aa}{\sqrt{PR^2 - Aa^2}} \right|$ both impossible, as including negative Squares.

But the elliptical Sector, now $Aak = \frac{Aa \times AP}{2} \left| \frac{AP \cdot aQ}{\sqrt{AP^2 + aQ^2}} \right|$, or $\frac{Aa \times AP}{2}$

$\left| \frac{PR \cdot Aa}{\sqrt{PR^2 + Aa^2}} \right|$

368. Cor. 2. In this last Case, viz. when the Points q and r fall in Q and R , with the semitransverse and semiconjugate Axes AP , Aa , describe PK a conjugate Hyperbola to the former aC ; then, by the same way of reasoning as before, the hyperbolical Sector $APK = \frac{Aa \times AP}{2} \left| \frac{Aa + PR}{\sqrt{Aa^2 - PR^2}} \right|$; or $\frac{Aa \times AP}{2} \left| \frac{aQ + AP}{\sqrt{aQ^2 - AP^2}} \right|$; and the elliptical Sector $APk = \frac{Aa \times AP}{2} \left| \frac{Aa - PR}{\sqrt{Aa^2 + PR^2}} \right|$; or $\frac{Aa \times AP}{2} \left| \frac{aQ - AP}{\sqrt{aQ^2 + AP^2}} \right|$.

369. Cor. 3. By the same way of reasoning it appears, that the hyperbolical Sector $AaC = \frac{Aa \times AP}{2} \left| \frac{Ar + Aq}{\sqrt{Ar^2 - Aq^2}} \right|$; and the elliptical Sector

$$Aac = \frac{Aa \times AP}{2} \left| \frac{Ar \cdot Aq}{\sqrt{Ar^2 + Aq^2}} \right|. \text{ And } APK = \frac{Aa \times AP}{2} \left| \frac{AQ + AR}{\sqrt{AQ^2 - AR^2}} \right| \text{ and } APk = \frac{Aa \times AP}{2} \left| \frac{AQ \cdot AR}{\sqrt{AQ^2 + AR^2}} \right|$$

370. Schol. Because the Measure of any Ratio or Angle is increased and diminished in the same Proportion as the Modulus is *, hence it appears that the Expressions $\frac{Aa \times AP}{2} \left| \frac{AP + aq}{\sqrt{AP^2 - aq^2}} \right|$, and $\frac{Aa \times AP}{2} \left| \frac{AP \cdot aq}{\sqrt{AP^2 + aq^2}} \right|$,

* Art. 350, 361. of this Explication.

are the same thing as $\frac{1}{2} Aa \times AP \left| \frac{AP + aq}{\sqrt{AP^2 - aq^2}} \right|$ and $\frac{1}{2} Aa \times AP \left| \frac{AP \cdot aq}{\sqrt{AP^2 + aq^2}} \right|$; in which last two Expressions, AP only is considered as the Modulus, and the Measure of the Ratio, or Angle, to that Modulus, is multiplied by $\frac{1}{2} Aa$; and so in other like Cases. Moreover it may be observed, that the Terms $AP \cdot aq \cdot \sqrt{AP^2 \pm aq^2}$ may be multiplied or divided by any Quantity, without altering the Measure of the Ratio or Angle.

371. I have insisted the longer upon this Affair of the Measures of Ratios and Angles, as it is connected with hyperbolical and elliptical Areas, that I might thereby shew the Connection there is betwixt our Author's Method of reducing curvilinear Areas, and consequently other analogous Fluents, to hyperbolical, elliptical and circular Areas; and the Method of the ingenious Mr. Cotes, by which he reduces such Fluents, (with many other besides) to the Measures of Ratios and Angles, in his *Harmonia Mensuratum*, published and enlarged by the learned Dr. Smith. So that what I have said, may serve as an Introduction to the young Geometrieian, to the reading and understanding Mr. Cotes's Method, in the Book just now mentioned. For a further Commentary upon which, the Reader may consult a Performanee entitled *Epistola ad Amicum de Cotesii Inventis*.

372. And

372. And now I shall illustrate this Connection of the two different Methods by an Example. Take the Curve belonging to Form third Species first of Sir Isaac Newton's second Table, defined by this Equation $\frac{d}{z}\sqrt{e+fx^n}=y$: the Quadrature of which was demonstrated, and explained at full Length already, according to our Author's Method *. The analogous Fluxion and Fluent are found in Form third of Mr. Cotes's first Table, viz. $dxz^{-1}\sqrt{e+fx^n}$, and $\frac{2}{n}dP - \frac{2}{n}dR \left| \frac{R+T}{S} \right|$: where the Values of P. R. T. S. are $P = \sqrt{e+fx^n}$, $R = \sqrt{e}$, $T = \sqrt{e+fx^n}$, $S = \sqrt{fx^n}$.

* Art. 276
-279.
of this Ex-
plication.

* Of this
Explica-
tion.

Now by what was demonstrated Art. 276 *, if (see Fig. 2. Tab. 2. p. 25.) with the Center A, and half transverse Axis $AP = \sqrt{f}$ and half Parameter $\frac{\sqrt{f}}{e}$; you describe the Hyperbola PGDS; and, upon the conjugate Axis AN, you take $AB = (x =) \frac{1}{\sqrt{x^n}}$, draw the Ordinate BD, and the Tangent DT meeting AB in T: then the Area of the Curve sought is $\frac{4de}{nf} \times TDB - APDB$. Join AD, and $\frac{1}{2}$ the conjugate Axis is $\sqrt{\frac{f}{e}}$; therefore if to $AO = \sqrt{f+ex^2}$, $OD = x$, and $AP = \sqrt{f}$, you find a fourth Proportional, viz. $\frac{x\sqrt{f}}{\sqrt{f+ex^2}} =$ (by inserting $\frac{1}{\sqrt{x^n}}$ for x) $\sqrt{\frac{f}{e+fx^n}}$: then you'll have, by what has been

* Art. 366.
of this Ex-
plication.

demonstrated *, the Sector $APGD = \frac{f}{2\sqrt{e}} \left| \frac{\sqrt{\frac{f}{e}} + \sqrt{\frac{f}{e+fx^n}}}{\sqrt{\frac{f}{e}} - \sqrt{\frac{f}{e+fx^n}}} \right| =$ (by

reducing) $\frac{f}{2\sqrt{e}} \left| \frac{\sqrt{e} + \sqrt{e+fx^n}}{\sqrt{fx^n}} \right|$: to which if you add the Triangle $ADB = \frac{x\sqrt{f+ex^2}}{2} = \frac{x^{-n}\sqrt{e+fx^n}}{2}$, the Sum $\frac{x^{-n}\sqrt{e+fx^n}}{2} + \frac{f}{2\sqrt{e}} \left| \frac{\sqrt{e} + \sqrt{e+fx^n}}{\sqrt{fx^n}} \right| = APDB$. But the Triangle TDB was shewn to be equal to $\frac{v^3}{2ex} = \frac{f+ex^2}{2ex} = \frac{f+ex^2\sqrt{f+ex^2}}{2ex} = \frac{f+ex^{-n}\sqrt{e+fx^n}}{2e}$: therefore $TDB - APDB = \frac{f+ex^{-n}\sqrt{e+fx^n}}{2e} - \frac{x^{-n}\sqrt{e+fx^n}}{2} - \frac{f}{2\sqrt{e}} \left| \frac{\sqrt{e} + \sqrt{e+fx^n}}{\sqrt{fx^n}} \right| = \frac{f}{2e}\sqrt{e+fx^n} - \frac{f}{2e}\sqrt{e} \left| \frac{\sqrt{e} + \sqrt{e+fx^n}}{\sqrt{fx^n}} \right| = \frac{f}{2e}P - \frac{f}{2e}R \left| \frac{R+T}{S} \right|$

$\left| \frac{R+T}{S} \right|$: wherefore $\frac{4de}{\eta f} \times \overline{TDB} - \overline{APDB} = \frac{2d}{\eta} P - \frac{2d}{\eta} R \left| \frac{R+T}{S} \right|$: as it is in Mr. Cotes's Table.

373. But if f be negative, so that it be $\frac{d}{z} \sqrt{e - fz^n} = y$: then by what was shewn at Art. 277 *, (see Fig. 3. Tab. 2. p. 25.) if, with the Center A, half Transverse $Aa = \sqrt{\frac{f}{e}}$ and half Conjugate $= \sqrt{f}$, * Of this Explication.

you describe the Hyperbola $aGDS$; take $AB = (x =) \frac{1}{\sqrt{z^n}}$, draw the Ordinate BD, the Line AD, and Tangent DT, then the Area of the Curve sought is equal to $-\frac{4de}{\eta f} \times aGDT$. But, by what has been demonstrated Art. 366 *, if to $AB = x$, $BD = v = \sqrt{-f + ex^2}$, and * Of this Explication.

$Aa = \sqrt{\frac{f}{e}}$, you find a fourth Proportional, viz. $\frac{\sqrt{-\frac{ff}{e} + fx^2}}{x}$.

$= \sqrt{\frac{ef - ffz^n}{e}}$, then you shall have the hyperbolical Sector $AaGD =$

$\frac{f}{2\sqrt{e}} \left| \frac{\sqrt{f} + \sqrt{\frac{ef - ffz^n}{e}}}{\sqrt{fz^n}} \right| = \frac{f}{2e} \sqrt{e} \left| \frac{\sqrt{e} + \sqrt{e - fz^n}}{\sqrt{fz^n}} \right|$: from which subtract the

Triangle ADT $= \left(\frac{AD \times BD}{2} = \frac{BA - BT}{2} \times BD = \frac{x - \frac{v^2}{ex}}{2} \times \sqrt{-f + ex^2} = \right.$

$\left. x + \frac{f - ex^2}{ex} \times \frac{\sqrt{-f + ex^2}}{2} = \right) \frac{f}{2e} \sqrt{e} + fz^n$, there remains $aGDT =$

$\frac{f}{2e} \sqrt{e} \left| \frac{\sqrt{e} + \sqrt{e - fz^n}}{\sqrt{fz^n}} \right| - \frac{f}{2e} \sqrt{e} + fz^n = \frac{f}{2e} R \left| \frac{R+T}{S} \right| - \frac{f}{2e} P$; wherefore $-\frac{4de}{\eta f} \times aGDT = \frac{2}{\eta} dP - \frac{2}{\eta} dR \left| \frac{R+T}{S} \right|$, the same as in Mr. Cotes's Table.

374. Finally, if it be $\frac{d}{z} \sqrt{-e + fz^n} = y$: then, by what was demonstrated Art. 278 * (see Fig. 4. Tab. 2. p. 25.) if with the Center A, Semitransverse $Aa = \sqrt{\frac{f}{e}}$, and Semiconjugate $AP = \sqrt{f}$ (the same as in the preceding Figure) the Ellipse $aGDS$ be described; the Radius AD, and Tangent DT be drawn; and you take $AB = (x =) \frac{1}{\sqrt{z^n}}$, the Area of the Curve sought is $\frac{4de}{\eta f} \times aGDT$. But (by what

has been demonstrated Art. 366 *.) if to $AB = x$, $BD = \sqrt{f - ex^2}$, * Of this Explication.
M m and

and $Aa = \sqrt{\frac{f}{e}}$, you find a fourth Proportional, viz. $\frac{\sqrt{\frac{f^2}{e} - fx^2}}{x} = \sqrt{\frac{-ef + ffz^n}{e}}$, the elliptical Sector $aGDA = \frac{f}{2\sqrt{e}} \left| \frac{\sqrt{f} \cdot \sqrt{\frac{-ef + ffz^n}{e}}}{\sqrt{\frac{f^2}{e} - fx^2}} \right| = \frac{f}{2e} \sqrt{e} \left| \frac{\sqrt{e} \cdot \sqrt{-e + fz^n}}{\sqrt{fz^n}} \right|$: which being subtracted from the Triangle ADT $= \left(\frac{AT \times BD}{2} = \frac{\frac{f}{ex} \times \sqrt{f - ex^2}}{2} = \right) \frac{f}{2e} \sqrt{-e + fz^n}$, there remains $\frac{f}{2e} \sqrt{-e + fz^n} - \frac{f}{2e} \sqrt{e} \left| \frac{\sqrt{e} \cdot \sqrt{-e + fz^n}}{\sqrt{fz^n}} \right| = \frac{f}{2e} P - \frac{f}{2e} R \left| \frac{R \cdot T}{S} \right|$: wherefore $\frac{4de}{\eta f} \times aGDT = \frac{2}{\eta} dP - \frac{2}{\eta} dR \left| \frac{R \cdot T}{S} \right|$: the same as in Mr. Cotes's Table.

375. Schol. It may be observed that the same Notation, making allowance for the Difference of the Signs, serves all the Cases, in this Example: and the same happens in all others: when the Fluents are expressed by the Measures of Ratios, and Angles. In the last Case, where the Area is expressed by the elliptical Area; and the Fluent by the Measure of an Angle, we might have taken AB for the Radius, and $\sqrt{Aa^2 - AB^2}$, for the Tangent of the Angle to the same

Modulus as before: and so it would have been $\frac{f}{2\sqrt{e}} \left| \frac{x \cdot \sqrt{\frac{f}{e} - x^2}}{\sqrt{\frac{f}{e}}} \right| =$

$\frac{f}{2\sqrt{e}} \left| \frac{\sqrt{e} \cdot \sqrt{-e + fz^n}}{\sqrt{fz^n}} \right|$, for the Value of the Sector $aGDA$, as before. It may likewise be observed that the two Hyperbolas are conjugate to one another, the transverse Axis of the one being the conjugate Axis of the other; and the conjugate Axis the same with the transverse: and the Ellipse has the same Axes with the Hyperbolas; so as to be that which is inscribed within the conjugate Hyperbolas, as represented by the Fig. at Art. 367*.

* Of this
Explica-
tion.

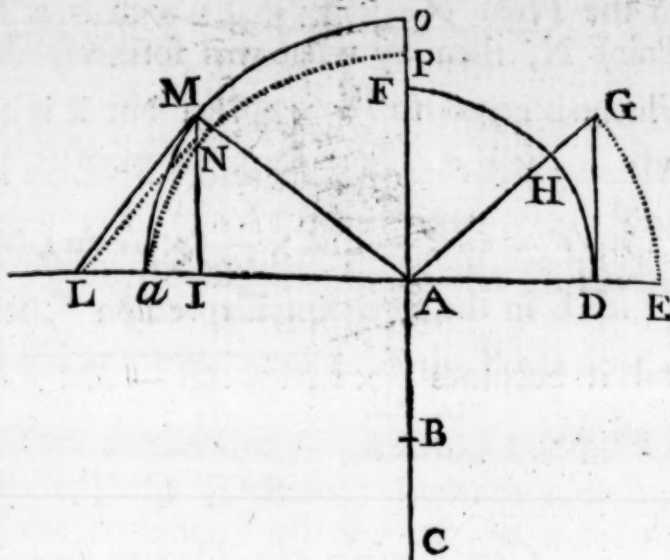
* Of this
Explica-
tion.

Moreover the Expressions for the Measures of the Ratios and Angles might have been deduced otherwise, by considering what was shewn Art. 367, 368, 369*.

376. If the Area of the Curve whose Equation is $\frac{d}{z} \sqrt{-e + fz^n}$

$= y$; or the analogous Fluent of the Fluxion $d\dot{z}z^{-1} \sqrt{-e + fz^n}$, be required in a linear Description by means of a Circle, it may be thus. Take some Quantity for linear Unity in the Construction, by which let

let the Dimensions of the several Terms be reduced to the just Number, by Multiplication and Division thereby: suppose it be f . Then take $AP = f$, and upon AP produced beyond A , take $AB = e$, and $AC = z^n$; and having described upon PB and PC as Diameters, two Semicircles; through A draw ADE perpendicular to AP , meeting the Semicircles in D and E : with the Radius AD describe the Quadrant DHF , draw the Tangent DG ; and with the Radius AE describe the Arch EG cutting the Tangent in G , join AG : and the Area, or Fluent required is equal to $\frac{2d}{n} \times DG - DH$.



For it is evident by the Construction, that $AD = \sqrt{e}$ and $AE = \sqrt{fz^n} = AG$, and therefore $DG = \sqrt{-e + fz^n}$: whence it follows that the Arch $DH = \sqrt{e} \left| \frac{\sqrt{e} \cdot \sqrt{-e + fz^n}}{\sqrt{fz^n}} \right|$; and $DG - DH = \sqrt{-e + fz^n} - \sqrt{e} \left| \frac{\sqrt{e} \cdot \sqrt{-e + fz^n}}{\sqrt{fz^n}} \right| = P - R \left| \frac{R \cdot T}{S} \right|$; therefore $\frac{2d}{n} \times DG - DH = \frac{2}{n} dP - \frac{2}{n} dR \left| \frac{R \cdot T}{S} \right|$, as it ought to be *.

*Art. 374. of this Explanation.

377. The Construction may be made otherwise thus. Supposing as before $AP = f = 1$, $AC = z^n$, $AB = e$, $AE = \sqrt{fz^n} = \sqrt{z^n}$, and $AD = \sqrt{e}$: produce EA towards L , upon which take AI a third Proportional to $AE = \sqrt{z^n}$ and $AP = 1$, so is $AI = \left(\frac{1}{z^n}\right) \times$ (see Table 2. Form 3. Quad. p. 25. and take Aa a third Proportional to $AD = \sqrt{e}$ and $AP = 1$, so is $Aa = \frac{1}{\sqrt{e}}$. With the Radius Aa describe the Quadrant aMO : through I draw IM perpendicular to Aa , cutting the Quadrant in M : draw the Radius AM , and the Tangent ML meeting Aa produced in L , then the Area of the Curve whose Ordinate is $\frac{d}{z} \sqrt{-e + fz^n}$, or the analogous Fluent whose Fluxion is $\frac{dz}{z} \sqrt{-e + fz^n}$ is $\frac{2d}{n} \times LM \times AB - aM \times AB$. For with the Semi-

M m 2

axes

*Art. 278.
of this Ex-
plication.

axes Aa , AP , describe the Quadrant of the Ellipse aNP , cutting IM in the Point N ; join NL , which is a Tangent to the Ellipse at the Point N , then by what was formerly demonstrated *, the Area or Fluent is equal to $\frac{4de}{nf} \times aNL$; but it is $aNL : aML :: (IN : IM ::)$ $AP = 1 : Aa = \frac{1}{\sqrt{e}}$, therefore $aNL = aML \times \sqrt{e} = \overline{LM - aM} \times \frac{Aa}{2} \times \sqrt{e} = \overline{LM - aM} \times \frac{1}{2\sqrt{e}} \times \sqrt{e} = \overline{LM - aM} \times \frac{1}{2}$: substitute this for aNL in the foregoing Expression $\frac{4de}{nf} \times aNL$, 1 for f , and AB for e , and it becomes $\frac{2d}{n} \times \overline{LM} \times AB - aM \times AB$ for the Area or Fluent required.

S E C T. XI.

Containing the Demonstration of Prop. 11.

378. **S**upposing the same Things as are supposed by our Author in this Proposition, (which see with the Figure belonging to it) only let us further suppose that the Areas of the Curves ADB , AEB , AFB , &c. are represented by α , β , γ , &c. and consequently that the Ordinates BE , BF , BG , &c. are $\frac{\alpha}{1}$, $\frac{\beta}{1}$, $\frac{\gamma}{1}$, &c. Then I say, if the Areas α , β , γ , &c. be terminated at the whole given Absciss $AC = t$, and at the Ordinate CI given in Position and infinitely produced, *i. e.* when they are the Areas $ADIC$, $AEKC$, $AFLC$, &c. and the Areas A , B , C , &c. be terminated at the same Absciss and Ordinate, then it shall be

$$1^{\circ}. ADIC = A.$$

$$2^{\circ}. AEKC = tA - B.$$

$$3^{\circ}. AFLC = \frac{t^2 A - 2tB + C}{2}.$$

$$4^{\circ}. AGMC = \frac{t^3 A - 3t^2 B + 3tC - D}{6}.$$

$$5^{\circ}. AHNC = \frac{t^4 A - 4t^3 B + 6t^2 C - 4tD + E}{24}.$$

&c.

That is to say universally, if ϵ represent the Area of any Curve in the Series α , β , γ , &c. at any Distance of Place from the first α ; and such indefinite Distance be called d , then, when ϵ and A , B , C , &c. are terminated at the whole given Absciss AC , and infinite Ordinate CI , it shall be

$\epsilon =$

$$\epsilon = \frac{t^d A - dt^{d-1} B + d \times \frac{d-1}{2} t^{d-2} C - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3} D + \&c.}{d \times d-1 \times d-2 \&c.}$$

Where the numeral Coefficients of the Terms in the Numerator, viz. 1, $-d$, $d \times \frac{d-1}{2}$ &c. are the same with those which belong to the Terms of a Residual, such as $a - b$, raised to the Power whose Exponent is d , i. e. of $a - b$ when actually involved: the Exponent of the Power of t in the first Term is d , in the second, $d - 1$, in the third, $d - 2$, &c. the same as the Exponents of the Powers of a the first Member of the Residual, and A, B, C, &c. are Factors of the first, second, third, &c. Terms respectively. And the Factors by whose Multiplication the Denominator is formed, diminish each by Unity from the first d , and are continued till the last be equal to Unity; or which is the same, till their Number be equal to d .

379. To demonstrate which, I shall prove that, when the Areas A, B, C, D, &c. and ϵ , have any common Absciss $z = AB$, the Relation of the Areas shall be expressed after the same manner, viz. thus.

$$\epsilon = \frac{z^d A - dz^{d-1} B + d \times \frac{d-1}{2} z^{d-2} C - d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-3} D + \&c.}{d \times d-1 \times d-2 \&c.}$$

For by Proposition 1, the Fluxion of

$$\frac{z^d A - dz^{d-1} B + d \times \frac{d-1}{2} z^{d-2} C - d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-3} D + \&c.}{d \times d-1 \times d-2 \&c.}$$

$$\begin{aligned} \text{is } & \frac{z^{d-1} \dot{z} A}{d-1 \times d-2 \&c.} + \frac{z^d \dot{A}}{d \times d-1 \times d-2 \&c.} \\ & - \frac{d-1 \times z^{d-2} \dot{z} B}{d-1 \times d-2 \&c.} - \frac{dz^{d-1} \dot{B}}{d \times d-1 \times d-2 \&c.} \\ & + \frac{d-1 \times \frac{d-2}{2} z^{d-3} \dot{z} C}{d-1 \times d-2 \&c.} + \frac{d \times \frac{d-1}{2} z^{d-2} \dot{C}}{d \times d-1 \times d-2 \&c.} \\ & - \frac{d-1 \times \frac{d-2}{2} \times \frac{d-3}{3} z^{d-4} \dot{z} D}{d-1 \times d-2 \&c.} - \frac{d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-3} \dot{D}}{d \times d-1 \times d-2 \&c.} \\ & \&c. \qquad \&c. \end{aligned}$$

But $\dot{A} = \dot{z}y$. $\dot{B} = (\dot{z}zy =) z\dot{A}$. $\dot{C} = (\dot{z}z^2y =) z^2\dot{A}$. $\dot{D} = (\dot{z}z^3y =) z^3\dot{A}$, &c. which Values of $\dot{B}$, $\dot{C}$, $\dot{D}$, &c. being put for them, in the

The Quadrature of CURVES explained.

in the Terms which constitute the second Column, it becomes

$$\frac{z^d \dot{A} - dz^d \dot{A} + d \times \frac{d-1}{2} z^{d-1} \dot{A} - d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-2} \dot{A} + \&c.}{d \times d-1 \times d-2 \&c.} \quad \text{or} \quad \frac{z^d \dot{A}}{d \times d-1 \times d-2 \&c.}$$

$\times 1 - d + d \times \frac{d-1}{2} - d \times \frac{d-1}{2} \times \frac{d-2}{3} + \&c.$ where the last Factor $1 - d + d \times \frac{d-1}{2} - \&c.$ being the numeral Coefficients of a Residual raised to the Power whose Exponent is d , is therefore equal to nothing: consequently all the Terms constituting the second Column, taken together are equal to nothing. Wherefore the Terms constituting the first Column, make up the Fluxion of the Quantity

$$\frac{z^d A - dz^{d-1} B + d \times \frac{d-1}{2} z^{d-2} C - d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-3} D + \&c.}{d \times d-1 \times d-2 \&c.}$$

Which therefore is

$$\dot{z} \times \frac{z^{d-1} A - \frac{d-1}{2} z^{d-2} B + \frac{d-2}{3} z^{d-3} C - \&c.}{d-1 \times d-2 \&c.}$$

Wherefore, if the first of these express the Area of a Curve whose Absciss is z , the second will express the Fluxion of that Area: and therefore, if it be divided by $\dot{z}$, it will express the Value of the Ordinate of that Curve. In which last Expression, the numeral Coefficients of the Terms in the Numerator are the same with those of the Power of a Residual, whose Exponent is $d-1$: the Indexes of the Powers of z are $d-1$, $d-2$, $d-3$, &c. and A , B , C , &c. are Factors of the first, second, third, &c. Terms respectively. Whence it appears that the Number of Terms in this Expression of the Ordinate, is one less than in the Expression of the corresponding Area: and further, the Denominator of the last Expression is equal to the Denominator of the former divided by d .

Moreover if d be 1; so that the Area of the Curve be $zA - B$, the Fluxion is $\dot{z}A + A\dot{z} - \dot{B} = (\dot{z}A + \dot{z}zy - \dot{z}zy) = \dot{z}A$, and consequently the Ordinate is A , by considering of which things it appears that in the Series

$$A \cdot zA - B \cdot \frac{z^2 A - 2zB + C}{2} \cdot \frac{z^3 A - 3z^2 B + 3zC - D}{6} \cdot \frac{z^4 A - 4z^3 B + 6z^2 C - 4zD + E}{24} \&c. \text{ in inf.}$$

If you take any two next adjoining Terms, the first of the two being the Ordinate of a Curve whose Absciss is z , the last of the two will

will be the Area. Or, which is the same, if you consider all the Terms as Areas of Curves, having the common Absciss z ; and suppose $\dot{z}$ or $\dot{z} \times 1 = 1$, then each Area will be the Fluxion, or, if you please, the Exponent of the Fluxion of the one immediately following. But $\alpha . \beta . \gamma . \delta$ &c. is a Series of Areas having the same Absciss, and Relation as the former: and the first Area A is equal to the first Area α , since they have the same Absciss z and Ordinate y : therefore all the succeeding Terms of the one Series are equal to all the succeeding Terms of the other, each to each. Now suppose the Absciss z to become equal to AC or t , then $\alpha, \beta, \gamma, \delta$, &c. become $ADIC, AEKC, AFLC$, &c. and therefore it will be

$$1^{\circ}. ADIC = A.$$

$$2^{\circ}. AEKC = tA - B.$$

$$3^{\circ}. AFLC = \frac{t^2A - 2tB + C}{2}.$$

$$4^{\circ}. AGMC = \frac{t^3A - 3t^2B + 3tC - D}{6}.$$

$$5^{\circ}. AHNC = \frac{t^4A - 4t^3B + 6t^2C - 4tD + E}{24}.$$

&c.

This was the first thing to be demonstrated.

380. The second thing to be demonstrated is, that, if $x = t - z$, and $P . Q . R . S$, &c. be a Series of Areas having x for their common Absciss, and y, xy, x^2y, x^3y , &c. for their Ordinates respectively, when the Areas $P . Q . R . S$ &c. as well as $A . B . C . D$ &c. are terminated at the whole given Absciss $AC = t$, and infinite Ordinate CI , it shall be

$$P = A.$$

$$Q = tA - B.$$

$$R = t^2A - 2tB + C.$$

$$S = t^3A - 3t^2B + 3tC - D.$$

&c. in inf.

Or universally, if d denote the Distance of Place of any Area from the first P , in the Series of Areas $P . Q . R . S$ &c. that Area shall be equal to

$$t^dA - dt^{d-1}B + d \times \frac{d-1}{2} t^{d-2}C - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}D, \text{ \&c.}$$

the same Series as that mentioned in the preceding Part of the Proposition, only wanting the Denominator.

For the Curve belonging to the Series $P . Q . R . S$ &c. which is at the Distance d from the first, has x^dy for it's Ordinate: but $x = t - z$,
or

or $x^d = (t - z)^d = t^d - dt^{d-1}z + d \times \frac{d-1}{2} t^{d-2}z^2 - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}z^3 + \&c.$ therefore $x^dy = t^dy - dt^{d-1}zy + d \times \frac{d-1}{2} t^{d-2}z^2y - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}z^3y + \&c.$

Now the Ordinates $y, zy, z^2y, \&c.$ belong to the Areas A, B, C, $\&c.$ consequently, since t is a given Quantity, the Area belonging to the compounded Ordinate $t^dy - dt^{d-1}zy + d \times \frac{d-1}{2} t^{d-2}z^2y - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}z^3y + \&c.$ will be $t^dA - dt^{d-1}B + d \times \frac{d-1}{2} t^{d-2}C - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}D + \&c.$ the Absciss being z : and the same is true, when the Absciss z grows into the whole given Absciss t . Therefore, since the Ordinate x^dy at every Point of the given Absciss AC $= t$, is equal to the compounded Ordinate, the Area described by the Ordinate x^dy drawn along the whole Absciss AC, is equal to the former compounded Area. Hence the second Part of the Proposition appears.

And therefore by considering what has been demonstrated in this, and the preceding Article, the Truth of what our Author asserts in this Proposition is manifest; and the Corollary, thence deduced, plainly appears from the Proposition itself.

S E C T. XII.

Containing an Explication of our Author's last Scholium; with the Application of the preceding Doctrine to the Solution of some Problems.

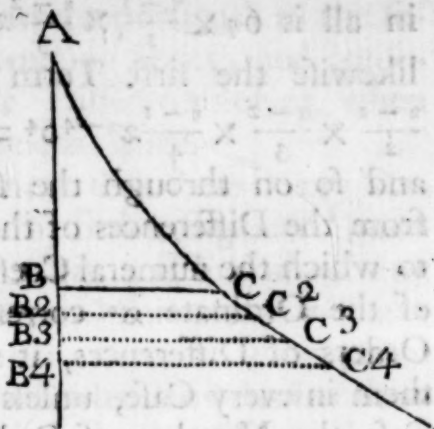
381. **T**HE Affair of the different Orders of Fluxions was fully explained at the Beginning of this Work in Sect. 2; and the Justness of the Conception vindicated from the Objections that have been raised against it by a late Author. In the same Place I likewise shewed how these different Orders of Fluxions may be expounded by the Ordinates of Curves, after such manner as is mentioned by our Author here in Art. 71—74*: so that it would be to no Purpose to spend more Time in explaining that Matter. But what is contained in Art. 70† must be illustrated a little: and so much the rather, that there seems to be some Mistake in the Text.

* Of the Quadrature of Curves.

† Of the Quadrature of Curves.

382. Let ABC be a Curve described by the Ordinate BC moving along the Absciss AB. Call $AB = z$, $BC = y$: and let the Curve be of that Nature that $y = z^n$. Let $BC, B_2C_2, B_3C_3, B_4C_4$, &c. be successive Positions of the Ordinate, so that the Distances $B B_2, B_2 B_3, B_3 B_4$, &c. be equal, any of which Distances call o . Then the successive Values of the Absciss are $z, z + o, z + 2o, z + 3o$, &c: to which the successive corresponding Values of the Ordinate are $z^n, z^n + o^n, z^n + 2o^n, z^n + 3o^n$. By throwing these Values of y into Series's, you'll have them expressed thus.

Art. 70.
of the
Quadrature
of
Curves.



1°. z^n .

2°. $z^n + n z^{n-1} o + \frac{n(n-1)}{2} z^{n-2} o^2 + \frac{n(n-1)(n-2)}{6} z^{n-3} o^3 + \frac{n(n-1)(n-2)(n-3)}{24} z^{n-4} o^4$, &c.

3°. $z^n + 2n z^{n-1} o + 4n \times \frac{n-1}{2} z^{n-2} o^2 + 8n \times \frac{n-1}{2} \times \frac{n-2}{3} z^{n-3} o^3 + 16n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} z^{n-4} o^4$, &c.

4°. $z^n + 3n z^{n-1} o + 9n \times \frac{n-1}{2} z^{n-2} o^2 + 27n \times \frac{n-1}{2} \times \frac{n-2}{3} z^{n-3} o^3 + 81n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} z^{n-4} o^4$, &c.

&c. in inf.

Where the first Term of each Series is the flowing Quantity z^n itself: the numeral Coefficients of the second Terms are 1. 2. 3. 4, &c; of the third Terms, 1. 4. 9. 16, &c; of the fourth, 1. 8. 27. 64, &c; of the fifth, 1. 16. 81. 256, &c. that is, the numeral Coefficients of the second, third, fourth, &c. Terms are Series's of the first, second, third, fourth, &c. Powers of the natural Numbers.

If you subtract the first Value of the Ordinate from the second, the second from the third, the third from the fourth, &c. you'll have the first Differences of the Ordinates $BC, B_2C_2, B_3C_3, B_4C_4$, &c. the first Term of which in them all is $n z^{n-1} o$. Again, by subtracting the first Differences from each other, you have the Differences of the Differences, or second Differences of the Ordinates, the first Term of each of which is $2n \times \frac{n-1}{2} z^{n-2} o^2 = n \times n-1 z^{n-2} o^2$. After the same

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manner, by subtracting the second Differences from each other, you'll find the third Differences of the Ordinates, the first Term of which in all is $6\eta \times \frac{\eta-1}{2} \times \frac{\eta-2}{3} z^{\eta-3} o^3 = \eta \times \eta - 1 \times \eta - 2 \times z^{\eta-3} o^3$. So likewise the first Term of the fourth Differences will be $24\eta \times \frac{\eta-1}{2} \times \frac{\eta-2}{3} \times \frac{\eta-3}{4} z^{\eta-4} o^4 = \eta \times \eta - 1 \times \eta - 2 \times \eta - 3 \times z^{\eta-4} o^4$; and so on through the subsequent Differences *in inf.* as is deduced from the Differences of the Series's of Powers of the natural Numbers, to which the numeral Coefficients of the Terms in the successive Values of the Ordinate z^η correspond. Now by considering these several Orders of Differences, it will appear that there are infinite Orders of them in every Case, unless when η is a positive Integer: in which last Case the Number of Orders is the same with η . And the first of the first Order is called the first Difference of z^η ; the first of the second Order is called the second Difference of z^η ; the first of the third Order, the third Difference of z^η , and so on. Only it may be observed that these various Orders of Differences of z^η , are generally understood by Writers upon this Subject, as meant of them when infinitely diminished: in which Case any one of the same Order of Differences of the Ordinates is the same with the first of that Order. And so it appears that the first Terms of the different Orders of Differences of z^η , are equal to the second Term; twice the third Term; six times the fourth Term; twenty-four times the fifth Term, &c. respectively, of the converging Series $z^\eta + \eta z^{\eta-1} o + \eta \times \frac{\eta-1}{2} z^{\eta-2} o^2 + \eta \times \frac{\eta-1}{2} \times \frac{\eta-2}{3} z^{\eta-3} o^3 + \eta \times \frac{\eta-1}{2} \times \frac{\eta-2}{3} \times \frac{\eta-3}{4} z^{\eta-4} o^4, \&c. = z + o^\eta$.

* Of the
Quadrature
of
Curves.

Now when the Increment of z , *viz.* o , which denotes the Distances of the Ordinates, is diminished more and more *in inf.* the prime or ultimate Ratios of the first Terms of the Differences, to the whole Differences respectively, will be a Ratio of Equality, by reasoning after the same Manner as in Art. 11 *, and in the Notes upon it: wherefore, hence it appears, that, in the converging Series $z^\eta + \eta z^{\eta-1} o + \eta \times \frac{\eta-1}{2} z^{\eta-2} o^2 + \eta \times \frac{\eta-1}{2} \times \frac{\eta-2}{3} z^{\eta-3} o^3 + \&c.$ the first Term z^η being the Fluent, the second, third, fourth, &c. Terms will be proportional to the first, second, third, &c. Differences of z^η , it being understood that the Quantities are taken in their prime or ultimate Ratio, *viz.* when these Differences are in their *nascent* or *evanescent* State. But from what was demonstrated in the Beginning of this Work, the Fluxion of z being uniform and constant, and put equal to

to Unity, the first, second, third, &c. Fluxions of z^n will be $n z^{n-1}$, $n \times n - 1 \times z^{n-2}$, $n \times n - 1 \times n - 2 \times z^{n-3}$, &c. Hence it appears, that these first, second, third, &c. Fluxions are proportional to the second, third, fourth, &c. Terms of the converging Series; and consequently to the first, second, third, fourth, &c. Differences of z^n , when these Differences are in their nascent or evanescent State.

But here you must observe, that I don't mean that there is any Ratio subsists betwixt Fluxions, represented here by finite assignable Quantities, and evanescent Differences: but only that a Fluxion of any Order is to a Fluxion of that same Order, as a Difference of the like Order is to another Difference of that Order, taken in their prime Ratio. And this I take to be the Meaning of what our Author here advances, which seems not to be expressed with that Clearness and Accuracy, which he was Master of upon other Occasions. For the second, third, fourth, &c. Terms of the converging Series are said to be equal to the first, second, third, &c. Increments or Differences; which certainly is not true. Therefore Dr. Keil (see *Commercium Epist. pag. 41.*) supposes that the Particle *ut* has stood originally betwixt the Word *erit* and Incrementum, so as to make the Text run, *tertius* $\frac{n^2-n}{2} 00z^{n-2}$ *erit ut ejus Incrementum secundum*: and again,

quartus $\frac{n^3-3n^2+n}{6} 0^3z^{n-3}$ *erit ut ejus Incrementum tertium*, &c. and that this Particle *ut* by some Accident or another hath been left out. But even when this Correction is made, I don't think the Text clear'd: because the plain and natural Meaning would still appear to be that the second Term of the converging Series will be the first Increment or Difference *absolutely*; and the third Term of the Series will be as the second Increment or Difference *absolutely*, &c. contrary to what has been shewn. But if we shall suppose that by first, second, third, &c. Increments or Differences of z^n , Sir Isaac Newton meant the second, third, fourth, &c. Terms of the converging Series $z^n + n0z^{n-1} + \frac{n^2-n}{2} 00z^{n-2} + \frac{n^3-3n^2+n}{6} 0^3z^{n-3}$, &c. (as he might have done)

the Addition of *cui nascenti* is superfluous. However I suppose I have explained the Author's Meaning: and the Omission, Inaccuracy, or Mistake, no way affects the Subject. There was no less than twenty-eight or thirty Years interven'd betwixt Sir Isaac's writing the former Part of the Treatise, and this Scholium, which was annex'd A°. 1704.

383. And hence by the by it appears how the differential Method of Mr. Leibnitz, and the foreign Mathematicians, will agree with the

Method of Fluxions in it's Conclusions; because the Ratios of the Differences are the same with the Ratios of the Fluxions, although the Principles they reason upon are not so just and *geometrical*.

In order to understand the subsequent Part of this Scholium: the Application of which is of very great Extent, there are several things to be taken Notice of, and a little explain'd, with respect to the Nature of *Fluents* and *Fluxions*, and the Constitution of *Equations* defining the *Relations* of the one or the other:

* Art. 95. of this Explanation. 384. It was observed in Section second, that every fluxional Equation must contain the Fluxions of two Quantities at least, either expressed or understood *: because Fluxion or Velocity being a relative Term, there must always be the Fluxions of two flowing Quantities at least, in every such Equation; even as there must be at least two flowing Quantities in every fluential Equation. Wherefore if any fluxional Equation be proposed, in which the Fluxion of one Quantity only is expressed, the Fluxion of some other flowing Quantity is understood, although it don't appear; by which all the Terms, that don't include the Fluxion of the first mentioned Quantity, must be supposed to be multiplied. Which other flowing Quantity is supposed to flow uniformly, and to have it's Fluxion equal to Unity, by which means it disappears in these Terms, and must be supplied, after the Manner taught in that Place *.

* Art. 98, 99. of this Explanation.

385. In the Doctrine of Fluxions, there are two different ways of expressing each of the two general Problems, that contain the *direct*, and *inverse* Method of Fluxions: which, although they may be reduced to one manner of Expression, yet that the young Geometrician may still preserve clear and distinct Ideas, I shall explain a little in this Place, to make the Way plainer to what follows.

In the direct Method, an Equation is sometimes given, which must contain at least two variable or flowing Quantities, and the Problem is expressed thus: *From the given Equation expressing the Relation of the Fluents, to find an Equation defining the Relation of the Fluxions.* Thus our Author expresses it in Prop. 1: and the Meaning of it was explained at full length, in the Notes upon that Prop. which see. Again, the same Problem is expressed otherwise thus: *Any Fluent being given, to find it's Fluxion.* Thus the flowing Quantity $ax^3 - 4abx^2 + 8bx^2 + ab$ being given, to find it's Fluxion: likewise, the flowing Quantity $x^3 - 3x^2y + 3xy^2 - y^3 + ab$ being proposed, to find it's Fluxion: and so for other Fluents, including more variable Quantities than two. The Meaning of all which must be conceived to be this. The Fluxions

ions or Velocities of flowing of the variable Quantities, which compose these Fluents, being supposed to be known, to find what are the Fluxions or Velocities with which the whole Fluents flow: or what Relation they bear to the Fluxion, or Fluxions of the flowing Quantity, or Quantities of which they are compounded. Thus the Fluxion of x^n is $nx^{n-1}\dot{x}$; the Fluxion of $ax^3 - 4abx^2 + 8bx^2 - ab$, is $3ax^2\dot{x} - 8abx\dot{x} + 16b\dot{x}x$; of $x^3 - 3x^2y + 3xy^2 - y^3 + ab$ is $3x^2\dot{x} - 6x\dot{x}y - 3x^2\dot{y} + 3\dot{x}y^2 + 6xy\dot{y} - 3y^2\dot{y}$: the Meaning of all which is, that, when Fluxions of x and y are $\dot{x}$ and $\dot{y}$, the Fluxions of the Fluents mentioned, are such, or bear such Relation to them, as the fluxional Expressions set down, import. Whence the two different Ways of expressing the second general Problem, containing the inverse Method of Fluxions, may be easily understood. One of which is this: *Any Equation being given, defining the Relation of the Fluxions of any flowing Quantities; to find an Equation, defining the Relation of the Fluents.* The other runs thus: *To find the Fluent of any fluxionary Expression proposed.* Which are the Converse of the former two Problems.

386. Now although these be different Ways of expressing the two general Problems, yet in effect they mean the same thing. For any Fluent being proposed, to find it's Fluxion, and conversely, you may put the flowing Quantity in the first Case, equal to some flowing Quantity expressed by a Letter or Symbol: and the fluxionary Expression in the second Case, to the Fluxion of some flowing Quantity expressed by the proper Notation of a fluxionary Symbol. Thus when it is proposed, as above, to find the Fluxion of $ax^3 - 4abx^2 + 8bx^2 - ab$; suppose the Fluent equal to z , so that it stand thus $ax^3 - 4abx^2 + 8bx^2 - ab = z$. And then from this Equation expressing the Relation of the Fluents x and z , you find an Equation expressing the Relation of their Fluxions. Again, when the fluxionary Expression $3x^2\dot{x} - 6x\dot{x}y - 3x^2\dot{y} + 3\dot{x}y^2 + 6xy\dot{y} - 3y^2\dot{y}$ is proposed, to find it's Fluent: you suppose it equal to the Fluxion of z , so as to stand thus, $3x^2\dot{x} - 6x\dot{x}y - 3x^2\dot{y} + 3\dot{x}y^2 + 6xy\dot{y} - 3y^2\dot{y} = \dot{z}$: and so it will amount to this, from the Equation just now mentioned, expressing the Relation of the Fluxions $\dot{x}$, $\dot{y}$ and $\dot{z}$, to find an Equation expressing the Relation of the Fluents: for the Import of which see Art. 89, 90*, so that the two different Ways of expressing the two general Problems, will in effect amount to one and the same thing.

* Of this
Explica-
tion.

387. But here something occurs that must be carefully observed, viz. that the Fluent belonging to any Fluxion, is either *pure* or *mixt*. Thus in one of the Examples above, it was shewn that $3ax^2\dot{x} - 8abx\dot{x}$

$8abxx + 16bxx$ was the Fluxion of $ax^3 - 4abx^2 + 8bx^2 - ab$: but it is also the Fluxion of the Expression $ax^3 - 4abx^2 + 8bx^2$, because $-ab$ is a constant Quantity, producing no Fluxion. The last, I call a *pure* Fluent; because it contains no Part but what flows, and leaves some Vestige of it in the Fluxion: whereas the other is called a *mixt* Fluent, as being made up of Parts, which don't all flow; there being a Part $-ab$ which is constant, producing no Fluxion, and therefore leaving no Vestige of itself in the fluxionary Expression, by which one might know that it was in the Fluent.

Now the finding the pure Fluents from the Fluxions given, is a determinate Problem, which in many Cases may be solved by the Quadrature of Curves, or the Method of finding their Areas taught by our Author in this Treatise: whereas, to find a mixt Fluent belonging to any proposed Fluxion is indeterminate, since there may be an infinite Number of such. However, there may be such Limitations either assigned, or arising from the Conditions and Circumstances of the Case, as may make this last Problem determinate; although otherwise it would not. Thus if you know that the Fluent ought to be equal to a certain known Quantity in certain Circumstances, you may thence find the *mixt* Fluent you want; provided you can first find the *pure* Fluent.

Thus if the Fluxion $\frac{axx}{b+cx^2}$ was proposed, in order to find a mixt Fluent, to which that Fluxion belongs, so circumstantiate or limited, as that, when $x=b$, the Fluent may be 0, or b ; or $\frac{bc}{a}$ or $b-a$; or any other known Quantity whatsoever: then I first seek a pure Fluent, which will be the same Expression with that, which exhibits the Area of a Curve, whose Absciss is x , and Ordinate $\frac{ax}{b+cx^2}$; as will be shewn more clearly below: this Area or pure Fluent you'll find (by Prop. 5. or Art. 147 *; or, more shortly, by comparing $\frac{ax}{b+cx^2}$ with the Ordinate of the Curve belonging to Form second, Table first) to be $\frac{ax^2}{2b^2 + 2bcx^2}$. Which found, suppose you take q for any given Quantity, to be just now determined: put $\frac{ax^2}{2b^2 + 2bcx^2} + q = 0$, or b , or $\frac{bc}{a}$, or $b-a$, or any other given Quantity, to which the mix'd Fluent must be equal, when $x=b$, that is, (by putting

* Of this
Explica-
tion.

putting $x = b) \frac{a}{z+2bc} + q = 0$, or b , or $\frac{bc}{a}$; or $b - a$, &c. Whence you find $q = -\frac{a}{z+2bc}$; or $q = b - \frac{a}{z+2bc}$; or $q = \frac{bc}{a} - \frac{a}{z+2bc}$ or $q = b - a - \frac{a}{z+2bc}$, &c. Therefore, in order to find the true Fluents answering to these several Cases or Conditions: to the pure Fluent formerly found, you must annex the corresponding Values of q , which will give $\frac{ax^2}{2b^2+2bcx^2} - \frac{a}{z+2bc}$; $\frac{ax^2}{2b^2+2bcx^2} + b - \frac{a}{z+2bc}$; $\frac{ax^2}{2b^2+2bcx^2} + \frac{bc}{a} - \frac{a}{z+2bc}$; $\frac{ax^2}{2b^2+2bcx^2} + b - a - \frac{a}{z+2bc}$, &c. for the true Fluents required, corresponding to the several Cases, or Suppositions put. And the same may be done in other Cases, when like Limitations, or Conditions of the Fluent are assigned or known.

388. It may likewise be observed, that there are sometimes two pure Fluents belonging to one and the same Fluxion: thus the fluxionary Expression $\frac{ax\dot{x}}{b+cx^2}$, mentioned above, has another pure Fluent,

viz. $\frac{-a}{2bc+2ccx^2}$; which is found by changing the Form of the given Fluxion $\frac{ax\dot{x}}{b+cx^2}$, into this other $\frac{ax^{-3}\dot{x}}{bx^{-2}+c}$; and finding the Area of a

Curve corresponding to the Ordinate $\frac{ax^{-3}}{c+bx^{-2}}$: which Area is the

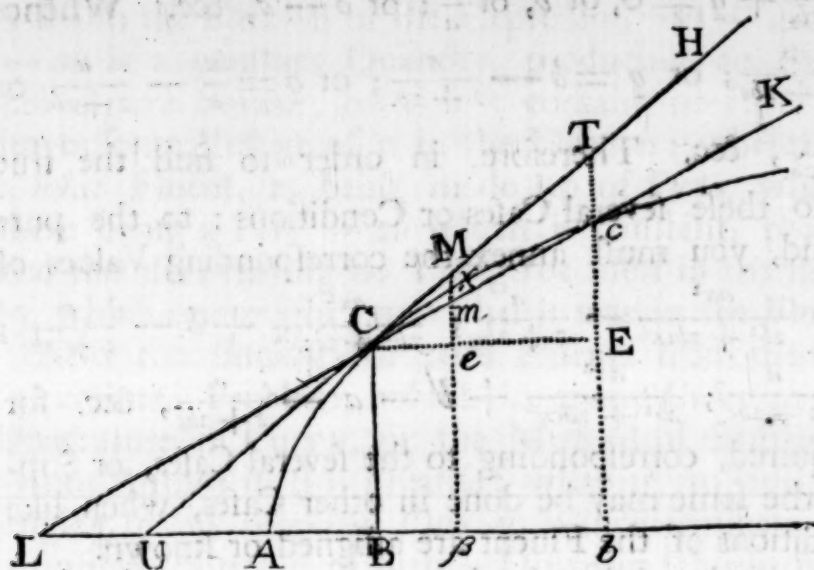
last mentioned Fluent, $\frac{-a}{2bc+2ccx^2}$; as you find by Prop. 5; or by consulting Form second of Table first. So that the same Rules, which were formerly given in Art. 147—176*, for determining when a proposed Curve is quadrable, or not; and when it is doubly quadrable, equally serve to find whether a proposed fluxionary Expression, has any *finite* Fluent, i. e. such as can be fully expressed, or not: and when there are two Fluents, and likewise to find what that Fluent or these Fluents are.

The two pure Fluents of any fluxionary Expression, including only one unknown or variable Quantity, with it's Fluxion multiplied into each Term, (when there are two such Fluents) correspond to the two Areas of a Curve, which lye upon the opposite Sides of the Ordinate: hence it follows that these two Fluents will be affected with opposite Signs.

389. Fur-

* Of this
Explica-
tion.

389. Further it is to be observed, that when any Equation, exhib-



iting the Relation of the Fluxions of any two flowing Quantities, is given, the Relation of the Fluents defin'd by the fluential Equation thence deriv'd, is the same, whether you suppose the one, or the other; or neither of the Quantities to flow with

* Art. 95.
of this Ex-
plication.

an uniform and constant Fluxion. For as these different Suppositions don't alter the Relation of the Fluxions *; so neither will they alter the Relation of the Fluents, which depends upon the Relation of the Fluxions. Thus in the curvilinear Figure ABC, whose Absciss is AB and Ordinate BC; and CV a Tangent meeting AB in the Point V; calling $AB = z$, $BC = y$, the Relation of z to y , is the same whether z flow uniformly, or y flow uniformly; or though neither of them do so, viz. that of VB to BC: and the Relation of the Fluents AB and BC in every Case is the same.

390. The finding the Fluents from the Fluxions given, by means of the Quadrature of Curves, depends upon this Principle or Position, *That any variable or flowing Quantity, which flows at the same Rate, or according to the same Law, that any curvilinear Area flows, may be represented or expounded by such curvilinear Area.* For whatever Relation a superficial Unit bears to the curvilinear Area, that same Relation will an Unit of any other Kind bear to a variable or flowing Quantity of that Kind, which flows in an analogous Manner. Hence it is that by the Quadrature of Curves, we may determine the *Lengths* of Curve-lines; the *superficial* and *solid* Contents of Bodies; the *Centers of Gravity*, *Per- cussion* or *Oscillation* of all Kinds of Magnitudes: and in general, Quantities of any Kind, considered as flowing at the same Rate, or according to the same Law, that any curvilinear Area flows: just after the same Manner that in the fifth Book of *Euclid's Elements*, Quantities of all Kinds are represented and expounded by right Lines.

Thus

Thus if the Fluxion $\sqrt{\dot{x}^2 + \frac{9x\dot{x}^2}{4a}}$ or $\dot{x}\sqrt{1 + \frac{9x}{4a}}$ was proposed, to find the Fluent: I consider $\dot{x}\sqrt{1 + \frac{9x}{4a}}$ as the Fluxion of a curvilinear Area, whose Absciss is x and Ordinate $\sqrt{1 + \frac{9x}{4a}}$: whatever be the Nature of the flowing Quantity, which flows with the proposed Fluxion. Then by the Quadrature of Curves, I find the Area of the Curve, *i. e.* I put $\sqrt{1 + \frac{9x}{4a}} = y$: and thence I find the Area by Prop. 5. Or thus: I consult the Rules laid down in Art. 147, * Of this 149 *; by which I find the Curve is quadrable, and that it's Area is Explication.

$\frac{8a+18x}{27}\sqrt{1 + \frac{9x}{4a}}$. The same thing you find more expeditiously, by

comparing the given Equation $\sqrt{1 + \frac{9x}{4a}} = y$, with the Equation belonging to the Curves of Species 1. Order 3. Table 1. Therefore

$\frac{8a+18x}{27}\sqrt{1 + \frac{9x}{4a}}$ is the Fluent belonging to the given Fluxion, by considering that Expression as no longer representing or denoting an Area of a Curve, but a Quantity of any other Kind, whether Line, Body, or any other, according to the Nature of the Fluent, whose Fluxion was proposed. More Examples of which may be seen below.

Wherefore I come now, as I promised at the Beginning, to shew how this Treatise of Quadratures is to be applied to the Solution of some of those Problems mentioned Art. 100 *, for I cannot here * Of this undertake to go through them all: it being sufficient to my Purpose Explication. to shew this in some Instances, by which the Reader may be enabled to form a Judgment concerning the rest.

P R O B.

To determine the Maxima and Minima of Quantities.

391. A variable or flowing Quantity is often of such a Nature, that it passes through a State where it neither increases nor decreases, but passes from increasing to decreasing, or from decreasing to increasing: in either of which Cases it arrives at an extreme Value, so as to be called a *Maximum*, in the first Case, and a *Minimum*, in the other. Now since the variable Quantity, at this Instant, neither flows backwards, nor forwards, therefore any Fluent being proposed, or an Equation defining the Relation of flowing Quantities, find the Fluxions,

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by

by our Author's first Proposition, and put the Fluxion of the Quantity which is to be determined to an extreme Value equal to nothing; and the Equation resulting, together with the Equation proposed, will help to discover and determine where and when the Quantity arrives at a *Maximum* or *Minimum*. The same thing is likewise done sometimes by putting the Reciprocal of the Fluxion of the Quantity equal to nothing: because in passing through a *Maximum* or *Minimum*, the Fluxion of the Quantity becomes infinitely great, in some Cases.

392. The Matter may be explained thus: let KCL (Fig. 1, 2, 3, 4.) be a Curve described with the Base AB and Ordinate BC: while the Absciss AB increases (Fig. 1, and 3.) the Ordinate BC increases together with it, until it arrive at the Position represented in these Figures, afterwards it decreases: therefore at that Instant of Time when BC is betwixt the increasing and decreasing, it attains a greater Length than in the Times or Positions preceding and succeeding, for some Space,

Fig. 1.

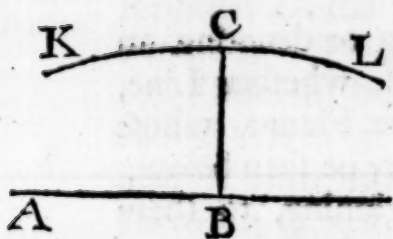


Fig. 2.

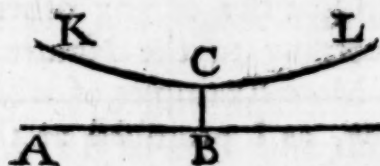


Fig. 3.

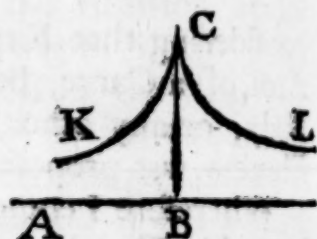


Fig. 4.

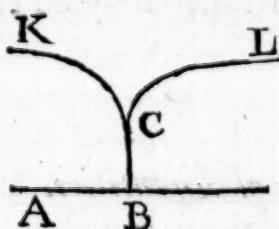
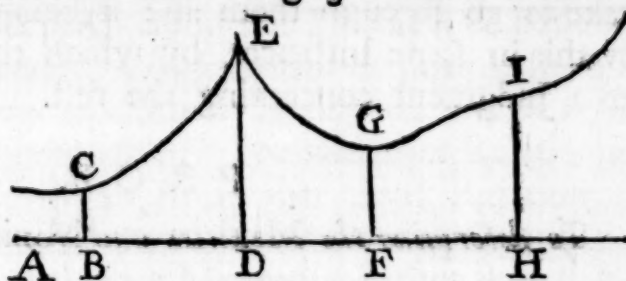


Fig. 5.



and therefore it is said to be a *Maximum*: for which end the Fluxion or Velocity with which BC flows must from being positive become negative: which it must do either by vanishing, as at Fig. 1, or by becoming infinitely great in comparison of the Fluxion of AB, as at Fig. 3. Again (at Fig. 2, and 4.) while AB increases BC diminishes until it arrive at the Position represented in these Figures, afterwards it increases, therefore in the intermediate Instant it is called a *Minimum*: at which Time the Fluxion of BC from being negative becomes

comes positive: to do which it must either vanish (as at Fig. 2.), or become infinitely great in comparison of the Fluxion of AB (as at Fig. 4.)

393. Sometimes in Curves of the higher Orders the Ordinate may arrive at a *Maximum* or *Minimum*, in different Places of the same Curve, as in Fig. 5. where the Ordinate is represented as a *Minimum* or *Maximum* in the several Positions BC, DE, FG. Whence it appears that by a *Maximum* or *Minimum* in this Doctrine, Geometricians don't mean a variable Quantity's attaining it's greatest or least Value it can possibly at any Time arrive at; but only such a Value as is greater or less than the Values for some Time or Space before or after that Instant at which it is said to be a *Maximum* or *Minimum*.

394. It is evident, by considering what has been said, that, if you call x the Absciss, and y the Ordinate of a Curve, when the Fluxion of y vanishes, the Tangent becomes parallel to the Absciss, as in Fig. 1, and 3; and when the Fluxion of y is infinitely great in comparison of the Fluxion of x , the Tangent falls in with the Ordinate, as in Fig. 2, and 4: but yet the Fluxion of y may vanish, or be infinite, and consequently the Tangent parallel to the Absciss, or coincident with the Ordinate, when the Ordinate is neither *Maximum* nor *Minimum*. For the Fluxion of y may vanish and the Tangent become parallel to the Absciss, by means of a Point of contrary Flexure, where the Ordinate doth not attain an extreme Value at that Place, as you may see with respect to the Ordinate HI at Fig. 5: such a Point of contrary Flexure may likewise make the Fluxion of y to be infinitely great in respect of the Fluxion of x , viz. when the right Line touching the Curve at the Point of contrary Flexure coincides with the Ordinate. The Fluxion of y may vanish, or be infinitely great likewise in another Case, when the Ordinate is no *Maximum* nor *Minimum*, viz. when at such Point where the Tangent becomes parallel to the Absciss; or coincident with the Ordinate, the Curve is reflected back again from the Ordinate, having no Branch upon the other Side of the Ordinate, at the Point where it meets with it: but either forming a Cuspis or turning back by a continued Curvature. But the Places where the Ordinate arrives at a *Maximum* or *Minimum* may be distinguished from these others, by considering that in the Case of a Point of contrary Flexure, the Ordinate upon the one Side of the Point of contrary Flexure, is greater; and that upon the other always less than the intermediate Ordinate which passes through the Point of contrary Flexure, or Inflection. Again when there is a Point of Reflec-

tion or Retrogression, the Ordinate upon the one Side of that Point, for some Space at least, is not to be found.

395. Wherefore an Equation being given including two unknown Quantities x and y , which you may conceive as the Absciss and Ordinate of a Curve, find the Relation of their Fluxions by the Author's first Proposition: then put $\dot{y}$ or $\frac{\dot{y}}{x} = 0$, and the Equation thence resulting together with the Equation proposed will serve to exterminate one of the unknown Quantities as y ; and give a new Equation including only the other x , whose Root or Roots will shew the Point or Points of the Absciss at which an Ordinate or Ordinates being applied will pass through a Point, or Points of the Curve, where the Tangent is parallel to the Absciss: and again by putting $\frac{\dot{x}}{y} = 0$, and proceeding the same way you may discover the Points of the Curve where the Tangent coincides with the Ordinate, having found these Values of x , insert $x + p$; and then $x - p$ instead of x in a Value of y arising from the Equation to the Curve, which will give two new Values of y , one on each Side of the Ordinate whose Fluxion is nothing or infinite: subtract the Value of y deduced from the given Equation, from these two new Values, which will give two Differences: in which Differences insert for x it's corresponding Value, then suppose p to be diminished infinitely, so as that all the Terms where p is found of any higher Dimensions, than is necessary for this Effect, be neglected: Which done, if both the resulting Differences be positive, y is there a *Minimum*; if both be negative, it is a *Maximum*: but if the one be positive and the other negative, there is a Point of contrary Flexure; and if one of the Differences include any possible or imaginary Expression, the Curve is reflected from the Ordinate. The Reason of all which will easily appear to the attentive Reader*: who must likewise know that in this Rule for determining where the Ordinate is a *Maximum* or *Minimum* by the Signs of the Differences, I suppose the Value of the Ordinate to be positive: for if it be negative (which may be found from the Expression of it's Value) two positive Differences denote a *Maximum*, and two negative Differences a *Minimum*. Moreover I only account that Ordinate a *Maximum* or *Minimum*, which has the Curve continued on both Sides of it. I shall now give some Examples of this Method.

*Art. 173.
of this Ex-
plication.

Ex. 1. Let $y + x^2 - 2ax - b^2 = 0$, be an Equation propos'd, and it is required to determine y to an extreme Value. The Relation of the Fluxions is $\dot{y} + 2\dot{x}x - 2a\dot{x} = 0$: put $\frac{\dot{y}}{x} = 0 = 2a - 2x$,
thence

thence you have $x = a$. To find whether y be at an extreme Value, and if so whether it be a *Maximum* or *Minimum*, put $x + p$, and then $x - p$ instead of x , in the Equation $y = b^2 + 2ax - x^2$; whence you have $1^{\text{mo}} y = b^2 + 2ax + 2ap - x^2 - 2px - p^2$. $2^{\text{do}} y = b^2 + 2ax - 2ap - x^2 + 2px - p^2$. From which two last Values of y subtract $b^2 + 2ax - x^2$, and the Differences are $1^{\text{mo}} - 2ap + 2px - p^2$. $2^{\text{do}} 2ap - 2px - p^2$: where if you insert for x , it's corresponding Value a , and throw out repugnant Terms, the resulting Differences are both $-p^2$, therefore y is a *Maximum*, since both Differences are negative.

Again, if you put y infinite in respect of x , or $\frac{x}{y} = 0$, thence you have $\frac{1}{2a-2x} = (\frac{x}{y} = 0)$, which can only happen when x is infinitely great, where no *Maximum* nor *Minimum* can take place.

Ex. 2. Let $abx - bx^2 - ay^2 = 0$, be an Equation defining the Relation of x the Absciss and y the Ordinate of a Curve, and it's required to determine y to an extreme Value. By proceeding as before you have $abx - 2bx^2 - 2ayy = 0$: whence by putting $y = 0$, you will find, by reducing, $x = \frac{1}{2}a$, therefore when $x = \frac{1}{2}a$, the Tangent is parallel to the Absciss. But to find whether y be in that Case a *Maximum* or *Minimum* put $x + p$, and then $x - p$ in place of x in a Value of y deduced from the given Equation, viz. $y = \sqrt{\frac{abx - bx^2}{a}}$, and you have $1^{\text{mo}} y = \sqrt{\frac{abx + abp - bx^2 - 2bxp - bp^2}{a}}$. $2^{\text{do}} y = \sqrt{\frac{abx - abp - bx^2 + 2bxp - bp^2}{a}}$: subtract $\sqrt{\frac{abx - bx^2}{a}}$, from these two last Values of y , insert for x it's Value $\frac{1}{2}a$, and throw out repugnant Terms, hence you'll have $1^{\text{mo}} \sqrt{\frac{ab}{4} - \frac{bp^2}{a}} - \sqrt{\frac{ab}{4}}$. $2^{\text{do}} \sqrt{\frac{ab}{4} - \frac{bp^2}{a}} - \sqrt{\frac{ab}{4}}$, for the Differences, both the same, and negative, whatever be the Value of p , as is evident, therefore y is a *Maximum*.

Again, if you put $y = \text{inf.}$ or $\frac{x}{y} = 0$, thence you find $\frac{2ay}{ab - 2bx} = (\frac{x}{y} = 0)$: which can only happen when it is $y = 0$; that is (by putting 0 for y in the Equation to the Curve $abx - bx^2 - ay^2 = 0$) when $abx - bx^2 = 0$: in which case you have $x = 0$, and $x = a$: therefore when the Absciss vanishes, and when it is equal to a , the Tangent is coincident with the

the Ordinate. Now if you proceed as before, to insert $x + p$ and $x - p$ instead of x , in the Value of y , and then put 0 for x which is the one Value, and suppose p infinitely diminished, the resulting Values of y are $\sqrt{bp}$ and $\sqrt{-bp}$, which last being an imaginary Quantity, shews that the Curve is not found above the Beginning of the Absciss, but is reflected back and there touches the Ordinate. After the same manner, by inserting a for x , you would find that the Curve is reflected back from the Ordinate by touching it, when the Absciss is equal to a .

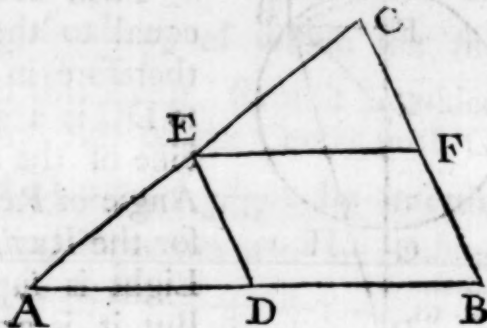
N. B. It may be remarked that when $x + p$, and $x - p$ are substituted for x in the Value of y , and the two Differences found, there is no Odds betwixt these two Differences but that the Terms where p is of an odd Number of Dimensions, are affected with opposite Signs, as is evident, which may help to abridge the Work.

Ex. 3. Let it be $x^3 - ax^2 + \frac{1}{3}a^2x - \frac{1}{3}a^3 = y$; and it is required to determine when y is a Maximum or Minimum, if ever it can become such: wherefore putting $\frac{y}{x} = 0$, you find $3x^2 - 2ax + \frac{1}{3}a^2 = (\frac{y}{x} =) 0$: whence, by reducing the Equation, you obtain one Value of x only, viz. $x = \frac{1}{3}a$. Wherefore in the given Equation insert $x + p$ and $x - p$ instead of x , and you obtain two new Values of y , viz. $x^3 \pm 3x^2p + 3xp^2 \pm p^3 - ax^2 \mp 2axp - ap^2 + \frac{1}{3}a^2x \pm \frac{1}{3}a^2p - \frac{1}{3}a^3$: from which subtracting $x^3 - ax^2 + \frac{1}{3}a^2x - \frac{1}{3}a^3$, and inserting the corresponding Value of x , viz. $\frac{1}{3}a$, you'll find, by throwing out repugnant Terms, the resulting Differences to be $\pm p^3$: which being affected with opposite Signs, shews that the Ordinate passes through a Point of contrary Flexure, and therefore is not a Maximum nor Minimum.

396. It may be remarked, that when y is of more Dimensions than one in the Equation proposed, the several Values of y must be found by resolving the Equation, in order to determine whether all or any of these Values of y become a Maximum or Minimum, as easily appears from what has been said. However it may be proper to observe likewise that, if the Equation by which the Value of x is determined, from putting $\dot{y} = 0$, or infinite, be only a simple Equation, y must be a Maximum or Minimum. But I must not insist upon these things. Whoever inclines to see more of this Subject, may consult the learned Mr. Maclaurin's Treatise of Fluxions, where he has explained this Affair more fully and accurately than any other Writer upon the Doctrine of Fluxions: by which, and by the Whole of that excellent Performance, he has discovered how just and comprehensive Views he

he has of these Matters. Wherefore all that I propose further to do upon this Subject, is to shew the Application of the general Doctrine concerning the Maxima and Minima to the Solution of a Problem or two.

397. Let it be required to inscribe in any given Triangle ABC the greatest possible Parallelogram DF. Call $AB = a$, $BC = b$, $AD = x$, therefore $DB = a - x$, and because AB, BC, AD and DE are proportional, hence $DE = \frac{bx}{a}$. Now since the Angles of the Parallelogram are given, hence it is a Maximum when $DB \times DE$, i. e. $\frac{abx - bx^2}{a}$



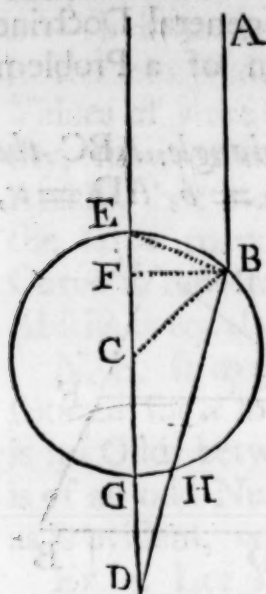
is a Maximum; or (because $\frac{b}{a}$ is a given Quantity) when $ax - x^2$ is a Maximum: wherefore suppose $ax - x^2 = y$, where you may imagine x and y to be related to one another as the Absciss and Ordinate of a Curve, in which you are to determine y to an extreme Value. Therefore you'll have $\frac{y}{x} = 0 = a - 2x$, whence arises $x = \frac{1}{2}a$, i. e. $AD = \frac{1}{2}AB$, in order to make the inscribed Parallelogram a Maximum. And this coincides with Prop. 27. Lib. 6. Elem.

And after the same manner you might determine the inscrib'd Parallelogram to a Maximum, if ABC had been a Segment of any known Curve: thus if it had been a Portion of a Parabola, having AB for the Diameter and BC an Ordinate bounding the Figure, then because AB (a), AD (x), BC² (b^2), DE² are proportional, hence $DE = b\sqrt{\frac{x}{a}}$: therefore put the Fluxion of $DE \times DB$, i. e. $a - x \times b\sqrt{\frac{x}{a}}$ or, which is the same (by dividing by the given Quantity $\frac{b}{\sqrt{a}}$) of $ax^{\frac{1}{2}} - x^{\frac{3}{2}}$ equal to nothing, that is $\frac{1}{2}ax^{-\frac{1}{2}}x - \frac{3}{2}x^{\frac{1}{2}}x = 0$: whence, by reducing you find $x = \frac{1}{3}a$: which shews that the inscribed Parallelogram is in this case a Maximum, when AD is $\frac{1}{3}$ of AB. And so you proceed in other like Cases.

398. Let it be proposed to find such a part of a spherical Superficies, as can be illuminated in it's farther Part, by Light coming from a great Distance, and refracted by the nearer Hemisphere.

To do which, let AB be a Ray of Light coming from a great Distance, falling upon the nearer Hemisphere at B: let BEGH be a
Section

Section of the Sphere through the Center. C: draw the Diameter



ECG parallel to AB: let the Ray be refracted at B into BHD, meeting the Circumference of the Circle in H, and the Diameter EG produced, if need be, in D. From B draw BF perpendicular to EG at F, join BE, BC, and GH.

Then because the Angles BCE and CBD are equal to the Angles of Incidence and Refraction, therefore in the Triangle BCD, the Ratio of BD to DC is a given invariable Ratio, being that of the Sine of the Angle of Incidence to the Sine of the Angle of Refraction, wheresoever the Point B falls, for the Ray AB is always parallel to EG, since the Light is suppos'd to come from a great Distance.

But it is evident that the right Line GH is the Chord of half the spherical Segment, the Circumference of whose Base is enlightened by all the Rays which fall upon the nearer Hemisphere at the same Distance from the Point E as AB falls: when the Point B is very near to E, GH is very small; when EB increases, GH increases with it for some time, but afterwards decreases: therefore the Problem will come to this *to determine when GH is a Maximum?* To do which call the Radius $EC = 1$, $EF = x$, $CD = z$: whence because GE, EB, EF are proportional, you have $EB = \sqrt{2x}$: moreover, if a express the Ratio of the Sine of the Angle of Incidence to the Sine of the Angle of Refraction, then $BD = az$: likewise from the similar Triangles DBE, DGH, you have $DB (az) : BE (\sqrt{2x}) :: DG (z - 1) : GH$, whence $GH = \frac{z-1}{az} \sqrt{2x}$: therefore $\frac{z-1}{az} \sqrt{2x}$, or, which is the same (by dividing by the given Quantity $\frac{\sqrt{2}}{a}$) $x^{\frac{1}{2}} - \frac{z}{z}$, must be determined to a Maximum: wherefore by putting it's

Fluxion equal to nothing, you have $\frac{\dot{x}}{2\sqrt{x}} - \frac{\dot{z}}{2z\sqrt{x}} + \frac{\dot{z}\sqrt{x}}{z^2} = 0$, that is (by multiplying by $2z^2\sqrt{x}$) $z^2\dot{x} - z\dot{z} + 2xz\dot{z} = 0$.

Now in order to eliminate one of the unknown Quantities, and it's Fluxion, you must find an Equation expressing the Relation of x and z , which you may deduce from the right-angled Triangle DBF, where $DB^2 (a^2z^2) = DF^2 (z^2 + 2z - 2xz - 2x + x^2 + 1) + BF^2 (2x - x^2)$ that is $a^2z^2 = z^2 + 2z - 2xz + 1$: from which you have this fluxional Equation $2a^2z\dot{z} = 2z\dot{z} + 2\dot{z} - 2x\dot{z} - 2z\dot{x}$: whence

whence $\dot{z} = \frac{zx}{z - a^2z - x + 1}$: which Value of $\dot{z}$ being inferted for it in the preceding fluxional Equation, viz. $z^2\dot{x} - z\dot{z} + 2xz = 0$, and the Equation reduced, you'll have $z^2 - a^2z^2 - xz + a^2z + 3x - 1 = 0$: by means of which, and the preceding Equation, viz. $a^2z^2 = z^2 + 2z - 2xz + 1$, if you throw out x , you'll obtain this cubical Equation $z^3 - a^2z^3 - a^2z^2 + z^2 + 3z + 3$, whose Roots are $z = -1$. $z = -\sqrt{\frac{3}{a^2-1}}$. $z = \sqrt{\frac{3}{a^2-1}}$: of which the first two being negative, can be of no use in this Affair, since it is evident the Point D must fall upon the same Side of the Center with G: therefore GH is a Maximum when $CD = \sqrt{\frac{3}{a^2-1}}$. by inserting which Value of z for it in the preceding Equation $a^2z^2 = z^2 + 2z - 2xz + 1$, and reducing, you'll get $x = \frac{1}{2\sqrt{\frac{3}{a^2-1}}} + 1 - \frac{1}{2}aa - \frac{1}{2}$

$\sqrt{\frac{3}{a^2-1}} = EF$. Thus, for Example, suppose the Globe was Glas, in which Case the Sine of the Angle of Incidence to the Sine of the Angle of Refraction is nearly as 3 to 2, therefore by inserting $\frac{3}{2}$ for a , you'll have $CD = \sqrt{\frac{3}{\frac{9}{4}-1}} = \sqrt{2.4} = 1.5492$, Sc. of CG; and $EF = 0.3545$, Sc. of CE. And since it was $GH = \frac{z-1}{az}\sqrt{2x}$: hence by substituting for z and x their Values, it is $GH = 0.198$, Sc. which is nearly the Subtense of an Arch $11^\circ 22'$ in a Circle whose Rad. is 1: therefore the Part of the farther Hemisphere which is enlightened, is a Segment, the Diameter of whose Base subtends an Arch of $22^\circ 44'$ nearly: and in this Case EB is nearly an Arch of $49^\circ 48'$.

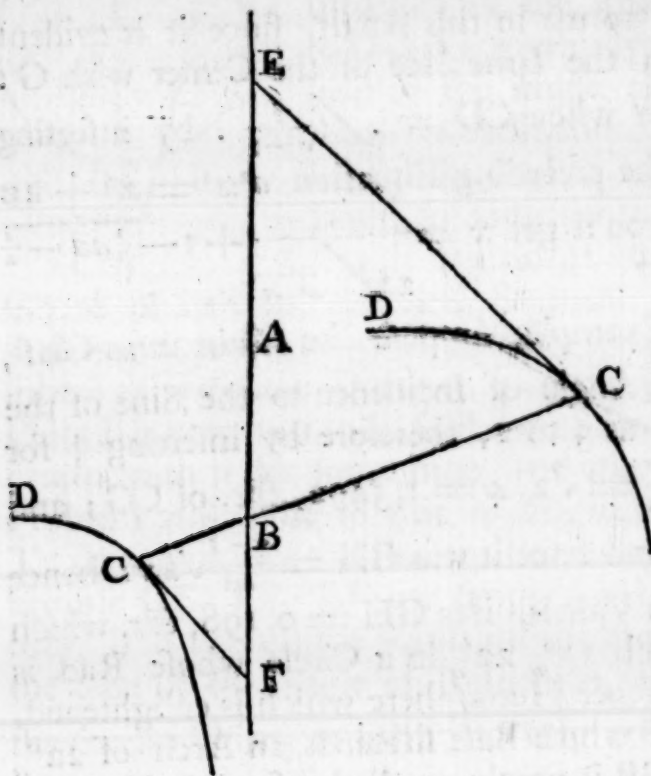
399. Schol. By means of the above Equation $a^2z^2 - z^2 - 2z + 2xz - 1 = 0$, defining the Relation of $x = EF$ and $z = CD$, you may determine several things in this Affair: e. g. by putting $z = 1$, you may find the Distance of those Rays from E, which after Refraction will pass through C. By putting $x = 1 = \text{Rad.}$ you may find where those Rays that touch the Globe after Refraction at their Entrance would cut CG. By putting $x = 0$ the Value of z will give the Focus of parallel Rays after Refraction at the first Surface, Sc. And if any two corresponding Values of x and z are given, you may thence find a , that is the Law of Refraction.

P R O B.

To draw Tangents to all sorts of Curves.

* Of the
Quadrature of
Curves.
† Of this
Explication.

400. The Principle upon which the Solution of this Problem depends is laid down by our Author Art. 4 *; and was explain'd at full Length Art. 41, 42 †. which amounts to this: if the Curve DC be described by the Ordinate moving along the Absciss AB; and CE be



a Tangent to the Curve at the Extremity of the Ordinate, meeting the Absciss in E, the Sides BE, BC and CE of the Triangle BCE are proportional to the Fluxions of AB, BC and DC, the Absciss, Ordinate and Curve-line: and it is the same thing whether BC be a rectangular Ordinate; or an Ordinate applied in any other given Angle. Wherefore calling $AB = x$, $BC = y$, and $BE = s$ (which last is termed the Subtangent) you have this Proportion $y : x :: y : s$, or $y \times \frac{x}{y} = s$: which

is the general Formula for finding the Subtangent BE. Therefore if an Equation be given defining the Relation of x and y , we may thence find the Relation of their Fluxions, and consequently from BC or AB given you find the Point E, so that EC being join'd may be a Tangent at the Point C.

401. Hence it is evident, that if, while the Absciss increases, the Ordinate increase at the same time, then $\frac{x}{y}$ is positive, and consequently the Subtangent is positive: in which Case BE lyes upon the same Side of the Ordinate that the Point A the Beginning of the Absciss does: if y become infinitely great in comparison of x , then $\frac{x}{y}$ vanishes, and consequently EB or s , if y be finite, and the Tangent falls in with the Ordinate: if y vanish, $\frac{x}{y}$ becomes infinitely great, consequently EB is infinite; and the Tangent becomes parallel to AB:

but

but if BC diminish while AB increases, and consequently have a negative Fluxion, then $\frac{\dot{x}}{y}$, and consequently $y \times \frac{\dot{x}}{y} = s$, is negative, in which Case BE lyes upon that Side of the Ordinate which is opposite to A.

Ex. 1. Let $y^n = a^{n-1}x$ define the Nature of the Curve: then, by Prop. 1, $ny^{n-1}\dot{y} = a^{n-1}\dot{x}$, or $\frac{\dot{x}}{y} = \frac{ny^{n-1}}{a^{n-1}}$, therefore $\frac{ny^n}{a^{n-1}} = (y \times \frac{\dot{x}}{y}) = s$ the Length of the Subtangent: or if you would have the Subtangent expressed in Terms of the Absciss, insert for y^n it's Value $a^{n-1}x$ from the Nature of the Curve, and you have $s = nx$: wherefore if you take $BE : AB :: n : 1$, and join EC it is the Tangent required. Thus if it be $n = 2$, the Equation belongs to the common Parabola, and so $EB = 2AB$. And if n be 3, 4, 5, &c. which gives a Series of Parabolas of different Orders, you have EB equal 3AB, 4AB, 5AB, &c. respectively. Again, if it be $n = -1$, then the Equation becomes $y^{-1} = a^{-2}x$ or $yx = x^2$, which is to the common Hyperbola, having the Center A, one of it's Asymptotes AB, and the other parallel to BC: here you have $s = -x$, therefore you must take $EB = AB$ upon the other Side of the Ordinate. After the same manner if you take $n = -2, -3, -4$, &c. you'll have different Orders of Hyperbolas, whose Subtangents are $-2x, -3x, -4x$, &c. i. e. 2AB, 3AB, 4AB, &c. taken upon the Absciss produced beyond the Ordinate.

Ex. 2. Let $abx - bx^2 - ay^2 = 0$ express the Relation of AB and BC, in which case the Curve DC is an Ellipse, A the Vertex of the Diameter, a the Diameter, b , its Parameter: then by taking the Fluxions, it is $ab\dot{x} - 2bxx\dot{x} - 2ay\dot{y} = 0$; whence $\frac{\dot{x}}{y} = \frac{2ay}{ab - 2bx}$, therefore $s = \frac{2ay^2}{ab - 2bx} =$ (by substituting $abx - bx^2$ for ay^2 from the Equation to the Curve) $\frac{2abx - 2bx^2}{ab - 2bx} = \frac{ax - x^2}{\frac{1}{2}a - x} = BE$: whence BE is a fourth Proportional to $\frac{1}{2}a - x, a - x$, and x , as is known from other Principles. Hence it appears that, since the Value of BE the Subtangent, doth not include the Parameter b , any two Diameters of any two Ellipses, and their Abscisses, being equal, the Subtangents are equal: and therefore if there be any Number of Ellipses having a common Axis, including also the Circle, the Tangents drawn from the Extremities of the Ordinates applied to the same Point of the common Axis, will meet that Axis in the same Point: If the Equation had been abx

$+bx^2 - ay^2 = 0$, it would have been to the Hyperbola, and $\frac{ax+x^2}{\frac{1}{2}a+x} = s = BE$.

Ex. 3. Let AGE be the Cissoïd of *Diocles* (see Fig. Art. 321 *) belonging to the Circle ADQ, the same things being supposed as in that Article, call $AQ = a$, $AC = x$, $CE = y$: and it is required to draw a Tangent to the Curve at the Point E.

* Of this
Explica-
tion.

The Equation to the Curve is $y = \frac{x^2}{\sqrt{ax-x^2}}$; or, by squaring, $ay^2 - xy^2 = x^3$, therefore, by taking the Fluxions, $2ay\dot{y} - 2xy\dot{y} - \dot{x}y^2 = 3x^2\dot{x}$, whence $\frac{\dot{x}}{y} = \frac{2ay - 2xy}{y^2 + 3x^2}$: wherefore $\frac{\dot{x}}{y} \times y = \frac{2a-2x}{y^2+3x^2} \times y^2$ (by inserting $\frac{x^2}{a-x}$ for y^2 , and reducing) $\frac{2ax-2x^2}{3a-2x} = \frac{CD^2}{\frac{1}{2}AQ+QC}$, which points out this Construction: produce AQ to I, so that QI be equal to the Radius of the Circle ADQ, join ID, draw DK perpendicular to it, cutting the Absciss AC in K, join KE, and it is the Tangent required.

Ex. 4. Let PEH be the Conchoid of *Nicomedes* (see the Fig. Art. 322 *) and supposing the same things as there mentioned, only consider AB as the Absciss, and BE as the rectangular Ordinate, which call x and y , the other Symbols remaining the same; it is required to draw a Tangent to the Curve at the Point E.

* Of this
Explica-
tion.

The Equation to the Curve is $xy = b + y\sqrt{cc-yy}$: therefore by taking the Fluxions, you have $\dot{x}y + x\dot{y} = y\sqrt{cc-yy} + b + y \times -\frac{y\dot{y}}{\sqrt{cc-yy}}$ *

* Art. 87.
of this Ex-
plication.

whence $\frac{\dot{x}}{y} = \frac{\sqrt{cc-yy} - \frac{by+y^2}{\sqrt{cc-yy}} - x}{y}$: therefore $\frac{y\dot{x}}{y} = \sqrt{cc-yy} - \frac{by+y^2}{\sqrt{cc-yy}} - x = CD - \frac{GC \times AC}{CD} - AB = s$ the Subtangent: which Value being always negative, in regard AB always exceeds CD, shews that it is to be taken upon the Absciss AB produced beyond the Ordinate: wherefore if you take the Point T in the Asymptote AB, upon the further Side of the Ordinate equal to $AL + \frac{GC \times AC}{CD}$, and join TE, it shall be the Tangent required.

402. This Curve affords us an Example of a Point of contrary Flexure, which separates betwixt the Part of the Curve that is concave to the Asymptote AB and the Part that is convex; which may be thus determined from the Consideration of the Tangent. When AB is nothing, and the Point E coincident with P the Vertex of the Axis, the Tangent

Tangent ET is parallel to the Affymptote AT, in which case the Subtangent BT is infinite; and if you suppose ET produced till it meet the Axis AP in F, the Part PF vanishes, when AB and consequently PC is of any finite Magnitude BC and PF are finite Magnitudes: so that, supposing the Absciss AB arising from nothing to increase, the Line AT will decrease and AF increase, until such time, as the Ordinate BE fall upon the Point of contrary Flexure: after which, AB still increasing, AT will begin to increase and AF to diminish. Hence it is evident that AT and AF attain an extreme Value, the first becoming a Minimum, the other a Maximum, when the Ordinate falls upon the Point of contrary Flexure. Therefore if you take the Value of any of these Lines, and put it's Fluxion equal to nothing, the resulting Equation will determine the Point of contrary Flexure.

Now since we found $BT = \sqrt{cc - yy} - \frac{by + y^2}{\sqrt{cc - yy}} - x$, hence calling $AT = t$, we shall have $-\sqrt{cc - yy} + \frac{by + y^2}{\sqrt{cc - yy}} + 2x = t$. From which find an Equation expressing the Relation of the Fluxions, in which suppose $t = 0$ *; for x and $\dot{x}$ insert their Values $\frac{b+y}{y}\sqrt{cc - y^2}$, and $\frac{\dot{y}}{y}\sqrt{cc - y^2} - \frac{by + yy}{\sqrt{cc - yy}} - \frac{xy}{y}$, from the Equation to the Curve, and reduce the resulting Equation, and it will give you $y^3 - 3by^2 - 2bc^2 = 0$. Therefore if this Equation be constructed, the Length of BE will be thereby found: to which if you take AC equal, and draw CE parallel to AB, the Point E shall be the Point of contrary Flexure required. And after the same manner the Points of contrary Flexures, in other Curves where there are such Points, may be investigated.

* Art. 391.
of this Ex-
plication.

403. Although the Curves which are proposed, be *mechanical*, Tangents may be drawn to them by the like means as if they were *geometrical*. All that is necessary is to find the Relation of the Fluxions of the Absciss and Ordinate, expressed by finite Lines: which may be done although there be no finite Equation that can define the Relation of the Absciss and Ordinate.

Ex. Take the common Cycloid or Trochoid ACD (see the Fig. Art. 332 *) and suppose the same things as are there mentioned, only call $AB = x$ and $AHG = z$; and let it be required to draw a Tangent to the Curve at the Point C.

* Of this
Explica-
tion.

To do which, from the Property of the Curve, you have $BC = BG + AHG$, that is $y = v + z$: therefore, by taking the Fluxions, you

* Of this
Explica-
tion.

you have $\dot{y} = \dot{v} + \dot{z} =$ (because $\dot{z} = \sqrt{\dot{x}^2 + \dot{v}^2}$ by Art. 42 *) $\dot{v} + \sqrt{\dot{x}^2 + \dot{v}^2}$: but from the Property of the Circle, viz. $v^2 = dx - x^2$, by taking the Fluxions, you get $\dot{v} = \frac{\frac{1}{2}d-x}{\sqrt{dx-xx}} \dot{x}$: which Value of $\dot{v}$ being substituted in the preceding Equation $\dot{y} = \dot{v} + \sqrt{\dot{x}^2 + \dot{v}^2}$, and it duly reduced, you have $\dot{y} = \dot{x} \sqrt{\frac{d-x}{x}}$, whence $\frac{\dot{x}}{\dot{y}} = \sqrt{\frac{x}{d-x}}$, and $\frac{\dot{x}}{\dot{y}} y = \frac{y \sqrt{x}}{\sqrt{d-x}} = BC \times \frac{AB}{BG} = s = BT$. Whence it appears that the Subtangent BT is a fourth Proportional to BG, AB and BC; so that the Tangent CT is parallel to the Chord AG: which Property we assumed at Art. 326 *, in order to abridge the Demonstration.

* Of this
Explica-
tion.

404. And by the like Methods may Tangents be drawn to other mechanical Curves, which are related after any other manner to geometrical Curves: but whereas it is not so evident how to proceed in the Case of Spirals, I shall shew the young Geometrician how Tangents may be drawn to them.

* See Fig.
Art. 335.
of this Ex-
plication.

Let ACM be a Spiral *, MEGM a Circle describ'd from the Center A with a given Radius AM: and let the Radius ACG beginning at AM revolve upon A as a Pole, and in the mean time the Point C, moving along AC, describe the Spiral ACFM: draw ATS perpendicular to ACG, and let CT be a Tangent to the Spiral at C meeting ATS in T: through G draw GS parallel to CT, meeting ATS in S. Suppose ACG to move forward into the new Position A γ , with the Center A and Distance AC draw the small Arch C δ meeting A γ in δ : then by supposing the Angle GA γ to be just vanishing, the Triangle C δ is ultimately similar to the Triangle ACT *, so that it is $\delta x : \delta C :: AC : AT$. Again from similar Arches it is $\delta C : \gamma G :: (AC : AG ::) AT : AS$, therefore *ex æquo* $\delta x : \gamma G :: AC : AS$; but δx is to γG as the Fluxion of AC to the Fluxion of MEG *. Therefore an Equation being given from the Nature of the Spiral, defining the Relation of AC and MEG, thence find the Ratio of their Fluxions as formerly: and take AS to AC in the Ratio of the Fluxion of MEG to the Fluxion of AC, so shall SG be parallel to the Tangent at C: therefore SE being join'd, through C draw CT parallel to it, and it shall be a Tangent to the Spiral at C.

* Art. 41.
of this Ex-
plication.

* Art. 28.
of this Ex-
plication.

Ex. I. Call MEG = x , and AC = y , which you may consider as Absciss and Ordinate; and let it be $x : y :: a : b$, a given Ratio, so that the Spiral shall be that of Archimedes: then $bx = ay$, whence $bx = ay$, so that $y : x :: b : a :: y : \frac{ay}{b} = AS$: therefore take AS a fourth Proportional

tional

tional to b , a , and AC, join GS, and CT drawn parallel to it touches the Spiral at C. If the given Circle MEGM be that which bounds the first Spiral, then a is to b as the Circumference of a Circle to its Radius: which therefore will be the Ratio of AS to AC: consequently when the Points C and G coalesce at M, and consequently T with S, the Subtangent AT is equal to the Circumference of the Circle MEGM, as Archimedes has demonstrated.

Ex. 2. Let the Equation to the Spiral be generally expressed thus, $ay^m = bx^n$: then by taking the Fluxions it is $may^{m-1}\dot{y} = nbx^{n-1}\dot{x}$:

whence $\dot{y} : \dot{x} :: nbx^{n-1} : may^{m-1} :: y : AS$, therefore $AS = \frac{may^m}{nbx^{n-1}}$

= (by inserting $\frac{ab^{\frac{1}{n}}y^m}{a^{\frac{1}{n}}by^{\frac{m}{n}}}$ for x^{n-1} , deduced from the given Equation and

reducing) $\frac{ma^{\frac{1}{n}}y^{\frac{m}{n}}}{nb^{\frac{1}{n}}}$.

Ex. 3. Let the Equation $x^2 - 2ax + by^2 = 0$ express the Relation of x and y : then by taking the Fluxions, you have $2x\dot{x} - 2a\dot{x} + 2by\dot{y} = 0$, whence $2a - 2x : 2by :: (\dot{y} : \dot{x} ::) y : AS = \frac{2by^2}{2a - 2x}$

= (by substituting $a - \sqrt{a^2 - by^2}$ for x , deduced from the given Equation) $\frac{by^2}{\sqrt{a^2 - by^2}}$: therefore if you call $AM = r$, then since $r :$

$y :: AS : AT$, it will be $\frac{by^2}{r\sqrt{a^2 - by^2}} = AT$, the Subtangent. And so you may proceed in all other like Cases.

405. These two general Problems, which we have been explaining, are an Application of our Author's first Proposition, and belong to the direct Method of Fluxions, because they only require the finding the Fluxions from the Fluents given. But the far greater Number of Problems to which the Doctrine of Fluxions is applied, require the finding the Fluents from the Fluxions given, in order to their Solution. This cannot always be done by the Quadrature of Curves; but may be done by it in many Cases, of which these that follow are Examples.

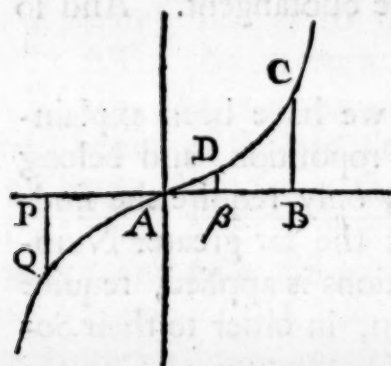
P R O B.

To find the Length of Curve-lines.

* Of this
Explica-
tion.

406. The Principle, upon which the Solution of this Problem depends, was explained and demonstrated in Art. 41 and 42 * (see the Fig. belonging to that Place) which is this: if you put the Absciss $AB = x$, the rectangular Ordinate $BC = y$, and the Curve-line terminated at the Extremity of the Ordinate as $AC = z$, then it is $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: where you must observe that it is not necessary to suppose the Curve-line to begin at A or even to pass through A, although it do so in this Figure: but the initial Limit must be determined from the Nature of the Curve, as will appear afterwards. Therefore, if, from the Nature of the Curve, an Equation be given, expressing the Relation of x and y , thence by Prop. 1. you may find the Relation of their Fluxions in another Equation: from which two Equations you may find the Value of the Fluxion of y , in Terms made up of the Fluxion of x , x itself, and known Quantities: which Value of the Fluxion of y being substituted for it in the Equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, the second Member shall contain no variable Quantity but x , and no Fluxion but that of x ; and so the Fluent z , that is the Length of the Curve-line, may be found by the Quadrature of Curves, as will appear by what follows.

407. Let $x^n = ay^{n-1}$ be an Equation to a Parabola of any Order, as AC, where n represents any positive Integer: and it is required to find the Length of the Curve, which call z .



Then you have $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: but from the Equation to the Curve, viz. $x^n = ay^{n-1}$, by taking the Fluxions, it is $nx^{n-1}\dot{x} = (n-1)ay^{n-2}\dot{y}$, whence $\dot{y} = \frac{nx^{n-1}\dot{x}}{(n-1)ay^{n-2}}$

that is (by inserting $\frac{x^{n-1}}{a^{n-1}}$ for y^{n-2} , de-

duced from the given Equation) $\dot{y} = \frac{nx^{n-1}\dot{x}}{(n-1)a^{n-1}}$, or by squaring,

$$\dot{y}^2 = \frac{n^2 x^{2n-2} \dot{x}^2}{(n-1)^2 a^{2n-2}} : \text{ now if this Value of } \dot{y}^2 \text{ be put in place of}$$

of it in the fluxional Equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, hence will arise $\dot{z} =$

$$\dot{x} \sqrt{1 + \frac{n^2 \times x^{n-1}}{n^2 - 2n + 1 \times a^{n-1}}}. \quad \text{But it appears that this Expression}$$

denotes the Fluxion of a curvilinear Area, whose Absciss is x , and

rectangular Ordinate the radical $\sqrt{1 + \frac{n^2}{n^2 - 2n + 1 \times a^{n-1}} x^{n-1}} :$

therefore the Curve-line z , and such curvilinear Area, flow according to the same Law; the one as a Line, the other as a plane Superficies, consequently the two Fluents will be analogous, supposing them to begin at the same time: wherefore the curvilinear Area having such Absciss and Ordinate, being found by the Quadrature of Curves, apply it to linear Unity, and that shall exhibit the Length of z required; or the curvilinear Area being found in an algebraical Expression, consider such Expression no longer as denoting a Superficies, but a Length, the same algebraical Expression serving for both, since the Division by Unity makes no Alteration upon it. Wherefore if the Value of n be

such, that the Ordinate $\sqrt{1 + \frac{n^2}{n^2 - 2n + 1 \times a^{n-1}} x^{n-1}}$ belong to a

Curve that is quadrable, the Length of the Curve-line is found exactly; if not, the Length of the Curve-line is found approximately, by Propositions fifth and sixth: or in this last Case, may be exhibited by the Area of a Conic Section; or perhaps of some other Curve applied to Unity. So that our Author's Tables of Quadratures serve for this Purpose, as well as for discovering the Areas of Curves: these Lengths likewise may sometimes be found by Logarithms and circular Arches, according to Sect. 10. For finding when these Curve-lines can be rectified exactly, or not consult Art. 147—149*.

Thus if it be $n = 3$, so that the Equation to the Parabola ABC be $x^3 = ay^2$, which is a Parabola of the second Kind: then $\dot{z} =$

$$\dot{x} \sqrt{1 + \frac{9x}{4a}}: \text{ which is analogous to the Fluxion of the Area of the}$$

Curve whose Ordinate is $\sqrt{1 + \frac{9x}{4a}}$, which may be squared by Art.

147—149*, and belongs to Species 1. Form 3. Table 1. which

stands thus $dx^{n-1} \sqrt{e + fz^n} = y$, by putting $n = 1$, $d = 1$, $e = 1$, $f =$

* Of this
Explica-
tion.

* Of this
Explica-
tion.

Qq

f =

$f = \frac{9}{4a}$: wherefore the Fluent $\frac{2d}{3\eta f} R^3$ in the Table, is by a proper Substitution of Values, $\frac{8a}{27} \times 1 + \frac{9x}{4a}$ or $\frac{8a+18x}{27} \sqrt{1 + \frac{9x}{4a}}$: which therefore denotes the Length of this Parabola. And after the same manner, the Lengths of all these Parabolas may be found where n is any positive odd Number greater than 3, as 5, 7, 9, &c. by help of the second, third, fourth, &c. Species's of Form third, Table first. The same thing might be done by having recourse to Prop. 5. See Art. 390 *.

* Of this
Explica-
tion.

408. But whereas it is not yet determined from what initial Limit, the Length of the Curve so found is to be computed, this may be done by putting the Quantity $\frac{8a+18x}{27} \sqrt{1 + \frac{9x}{4a}}$ equal to nothing, and the Value of x thence arising will determine the initial Limit; as was shewn formerly with respect to curvilinear Areas in Sect. 5. Thus if we put $\frac{8a+18x}{27} \sqrt{1 + \frac{9x}{4a}} = 0$, thence arises $\frac{8a+18x}{27} = 0$, or $1 + \frac{9x}{4a} = 0$, either of which gives $x = -\frac{4}{9}a$: which Value being negative, shews that if you take upon the Absciss AB produced beyond A, the Point P such that it be $AP = \frac{4}{9}a$, and draw the Ordinate PQ, the Part QAC = $\frac{8a+18x}{27} \sqrt{1 + \frac{9x}{4a}}$. If you would compute the Length of the Curve from the Beginning of the Absciss at A, then here is a known Condition or Limitation of the Fluent, viz. that it is nothing when $z = 0$,

* Of this
Explica-
tion.

therefore by Art. 387 *, $z = \frac{8a+18x}{27} \sqrt{1 + \frac{9x}{4a}} - \frac{8a}{27}$. More universally: if you would have any Part of the Curve-line as γC , lying betwixt any two given Ordinates $\beta\gamma$, BC, subtract the Value of z arising from putting $x = A\beta$ the lesser Ordinate, from the Value of z arising by putting $x = AB$ the greater Ordinate, the Difference is the Length of γC *: thus if you call $AB = \xi$ and $A\beta = x$, then you have $\gamma C = \frac{8a+18x}{27} \sqrt{1 + \frac{9x}{4a}} - \frac{8a+18\xi}{27} \sqrt{1 + \frac{9\xi}{4a}}$: where if ξ vanish, the Computation is from the Beginning of the Absciss, or rather the Ordinate passing through it.

* Art. 187.
of this Ex-
plication.

409. Suppose now that in the preceding general Equation $x^n = ay^{n-1}$, n is 4, so that the Property of the Parabola AC be expressed

thus $x^4 = ay^3$; then it will be $z = \dot{x} \sqrt{1 + \frac{16x^2}{9a^3}}$, by comparing this with

with Art. 149 *, you'll find the radical Expression denotes the Ordinate of a Curve that cannot be squared: therefore I next consider whether it may be reduced to any Ordinate found in our Author's second Table of Quadratures; which as it stands, it cannot; but if

* Of this Explication.

you reduce it to this other Form $x^{\frac{1}{3}} \sqrt{\frac{10}{9a^{\frac{2}{3}}} + x^{-\frac{2}{3}}}$ *, you find it to belong to Species third, Order third of Table second, viz. $y = \frac{d}{z^{2n+1}} \sqrt{e + fz^n}$, by putting $n = -\frac{2}{3}$, $d = 1$, $e = \frac{10}{9a^{\frac{2}{3}}}$, $f = 1$: by comparing which with Art. 298 *, you'll find the Conic Section, by

* Art. 127, and 317. of this Explication.

means of which the Area of the Curve, whose Ordinate is $\sqrt{1 + \frac{16x^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$,

* Of this Explication.

may be compared to be an Hyperbola belonging to Fig. third Table second, whose Latus Rectum is 1 and Lat. Transv. is $\frac{9a^{\frac{2}{3}}}{16}$. And if you put $a = 1$, or $x^4 = 1y^3$, for the Equation to the Curve proposed, you'll have the following Construction. (See Fig. third, Table second, Quad. p. 25.) With the Parameter a , and transverse Axe $Ka = \frac{9}{16}a$ describe the Hyperbola $aGDS$, take $aB = AB^{\frac{2}{3}}$, that is, take aB the second of two middle Proportionals betwixt a and AB , the Absciss of the given Parabola AC , draw the Ordinate BD , then

the Area of the Curve whose Ordinate is $\sqrt{1 + \frac{16x^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$, is $\frac{3}{2}aGDB$: consequently, the Area $aGDB$ being applied to the Lat. Rectum a , $\frac{1}{2}$ of the Line arising shall be equal to the Curve-line of the proposed Parabola AC , of the third Kind.

Now if we would know from whence the Beginning of the Curve-line is to be computed, it is evident, that, since $z = \frac{1}{2} \frac{aGDB}{a}$, when $aGDB = 0$, then $z = 0$: but $aGDB$ is nothing when the Absciss aB is nothing, and $aB = AB^{\frac{2}{3}}$, therefore z is nothing when AB the Absciss of the proposed Parabola AC is nothing: therefore the Length of the Curve-line found is to be computed from A , being AC .

410. If we would have the Length of the Curve-line expressed by Logarithms, according to what has been delivered in Sect. 10: then see Fig. Art. 355 *, which substitute in place of Fig. 3. Table 2. Quad. supposing it described with the same Latus Transversum, and Rectum, as before: then $z = \frac{\Delta ABD}{a} - \frac{\frac{1}{2}Aa \times aH}{a} \frac{aH}{DC}$ *, e. g. suppose the

* Of this Explication.

* Art. 355, 370. of this Explication.

Length of the Absciss $AB = x = a = 1$, then in the Hyperbola by making $aB = (AB^2 =) 1$, you'll find $aH = \frac{2}{3} = 0.6666$, &c. in inf. $BC = \frac{1}{2} = 1.41666$, &c. in inf. $BD = \frac{1}{4} = 1.25$, whence $DC = 0.16666$, in inf. therefore it is $z = 1.80569$, &c. — 0.296296 , &c. 0.6666 in inf. 0.1666 in inf. : wherefore, since the Modulus of the tabular Logarithm is 0.434294 , &c. * hence it follows that it is $z = 1.80569$, &c. — 0.296296 , &c. 0.434294 , &c. $\times \text{Log. } 0.6666 \text{ in inf.} - \text{Log. } 0.1666 \text{ in inf.} = 1.80569$, &c. — 0.682247 , &c. $\times \text{Log. } 4 = 1.80569$, &c. — 0.682247 , &c. $\times 0.6020600 = 1.80569$, &c. — 0.41075 , &c. = 1.4949 nearly the Length of the Curve-line AC required.

* Art. 349.
of this Ex-
plication.
* Art. 350.
of this Ex-
plication.

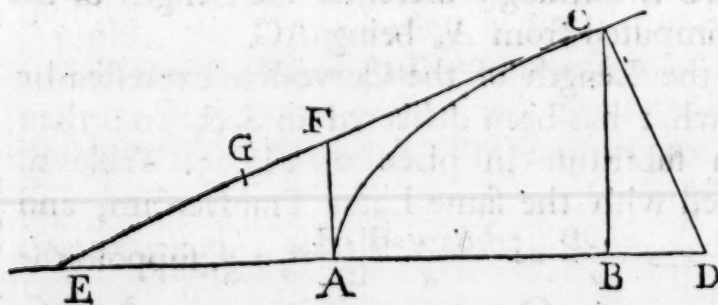
The same thing may be done by reducing the Value of the Area into a converging Series by Prop. 5. And after the like manner you may find the Lengths of the Curve-lines belonging to the other Orders of Parabolas, when n is 6, 8, 10 or any other even Number.

* Of this
Explica-
tion.

411. Ex. 2. Let it be proposed to find the Length of the Cycloid (see the Fig. at Art. 332 *). Use the same Symbols as in that Article, only call the Absciss $AB = x$, and the Curve-line z .

Then by what is there demonstrated, you have $y = \frac{x\sqrt{dx - xx}}{x}$, or $y^2 = x^2 \times \frac{d - x}{x}$: insert this Value of y^2 in the general Formula $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, and it stands thus $\dot{z} = \sqrt{\dot{x}^2 + \frac{d - x}{x} \dot{x}^2} = \dot{x} \sqrt{\frac{d}{x}}$ or $d^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$: which belongs to Form first, Table first, Quad. viz. $dx^{\frac{1}{2}} = y$: so that if you substitute $d^{\frac{1}{2}}$ for d , $\frac{1}{2}$ for n , and x for z , the Fluent is $2\sqrt{dx} = 2$ Chord AG : which therefore denotes the Length of the Arch of the Cycloid AC : for it's initial Limit is at A, since $2\sqrt{dx}$ is nothing when $x = AB$ is nothing. Hence it follows that $ACD = 2AE$, twice the Diameter of the generating Circle.

412. Ex. 3. Let the common Parabola ABC be proposed, and it is required to find the Length of the Part AC of the Curve-line, AB being the Axis, and BC it's Ordinate.



To do which, using the same Symbols for Absciss AB, Ordinate BC and Curve-line as formerly ; call the Parameter a : then since it is $ax = y^2$; hence by taking

taking the Fluxions you $ax = 2yy$, that is $y = \left(\frac{ax}{2y}\right) = \frac{a^2x}{2x^2}$, or $y^2 =$

$\frac{ax^2}{4x}$ insert this Value of y^2 in the general Formula $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, and

it becomes, by a due Reduction, $\dot{z} = \dot{x} \times \frac{1}{2} \sqrt{4 + ax^{-1}}$: which being the same with the Fluxion of a Curve that is not quadrable (by Art. 147, 149 *.) I find it belongs to Species second, Order third, Table

second, viz. $\frac{d}{z^{n+1}} \sqrt{e + fz^n} = y$, by putting $n = -1$, &c. By which

* Of this
Explica-
tion.

means it may be found, by applying an hyperbolical Space to linear Unity, as was shewn before in the Case of a Parabola of the third Order.

413. Moreover it may be observed, that as we arrive at the Lengths of Curves by substituting the Value of y^2 in Terms made up of x and it's Fluxions, in the general Formula $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; so you may substitute $\dot{x}^2$ it's Value made up of y and it's Fluxion, and by that means do the same: thus because it is in this Example, $ax = 2yy$, substitute $\frac{2yy}{a}$ for $\dot{x}$, or $\frac{4y^2y^2}{a^2}$ for $\dot{x}^2$,

thence it will be $\dot{z} = \sqrt{\frac{4y^2y^2}{a^2} + \dot{y}^2} = \dot{y} \sqrt{1 + \frac{4y^2}{a^2}}$: which agrees with the general Expression for all Parabolas, as in Art. 407 *, viz. $\dot{z} =$

* Of this
Explica-
tion.

$$\dot{z} \sqrt{1 + \frac{n^2 x^{n-1}}{n^2 - 2n + 1 \times a^{n-1}}}, \text{ with this only Difference, that you}$$

have y here for x there, arising from this Circumstance, that the Absciss in the former Case was taken upon a Line perpendicular to the Axis upon which the Absciss is taken in this Example, as is evident.

Therefore now that it is $\dot{z} = \dot{y} \sqrt{1 + \frac{4y^2}{aa}}$ or $\dot{z} = \dot{y} y \sqrt{\frac{4}{a^2} + y^{-2}}$, which belongs to Species second, Order third, Table second, viz.

$\frac{d}{z^{n+1}} \sqrt{e + fz^n} = y$, by putting $n = -2$, $z = y$, &c. it will appear

by consulting Art. 295 *, and comparing it with the Expressions $\frac{1}{z^2} = x^2$, $\sqrt{f + ex^2} = v$, and $-\frac{2d}{z} s = t$, in the Table, that, assum-

* Of this
Explica-
tion.

ing any Line for linear Unity, if you take (see Fig. 2. Table 2. p. 25.) $AP = 1$, and with it for the half transverse Axis, and $\frac{1}{2} \frac{aa}{AP}$ for the Latus Rectum, describe the Hyperbola PGDS, take upon AN the indeterminate Axis, AB equal to BC the Ordinate of the pro-

posed

posed Parabola ABC, and draw the Ordinate BD, then you shall have $\frac{ABDP}{1} = z = AC$ the parabolical Arch required.

414. Whence we learn to express the Length of the parabolical Arch AC by the Measures of Ratios thus. Draw the Tangent CE meeting the Axis in E, through A draw AF perpendicular to AB, meeting the Tangent in F, likewise CD perpendicular to the Tangent meeting the Axis in G: then take FG the Measure of the Ratio of AE + EF to AF to the Modulus $\frac{1}{2}BD$, so shall it be CG = Arch AC: which according to the Notation in Sect. 10. is expressed thus; $AC = CF + \frac{1}{2}BD \left| \frac{AE + EF}{AF} \right|$.

For since it has been shewn that $AC = \frac{ABDP}{1}$ in the preceding Art. and $ABDP = \triangle ABD + \text{Sector APGD}$; and moreover that in the Hyperbola you have $AP = 1$, Lat. Rect. $= \frac{1}{2}aa$, $AB = y$ and $BD =$

* Of this
Explica-
tion.

$\sqrt{1 + \frac{4}{a^2}y^2}$, it will follow by considering Art. 344 *, that the hyperbolical Sector APGD $= \frac{1}{4}a \left| \frac{\sqrt{\frac{1}{4}aa + yy} + y}{\frac{1}{2}a} \right| =$ (by the Construction above in the Parabola) $\frac{1}{2}BD \left| \frac{CD + BC}{BD} \right|$, or (from similar Triangles) $\frac{1}{2}BD \left| \frac{FE + EA}{AF} \right|$ and the Triangle ABD in the Hyperbola is equal

to $\frac{y\sqrt{1 + \frac{4}{a^2}y^2}}{2} = \frac{1}{2} \times \frac{y\sqrt{\frac{1}{4}aa + yy}}{\frac{1}{2}a} =$ (in the Parabola) $\frac{1}{2} \times \frac{BC \times CD}{BD} =$

CF. Therefore the parabolical Arch $AC = (CF + \frac{1}{2}BD \left| \frac{FE + EA}{AF} \right| = CF + FG) = CG$. As was to be shewn. And this shall suffice for explaining and solving this general Problem: *to find the Length of Curve-lines.*

415. Schol. Here it is proper to observe that you may find as many Curves as you please, whose Lengths are equal to known right Lines; or may be compared to the Areas of known Curves applied to Unity, like to what was shewn formerly with respect to the Areas of Curves. See Art. 100, 225, 226 *, e. g. let it be required to find the Curve whose Length $z = \frac{2}{3}x^{\frac{3}{2}}$.

* Of this
Explica-
tion.

Therefore by taking the Fluxions, it is $\dot{z} = x^{\frac{1}{2}}\dot{x}$ and by squaring, $\dot{z}^2 = x\dot{x}^2$: but it is $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, therefore by Substitution, $x\dot{x}^2 = \dot{x}^2 + \dot{y}^2$, or $x - 1 \times \dot{x}^2 = \dot{y}^2$, that is by extracting the Square Root $\dot{x}\sqrt{x-1}$

$\dot{x}\sqrt{x-1}=\dot{y}$; therefore it appears from what has been said, that the Area of a Curve whose Absciss is x and Ordinate $\sqrt{x-1}$, applied to Unity, will give the Length of y the Ordinate of the Curve required: which you'll find, by comparing the Expression $\sqrt{x-1}$ with Species 1. Order 3. Table 1. to be $\frac{2}{3} \times x - 1)^{\frac{1}{2}}$: therefore $y = \frac{2}{3} \times x - 1)^{\frac{1}{2}}$ defines the Property of the Curve sought.

And thus you may find as many other Curves as you please, which may be rectified; or compared with the Areas of known Curves applied to Unity, by assuming or feigning any Relations of x and z you please.

P R O B.

To find the Areas of Curve Surfaces.

416. The Principle upon which the Solution of this Problem depends, was explained and demonstrated Art. 50 *, which amounts to this: if any plane Figure ABC (see the Fig. belonging to that Art.) revolve upon AB as an Axis of Rotation, and thereby generate a Solid, call $AB=z$, it's rectangular Ordinate $BC=y$, the Surface generated by the revolving of the Curve-line $AC=s$; then supposing AB, BC and AC to flow, it is $\dot{s} = \frac{c}{r} \sqrt{\dot{z}^2 + \dot{y}^2}$: where c denotes the Circumference of any Circle whose Radius is r . Therefore if from the Equation to the generating Curve ABC, defining the Relation of z and y , you find a new fluxional Equation containing the Relation of their Fluxions, by means of this last Equation, and the Equation to the Curve, you may always find such a Value of $\dot{s}$, as shall only include one of the unknown Quantities z and y , and it's Fluxion, with known Quantities: so that by the Quadrature of Curves the Value of s may be found, after the same Manner as in the Solution of the preceding Problem, as will appear by the following Examples.

* Of this
Explica-
tion.

417. Ex. 1. Let it be proposed to find the superficial Content of a Solid generated by the Rotation of a Segment or Frustum of the common Parabola round it's Axis.

Then from the Property of the Parabola (if you put a for the Parameter) it is $az=y^2$, whence $a\dot{z}=2y\dot{y}$: but by the preceding Article $\dot{s} = \frac{c}{r} \sqrt{\dot{z}^2 + \dot{y}^2}$: wherefore, by inserting $\frac{4y^2\dot{y}^2}{aa}$ for $\dot{z}^2$, deduc'd

from the preceding fluxional Equation, it becomes $\dot{s} = \left(\frac{cy}{r} \sqrt{\frac{4y^2\dot{y}^2}{aa} + \dot{y}^2} \right)$

* Of this
Explica-
tion.

$=) \dot{y} \times \frac{y}{r} \sqrt{1 + \frac{4}{aa} y^2}$. Now the Expression $\frac{y}{r} \sqrt{1 + \frac{4}{aa} y^2}$ be-
longs to a Curve that may be squared (as appears by Art. 147, 149 *.)
and is the same with the Ordinate of a Curve belonging to Species 1.
Order 3. Table 1. viz. $dz^{\eta-2} \sqrt{e+fz^{\eta}}$, whose Area is $\frac{2d}{3\eta f} \times e + fz^{\eta}$;
so that by substituting y for z , 2 for η , $\frac{c}{r}$ for d , 1 for e , and $\frac{4}{aa}$
for f , the Fluent analogous to the Area of the Curve, is $\frac{a^2 c}{12r} \times$
 $\sqrt{1 + \frac{4}{aa} y^2}$. But to ascertain the true Value of the Fluent we want,

* Of this
Explica-
tion.

it must be computed from the Vertex of the Paraboloid as it's initial
Limit, which corresponds to the Computation of the curvilinear Area
from the Beginning of the Absciss, where $y = 0$. Here then is a
Condition limiting the Fluent, viz. that it must be nothing when y
is nothing: therefore by considering Art. 387 *, the true Fluent is
 $\frac{a^2 c}{12r} \times \sqrt{1 + \frac{4}{aa} y^2} - \frac{a^2 c}{12r}$: for this only is the Fluent that vanishes when
 y vanishes: and therefore $\frac{a^2 c}{12r} \times \sqrt{1 + \frac{4}{aa} y^2} - \frac{a^2 c}{12r} = s$ the superficial
Content of the parabolical Conocoid required.

418. Ex. 2. To find the Surface of a spheroidical Segment, the
Spheroid being that which is formed by the Rotation of an Ellipse
round it's longer Axis, called an oblong Spheroid.

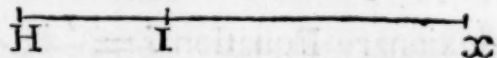
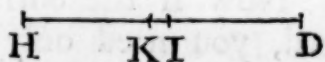
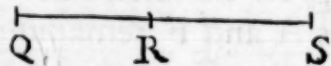
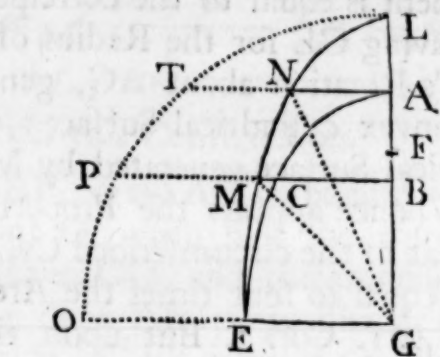
Let c, r, z, y, s denote the same things as formerly: moreover
let k and l denote the Latus Transversum and Latus Rectum of the
Ellipse. Then from the Property of the Ellipse, you have $y^2 =$
 $\frac{klz - lz^2}{k}$: whence, by taking the Fluxions, $2yy\dot{y} = \frac{kl\dot{z} - 2l\dot{z}z}{k}$, that is $\dot{y}^2$
 $= \dot{z}^2 \times \frac{k^2 l - 4klz + 4lz^2}{4k^2 z - 4kz^2}$; by inserting of which Value of $\dot{y}^2$ instead of
it in the general Formula $s = \frac{y}{r} \sqrt{\dot{z}^2 + \dot{y}^2}$, and $\sqrt{\frac{klz - lz^2}{k}}$
for y , and making the proper Reduction, you shall obtain $\frac{c}{r}$

* Of this
Explica-
tion.

$\dot{z} \sqrt{\frac{1}{4} ll + \frac{kl - ll}{k} z - \frac{kl - ll}{k^2} z^2} = \dot{s}$. Now by comparing this ra-
dical Expression with the Form of the Ordinate Art. 302 *, viz. $v =$
 $\sqrt{\frac{apt - a^2 p}{t} + \frac{pt - 2ap}{t} x + \frac{p}{t} x^2}$, you will find that it denotes the
Ordinate of an Ellipse: in which, by comparing the homologous
Terms

Terms by the Rules of Algebra, you'll get 1^{mo}. $t = k\sqrt{\frac{k}{k-l}}$ 2^o. $p = l\sqrt{\frac{k}{k-l}}$ 3^o. $a = \frac{1}{2}k\sqrt{\frac{k}{k-l}} - \frac{1}{2}k$ or $\frac{1}{2}t - \frac{1}{2}k$. By considering which you may deduce the following Construction.

Let AGE be a Quadrant of the generating Ellipse, AG half the longer and GE half the shorter Axis, BC an Ordinate to the longer Axis, F the Focus. Then upon GA produc'd take GL a third Proportional to GF and GA: with the semitransverse GL and femiconjugate GE, describe the Ellipsis LE, draw AN perpendicular to AG, meeting the Ellipse LE in the Point N, and produce BC till it meet it in M: then the Area ABMN is to the Surface of the spheroidal Segment generated by the Rotation of ABC, as r to c , that is as Radius of a Circle to the Circumference.



For since it is $GA = \frac{1}{2}k$, and half Parameter $= \frac{1}{2}l$, hence it is $GE = \frac{1}{2}\sqrt{kl}$; therefore $GF^2 = (GA^2 - GE^2) = \frac{1}{4}k^2 - \frac{1}{4}kl$, or $GF = \sqrt{\frac{1}{4}k^2 - \frac{1}{4}kl}$, consequently $GL = \left(\frac{GA^2}{GF}\right) = \frac{1}{2}k\sqrt{\frac{k}{k-l}}$: but from the Property of the Ellipsis LGE, $BM^2 : GE^2 (\frac{1}{2}\sqrt{kl}) :: GL^2 - GB^2 (\frac{1}{4}k^2 - \frac{1}{2}k - z)^2 : GL^2 (\frac{1}{4}k^2)$: from whence, by a proper Reduction, you have $BM =$

$\sqrt{\frac{1}{4}ll + \frac{kl-ll}{k}z - \frac{kl-ll}{k^2}z^2}$. Wherefore the Fluxion of the elliptical

Space ABMN is $\dot{z}\sqrt{\frac{1}{4}ll + \frac{kl-ll}{k}z - \frac{kl-ll}{k^2}z^2}$: but it was shewn above

that $\dot{z} = \frac{cz}{r}\sqrt{\frac{1}{4}ll + \frac{kl-ll}{k}z - \frac{kl-ll}{k^2}z^2}$, therefore the Fluxions are to one another as r to c , a given Ratio, and the Fluents have both the same initial Limit at AN, therefore ABMN : the Surface of the spheroidal Segment :: $r : c$.

419. Cor. 1. Hence it appears that the whole spheroidal Surface is to 4AGEN, as the Periphery of a Circle to it's Radius.

420. Cor. 2. If GE remaining the same, GA grow shorter and shorter, GL will thereby grow longer and longer, until GA become
R r equal

by comparing of which with Art. 294 *, the radical Expression will be found to denote an Ordinate to the second or conjugate Axis of an Hyperbola. And if the Terms of the radical Expression

$\sqrt{\frac{1}{4}ll - \frac{ll-k^2}{k}z + \frac{ll-k^2}{k^2}z^2}$ be compared with the homologous Terms

of the hyperbolic Ordinate in that Art. viz. $\sqrt{\frac{a^2t + \frac{1}{2}pt}{p} - \frac{2at}{p}x + \frac{1}{p}x^2}$, as formerly, you will obtain these Values of t , p and a , viz. $t = \sqrt{kl}$, $p = \frac{k^2}{l-k}\sqrt{\frac{k}{l}}$ and $a = \frac{1}{2}k$. Whence arises this Construction.

F being the Focus of the Ellipse, join AF, draw AO perpendicular to AF, meeting EG produced in the Point O: with half the transverse Axis GE, and half the conjugate GL = GO, and the principal Vertex E, describe the Hyperbola EMN; draw AN perpendicular to AG meeting the Hyperbola in N; produce BC till it meet the Hyperbola in M: then the hyperbolic Area ABMN is to the Surface of the Segment of the oblate Spheroid generated by the Revolution of ABC as r to c .

For $GF = (\sqrt{AF^2 - AG^2}) = \sqrt{\frac{1}{4}kl - \frac{1}{4}k^2}$, and $GL = (\frac{AG^2}{FG} = \frac{\frac{1}{4}k^2}{\sqrt{\frac{1}{4}kl - \frac{1}{4}k^2}})$: but from the Property of the Hyperbola it is $BM^2 - GE^2 : BG^2 :: GE^2 : GL^2$, or by Alternation and Composition, $BM^2 : GE^2 (\frac{1}{4}kl) :: BG^2 + GL^2 (\frac{1}{4}k - z)^2 + \frac{1}{4}k^2 : GL^2 (\frac{1}{4}k)$, whence $BM = \sqrt{\frac{GE^2 \times BG^2 + GL^2}{GL^2}} =$ by a due Reduction $\sqrt{\frac{1}{4}ll - \frac{ll-k^2}{k}z + \frac{ll-k^2}{k^2}z^2}$:

whence it follows as in the preceding Construction, that the hyperbolic Space ABMN is to the Surface of the Segment of the oblate Spheroid, as Radius of a Circle to it's Circumference.

423. Schol. By comparing these two Cases of the oblong and oblate Spheroid together, and by supposing the one to pass through the Sphere into the other, you see the gradual Progression of the Values of the Surfaces, by the Line EMN passing from an Ellipse, concave to AB, into a right Line parallel to it, and then into an Hyperbola convex to it: and that the revolving Axis GE is always one Axis of the Conic Section EMN, and the other GL a third Proportional to GF and GA.

In the foregoing Investigations of the Surfaces of the oblong and oblate Spheroid, if you had supposed $GB = z$, so that the Absciss

425. Schol. If the Points A and F continuing, the Center G be supposed to remove to an infinite Distance, AC becomes a Parabola, and LNM another, having both the same Latus Rectum, and $AL = AF$: for by supposing k infinite, the fluxional Equation $\dot{s} =$

$\frac{cz}{r} \sqrt{\frac{1}{4}ll + \frac{ll+lk}{k}z + \frac{ll+l^2}{k^2}z^2}$ becomes $\dot{s} = \frac{cz}{r} \sqrt{\frac{1}{4}ll + lz}$: where l denotes the Latus Rectum of the Axis of the revolving Parabola. Wherefore since it is $AL = (AF =) \frac{1}{4}l$, you have $LB = \frac{1}{4}l + z$, multiply this by l , and it produces $\frac{1}{4}ll + lz$, which is the Square of the Ordinate BM, therefore LNM is a Parabola having the same Latus Rectum with the revolving Parameter AC: so that the Surface of the Segment of the Paraboloid, generated by the Rotation of the Parabola ABC, is to the parabolical Space ABMN, as c to r .

426. Finally, if the Hyperbola AC be supposed to turn round upon the conjugate Axis GE, and thereby generate a Surface convex towards GE; take upon GE, produced if need be, any Absciss $G\beta$, and draw an Ordinate βx : then calling $GB = z$, and $\beta x = y$, you may find as before another Hyperbola giving an hyperbolical Space, which shall be to the Surface generated by the Rotation of Ax , as r to c . This other Hyperbola has GA for it's half Transverse, and a third Proportional to GF and GE for it's half Conjugate: the same Property as the Conic Sections formerly found.

427. If the Area ABMN in all the preceding Cases be applied to the Absciss AB, and the Line resulting be made the Altitude of a Rectangle whereof AB is the Base; and such Rectangle, by revolving round AB as an Axis, generate a right Cylinder, the convex Surface of the Cylinder is equal to the Curve-Surface of the Segment of the Spheroid or Conoeid generated by the Rotation of ABC about AB. For such cylindrical Surface is to the Area ABMN as c to r : as is evident. Understand the same of the Surface formed by the Rotation of Ax about $A\beta$, when the hyperbolical Area arising is applied to $G\beta$.

428. By considering what has been demonstrated, and comparing it with what was shewn formerly in Sect. 10. you may know how to express these several things by the Measures of Angles and Ratios.

Thus in the Case of the oblong Spheroid (see Fig. Art. 418 *) if you imagine the two right Lines or Radius's GM, GN to be drawn, the Area $ABMN = LBMNL - LAN = \text{Sector LGM} - \triangle BGM - \text{Sector LGN} + \triangle AGN$: but the Sector LGM $= \frac{1}{2}GE \times GL$.

$\frac{GB \cdot \sqrt{GL^2 - GB^2}}{GL}$, and the Sector LGN $= \frac{1}{2}GE \times GL \frac{GA \cdot \sqrt{GL^2 - GA^2}}{GL}$; * Art. 363, 370. of this Explication.

* Of this Explication.

the

the $\Delta BGM = \frac{BG \times BM}{2}$, and $\Delta AGN = \frac{AG \times AN}{2}$: therefore the Area

$ABMN = \frac{1}{2}GE \times GL \left| \frac{GB \cdot \sqrt{GL^2 - GB^2}}{GL} - GL \right| \frac{GA \cdot \sqrt{GL^2 - GA^2}}{GL} + \frac{AG \times AN - BG \times BM}{2}$: wherefore if this Quantity be multiplied by $\frac{c}{r}$ $= 6.2831853$, &c. the Quantity arising will denote the Value of the Surface generated by the Rotation of AC about AB.

429. If the Surface generated by the Rotation of the Arch EC were required, then it is equal to $\frac{c}{r} \times BGEM$: but $BGEM = (\text{Sector}$

* Art.
363, 370.
of this Ex-
plication.

$EGM + \Delta BGM = \frac{1}{2}GE \times GL \left| \frac{\sqrt{GL^2 - GB^2} \cdot GB}{GL} * + \frac{BG \times BM}{2} \right|$: there-
fore if the Arch which measures the Angle whose Radius, Tangent
and Secant are to one another as $\sqrt{GL^2 - GB^2}$, GB, and GL, or
(which is the same) whose Radius is to it's Sine as GL to GB, be
multiplied into $\frac{1}{2}GE$, and the Triangle BGM added thereto, then that
Sum increased in the Ratio of c to r shall be equal to the Curve-
Surface required. Which may be constructed thus.

With the Center G and Radius GL describe the Quadrant LPO,
produce BCM till it meet it in P: take $QR : BG :: BM : GE$; to
QR add $RS = OP$, then the Surface generated by the Rotation of
the Arch CE is to the Circle described with the Radius GE as QS
to GE.

For the Surface generated by the Rotation of BC is $\frac{1}{2}GE \times GL$

$\left| \frac{\sqrt{GL^2 - GB^2} \cdot GB}{GL} + \frac{BG \times BM}{2} \right| \times \frac{c}{r} = \frac{1}{2}GE \times OP + \frac{GE \times QR}{2} \times \frac{c}{r} = \frac{1}{2}GE$
 $\times QS \times \frac{c}{r}$: but the Circle described with the Radius GE is $\frac{1}{2}GE^2 \times$
 $\frac{c}{r}$: therefore, dividing both by $\frac{1}{2}GE \times \frac{c}{r}$, they are to one another as
QS to GE. And this is the Construction given by Cotes, *Harmonia*
Mensurarum, p. 28.

And after the same manner, if you produce AN till it meet the
Circle in T, take $HI : GA :: AN : GE$ and $HK : GB :: BM : GE$;
and upon HI produced take $ID = PT$, then the Surface produced by
the Rotation of the Arch AC, shall be to the Circle described with
the Radius GE, as KD to GE.

Moreover, if you take HI the same as before, and add to it $Ix =$
OT, then the Surface generated by the Rotation of ACE, that is the
half

half of the Surface of the oblong Spheroid, is to the Circle described with the Radius GE, as HX to GE.

All which things may be easily found by the trigonometrical Canon, and the known Ratio of the Radius of a Circle to it's Circumference.

430. With respect to the oblate Spheroid (see Fig. Art. 422 *) the Surface generated by the Rotation of CE about BG is equal to $\frac{c}{r}$ * Of this Explication.

× BGEM : now if you draw the Radius GM to the Hyperbola NEM, it appears that the Area BGEM = Sector GEM + ΔBGM : but if you draw the Asymptote GPQ, cutting BM and AN in P and Q, the Sector GEM may be variously expressed by the Measure of a Ratio, according to Art. 344, 366, 367 *, as thus $\frac{1}{2}LG \times GE \left| \frac{BM+BP}{GE} \right.$ or $\frac{1}{2}LG$ * Of this Explication.

$$\times GE \left| \frac{GE}{MP} \right. \text{ or } \frac{1}{2}LG \times GE \left| \frac{BM+BP}{\sqrt{BM^2 - BP^2}} \right. \text{ \&c. or } \frac{1}{2}LG \times GE \left| \frac{BM + \sqrt{BM^2 - GE^2}}{GE} \right. :$$

wherefore BGEM = ΔBGM, together with the Measure of any of these Ratios to the Modulus GE multiplied into $\frac{1}{2}LG$: for which it might be easy to make a Construction, similar to that for the Surface of the oblong Spheroid, and like what Mr. Cotes has given in the Place already mentioned. And if the Lengths of the Lines GE, GA, GB be expressed in Numbers, you may easily compute the Surface of any Part or the Whole of the oblate Spheroid by means of a Table of Logarithms. The same things are to be understood of the other Solids which have been mentioned. Which I shall leave to the young Geometrician for his own Exercise : and pass on to the next Problem that I may hasten to a Conclusion.

P R O B.

To find the Contents of Solids generated by the Rotation of plane Figures.

431. The Principle upon which the Solution of this Problem depends, was fully explained Art. 43, 44 *, which amounts to this : if z denote the Fluxion of a Rectangular parallelopiped, whose Breadth and Thickness is 1, or having a square Base whose Side is 1 ; and the variable Length of the Parallelopiped, z , then the Fluxion of any Body, generated by the Rotation of any plane Figure, having z for it's Absciss and y for it's perpendicular Ordinate, will be equal to z multiplied into the Circle whose Radius is y . Now if r and c signify the same thing as before, then the Area of a Circle whose Radius is y , will be designed by $\frac{y^2}{2r}$: therefore if the solid Content of any such

such Body be called s , then you have this general Formula $\dot{s} = \frac{cy^2\dot{z}}{2r}$. Wherefore an Equation being given containing the Relation of z and y , you may by help of it throw y out of the Equation $\dot{s} = \frac{cy^2\dot{z}}{2r}$; or, if you find it more convenient, you may throw out $\dot{z}$, and retain a Value of $\dot{s}$ including only y and it's Fluxion: by either of which means you may find the Fluent by the Quadrature of Curves, by finding the Area of a Curve having an analogous Fluxion; which Area being found, and supposed to be multiplied by 1, gives the Content of the Solid. See Art. 390, 407 *.

* Of this
Explica-
tion.

432. Ex. 1. Let it be proposed to find the Solidity of a right Cone; which is generated by the Revolution of a right-angled Triangle round one of the Sides including the right Angle. Wherefore calling the Side about which the Triangle revolves a , and the other Side, which generates the circular Base, b ; let z and y denote an Absciss and Ordinate in that Triangle: then $z : y :: a : b$, or $y = \frac{bz}{a}$: wherefore in the general Formula, viz. $\dot{s} = \frac{cy^2\dot{z}}{2r}$, insert this Value of y , and it becomes $\dot{s} = \frac{cb^2z\dot{z}}{2ra^2}$, which may denote the Fluxion of a quadrable Curve belonging to Form 1. Table 1. or the Fluents may be found by the Converse of our Author's first Proposition; by either of which you'll have $s = \frac{cb^2z^3}{6ra^2} = (\text{by inserting } y^2 \text{ for } \frac{b^2z^2}{a^2}) \frac{cy^2z}{6r}$, the Value of the Cone having z for it's Altitude: which when z and y become a and b , is $\frac{cb^2a}{6r}$, i. e. a third Part of the Cylinder having the same Base and Altitude, which is $\frac{cb^2a}{2r}$.

* Of this
Explica-
tion.

433. Ex. 2. Let it be required to find the Solidity of any Segment of an oblong Spheroid, as that generated by the Rotation of ABC (see Fig. Art. 418 *) about the Axis AG. Let a and b denote the Halves of the longer and shorter Axis: call $AB = z$, $BC = y$, then the Equation to the Curve is $y^2 = \frac{2b^2z}{a} - \frac{b^2z^2}{a^2}$, therefore by inserting this Value of y^2 into the general Formula $\dot{s} = \frac{cy^2\dot{z}}{2r}$, it becomes $\dot{s} = \dot{z} \times$

* Of this
Explica-
tion.

$\frac{cb^2z}{ra} - \frac{cb^2z^3}{2ra^2}$: wherefore if you find (by Art. 145 *, or otherwise) the Area of a Curve whose Absciss is z , and Ordinate $\frac{cb^2z}{ra} - \frac{cb^2z^3}{2ra^2}$; it will give the Value of the Segment required, viz. $\frac{cb^2z^2}{2ra} - \frac{cb^2z^3}{6ra^2} = \frac{cb^2}{2r} \times$

$\times \frac{z^3}{a} - \frac{z^3}{3a^3}$ or $\frac{cz^3}{2r} \times \frac{b^2}{a} - \frac{b^2 z^3}{3a^3}$: the first of which Expressions denotes a Cylinder having b for the Radius of it's Base, and a third Proportional to a and z for it's Altitude, lessened by a Cylinder standing on the same Base whose Altitude is to that of the former as z to $3a$; or by a Cone standing upon the same Base, whose Altitude is to that of the Cylinder as z to a : and after the like manner may you resolve the other Expression.

433. Whence it follows that if z become equal to a , the former Expressions become $\frac{cb^2}{2r} \times \frac{2}{3}a$, and $\frac{ca^2}{2r} \times \frac{2}{3}\frac{b^2}{a}$: which denote $\frac{2}{3}$ of a Cylinder having the same Base and Altitude with the Half of the Spheroid; the other a Cone having half the longer Axe for the Radius of it's Base, and the Latus Rectum for it's Altitude. When a becomes equal to b , so that the Spheroid become a Sphere, it is easy to apply the Expressions to that Case: and likewise when a becomes shorter than b , so that the Spheroid become oblate: also to the Cases of the hyperbolical and parabolical Conocoids: particularly you'll find the parabolical Conocoid to be $\frac{1}{2}$ of the Cylinder having the same Base and Altitude.

434. Ex. 2. Let it be required to find the solid Content of a Body formed by the Rotation of the Cissoïd ACE (see Fig. Art. 321 *.) round it's Axis ACQ.

* Of this Explication.

Call $AC = z$, $CE = y$, then the Equation to the Curve $y^2 = \frac{z^3}{a-z}$: insert this Value of y^2 in place of it in the general Formula $s = \frac{cy^2 z}{2r}$, and you have $s = \frac{cz^3 z}{2r \times a-z}$, which I find to belong to Form 1. Tab. 2.

viz. the fourth Species, by which means therefore it may be found, by help of the equilateral Hyperbola; and consequently by Logarithms or Measures of Ratios: which I shall omit, and shew how to express it by applying directly to our Author's fifth Proposition. The Ordinate then of the Curve which flows according to the same Law with this Solid, is $\frac{cz^3}{2r \times a-z}$, which being compared with the general Form

for Binomials, viz. $y = az^{b-1} \times e^{\pm f z^n} \lambda^{-1}$, will give by Art. 147 *,

* Of this Explication.

for the Area $\frac{c}{2r} \times \frac{z^4}{4a} + \frac{z^5}{5a^2} + \frac{z^6}{6a^3} + \frac{z^7}{7a^4} + \frac{z^8}{8a^5} + \&c. \text{ in } \text{inf.}$ which therefore denotes the Value of the Solid generated by the Rotation of the Cissoïd.

435. And now I have shewn what way this Treatise of the Quadrature of Curves is applied to the Solution of some Problems: by which the young Geometrician may see how extensive this Doctrine is; and understand how it is to be applied in other Cases. In general it will serve for finding the Fluent of any given Fluxion (of whatever Kind the Fluent be) where the given Fluxion can be reduced to such an Expression as denotes the Fluxion of any curvilinear Area. And I shall now proceed to explain the several Cases of fluxional Equations, mentioned by our Author, from whence the Fluents can be found by the Quadrature of Curves.

* Of the Quadrature of Curves.

* Of this Explication.

436. The first Case is contained in Art. 75 *, and it is explained and illustrated thus. Let $ax - e^2\dot{z} = 2efx^2\dot{z} + f^2x^4\dot{z}$: where x is supposed to flow uniformly, and $\dot{z}$ is the first Fluxion of the other flowing Quantity, viz. z . I say the Relation of x and z may be found, so that x being given, z shall be given. In order to which, multiply the Term ax by $\dot{x} = 1$, according to Art. 335 *, by which the Equation stands thus $ax\dot{x} - e^2\dot{z} = 2efx^2\dot{z} + f^2x^4\dot{z}$: thence reduce the Equation so, as that $\dot{z}$ stand upon one Side alone, which gives this

$$\frac{ax\dot{x}}{e^2 + 2efx^2 + f^2x^4} = \dot{z} : \text{where if } x \text{ be supposed the Absciss of a Curve,}$$

and $\frac{ax}{e^2 + 2efx^2 + f^2x^4}$ the corresponding Ordinate: and the Area of that

Curve be found, it shall be equal to z ; whence z shall be determined. Now, in the present Example, if you compare the Ordinate

$$\frac{ax}{e^2 + 2efx^2 + f^2x^4}$$

with the Ordinate belonging to Order 2. Table 1. you'll obtain, by a proper Substitution of Values, $\frac{ax^2}{2ee + 2efxx} = z$; or

$$\frac{-a}{2ef + 2f^2x^2} = z, \text{ either of which gives the Value of } z, \text{ and expresses}$$

the Relation of x and z . Which you may try by Prob. 1, to see whether both these fluential Equations, will produce the given Relation of the Fluxions. The Relation of the Fluxions arising from the

$$\text{first, viz. } \frac{ax^2}{2e^2 + 2efx^2} = z \text{ is } \frac{2ax\dot{x} \times 2e^2 + 2efx^2 - 4efx\dot{x} \times ax^2}{4e^4 + 8efx^2 + 4e^2f^2x^4} *, \text{ or by re-}$$

* Art. 84. of this Explication.

ducing $\frac{ax\dot{x}}{e^2 + 2efx^2 + f^2x^4} = \dot{z}$, that arising from the second Equation,

$$\text{viz. } \frac{-a}{2ef + 2f^2x^2} = z, \text{ is } \frac{4af^2x\dot{x}}{4e^2f^2 + 8ef^3x^2 + 4f^4x^4} \text{ or } \frac{ax\dot{x}}{e^2 + 2efx^2 + f^2x^4} = \dot{z};$$

both the same with the fluxional Equation at first proposed, supposing $\dot{x} = 1$.

437. So likewise, if the Equation had been $ax - e^2\ddot{z} = 2efx^2\ddot{z} + f^2x^4\ddot{z}$: where you have the second Fluxion of z , and any other flowing Quantity x , whose Fluxion is equal to 1, you may find the Relation of the Fluents by the Quadrature of Curves. For supposing as before $x = AB$ the Absciss [see our Author's Figure] and now $\ddot{z}$ or $\frac{\ddot{z}}{x} = BD$ the Ordinate, and consequently $\dot{z} = BE$ or $\frac{ABD}{1}$, and $z = BF$ or $\frac{ABE}{1}$: from the given Equation reduced, suppose, to this Form

$\ddot{z} = \frac{ax}{e^2 + 2efx + x^2}$ or $\ddot{z} = \frac{axx^2}{e^2 + 2efx + x^2}$, find the Relation of $\dot{z}$ and x that is of BE and AB , by finding the Area of the Curve ABD , whose Absciss being x has it's Ordinate $BD = \frac{ax}{e^2 + 2efx + x^2} = \ddot{z}$: which is done as before. This gives $\dot{z}$, or the Relation of x and $\dot{z}$; the same as of x and z , in the preceding Case, viz. $\frac{ax^2}{2e^2 + 2efx^2} = \dot{z} = BE$. From which given Relation of AB and BE or x and $\dot{z}$, you find $z = BF$ or $\frac{ABE}{1}$, by finding the Area of the Curve ABE , whose Absciss is $AB = x$, and Ordinate $BE = (\dot{z} =) \frac{ax^2}{2e^2 + 2efx^2}$: which therefore exhibits the Relation of z and x sought. But whereas the Curve which has x for it's Absciss, and $\frac{ax^2}{2e^2 + 2efx^2}$ for it's Ordinate, cannot be squared *,

* Art. 139. of this Explication.

therefore the Expression of the Area, found by Prop. 5, runs out into an infinite Series: so that the Relation of z and x cannot be expressed by any other than infinite Equation, *algebraically*. Yet it may be exhibited *geometrically* by the Description of a Conic Section, since the Curve comes under Species second, Order second, Table second. And thus you proceed in all like Cases, where the given Equation is of that Sort mentioned by our Author in this Article.

438. Hence it appears that if the Curve ADB be quadrable, the Relation of the Fluents may be expressed by a finite Equation, when it is a first Fluxion that enters into the given Equation. If the Curves ADB and AEB be quadrable, the Relation of the Fluents may be expressed by a finite Equation, when it is a second Fluxion that is contained in the given Equation. If the Curves ADB , AEB and AFB can be squared, then the same thing is true with respect to the Fluents, when it is a third Fluxion that enters the given Equation, &c. but if these Curves cannot be squared, the Relations of the Fluents cannot be exhibited by finite Equations, in the Cases mentioned. And you know when these Curves may be squared by Prop. 2, viz.

S s 2

then,

then, when, calling $AB = z$ and $BD = y$, the Curves having the Absciss z and Ordinates y , zy , z^2y , &c. can be squared: from which Proposition you likewise discover what the Areas of the Curves ADB, AEB, AFB, &c. in that Case are. Again if these Curves ADB, AEB, AFB, &c. are capable of a geometrical Comparison with the Conic Sections; then the Relations of the Fluents may be exhibited *geometrically*, by the Description of the Conic Sections.

Art 76, 77
of the
Quadrature of
Curves.

439. What our Author says with respect to Equations that involve only one flowing Quantity and it's first Fluxion; or only any two next adjoining Orders of Fluxions of the same flowing Quantity, without the other flowing Quantity to which it is related, *viz.* that, in such Cases, the Fluents may be found by means of the Quadrature of Curves, may be further explained and illustrated, in the following manner.

*Art. 384.
of this Ex-
plication.

Let the Equation $av\dot{v} = av + v^2$ be proposed: then suppose any other flowing Quantity of the same sort with the Quantity v : call it z , and let it's Fluxion $\dot{z} = 1$. Wherefore, by completing the Dimensions *, the given Equation is $av\dot{v} = av\dot{z} + v^2\dot{z}$. Let v , $\dot{v}$ and z be represented by BE or $\frac{ABD}{1}$, BD and AB respectively. Now

*Art. 389.
of this Ex-
plication.

since the Equation $av\dot{v} = av\dot{z} + v^2\dot{z}$ exhibits the Relation of the Fluxions $\dot{v}$ and $\dot{z}$, when $\dot{z} = 1$; the Relation of the Fluents will be the same, if you change the Supposition, and imagine that v flows uniformly, making $\dot{v} = 1$ *. And therefore, bringing the Equation to this Form $\frac{avv}{va + vv} = \dot{z}$, it appears, that, if you find the Area of a

Curve whose Absciss is v , and Ordinate $\frac{aa}{av + vv}$, that Area is equal to z : so that hence you have the Relation of v and z , or BE and AB, as formerly: and therefore z being assumed, v is thence determined, either *algebraically* or *geometrically*.

440. Moreover, if the Equation $av\ddot{v} = a\dot{v} + \dot{v}\dot{v}$ was proposed: suppose z another variable Quantity, whose Fluxion $\dot{z} = 1$; then by the like Operation as in the preceding Article, supposing v , $\dot{v}$, $\ddot{v}$ and z to be expounded by BF, BE, BD and AB respectively; by squaring the Curve, whose Absciss is $\dot{v}$ and Ordinate $\frac{aa}{a\dot{v} + \dot{v}\dot{v}}$, you find the Area equal to z ; so that you obtain the Relation betwixt $\dot{v}$ and z , or BE and AB. Again from the given Relation of $\dot{v}$ and z or BE and AB, you find by the Quadrature of Curves, the Area of the Curve AEB, whence you have the Relation of AB and BF, that is z and v , since $BF = \frac{AEB}{1}$. And the Method of proceeding is the same

same, when the given Equation contains any two Fluxions of any the same flowing Quantity, of any two Orders next to each other, without including the other Fluent to which it is related.

441. Our Author shews in this Article how a fluxional Equation including three unknown Quantities may sometimes be reduced to another including only two unknown Quantities: in which Case, the Relation of the Fluents may be found by the Quadrature of Curves as before. Thus let $a - bx^m = cxy^n\dot{y} + dy^{2n}\dot{y}\dot{y}$ be proposed, including three unknown Quantities, viz. x , y and $\dot{y}$, and the Relation of the Fluents x and y is sought. Put $y^n\dot{y} = \dot{v}$, then the Equation stands thus $a - bx^m = cx\dot{v} + dv\dot{v}$, which includes only two unknown Quantities, viz. x and $\dot{v}$: from whence as in Art. 75*, you may find the Relation of x and v . For by reducing the Quadratic Equation, you

Art. 78.
of the
Quadrature of
Curves.

get $\dot{v} = \sqrt{\frac{a}{d} + \frac{c^2x^2}{4d^2} + \frac{b}{d}x^m - \frac{cx}{2d}}$, or by multiplying by $\dot{x} = 1$, $\dot{v}$

$= \dot{x} \times \sqrt{\frac{a}{d} + \frac{c^2x^2}{4d^2} - \frac{b}{d}x^m - \frac{cx}{2d}}$: whence it appears, that the Area

of a Curve whose Absciss is x and Ordinate $\sqrt{\frac{a}{d} + \frac{c^2x^2}{4d^2} - \frac{b}{d}x^m - \frac{cx}{2d}}$

$= \dot{v}$, is equal to v : whence, by the Quadrature of Curves, you obtain the Relation betwixt x and v . Again from the other Equation, viz. $y^n\dot{y} = \dot{v}$, we find the Relation of v and y by the Converse of Prop. 1, which is $\frac{1}{n+1}y^{n+1} = v$: but the Relation of x and v was found formerly: therefore you have the Relation of x and y , so that y is found when x is given.

442. Our Author having shewn in the preceding Article that the Fluents may be found from fluxional Equations including three unknown Quantities, when they are capable of being reduced to such as involve only two unknown Quantities, proceeds, in this Article, to shew that sometimes the Fluents may be obtained from such Equations, even when they cannot be reduced to Equations that have but

Art. 79.
of the
Quadrature of
Curves.

two unknown Quantities. Thus let the Equation $ax^m + bx^n\dot{y}^p = rex^{r-1}\dot{y}^s + sex^ryy^{r-1} - f\dot{y}y^t$ including the Fluents x and y together with the Fluxion of y , be proposed: and y is sought for x given. Multiply the Terms where $\dot{y}$ is not found by $\dot{x} = 1$, and you have $\dot{x} \times ax^m + bx^n\dot{y}^p = rex^{r-1}\dot{y}^s + sex^ryy^{r-1} - f\dot{y}y^t$: of which Equation the Fluent of the last Part, viz. $rex^{r-1}\dot{y}^s + sex^ryy^{r-1} - f\dot{y}y^t$, may be had by going back from the Fluxion to the Fluent, by a reverse

verse

verse Operation to that by which the Fluxion of a given Fluent is found, *viz.* by increasing the Indexes of the flowing Quantities, in the Terms where their Fluxions are, by Unity, and dividing the Terms by the Indexes so increased, and dashing out the fluxionary Expressions: after which, if there result any Quantities that are equal, dash all of them out but one: and the Terms remaining make the Fluent (if it can be found by the Converse of Prop. 1.) And to ascertain the Truth, try whether the Quantity so found produce the same Fluxion as that at first: which if it do, you have the right Fluent, otherwise not. By proceeding thus, you find from the last Side of the Equation, $ex^ry^s + ex^ry^s - \frac{1}{r+1}y^{r+1}$: where the Term ex^ry^s being twice found, dash out one of them, and so you'll have $ex^ry^s - \frac{1}{r+1}y^{r+1}$ for the true Fluent of that Side of the Equation. And the other Side of the Equation, *viz.* $\dot{x} \times \overline{ax^m + bx^n}^p$ expresses a Fluxion analogous to that of a Curve whose Absciss is x , and Ordinate $\overline{ax^m + bx^n}^p$: wherefore, if you find the Area of that Curve, the Expression of the Value of that Area will be equal to $ex^ry^s - \frac{1}{r+1}y^{r+1}$: and thence therefore you obtain the Relation of the Fluents x and y , so that y is given from x given.

Art. 80.
of the
Quadrature
of
Curves.

443. Suppose you have such an Equation as this $\dot{x} \times \overline{ax^m + bx^n}^p = \frac{dy^{n-1}}{\sqrt{e + fy^n}}$: which involves both the Fluents x and y , and their Fluxions (where, if one of the Fluxions had been wanting it had been the same thing, because it behov'd to have been supplied). Here, the two Sides of the Equation may be considered as the Fluxions of Curves, whereof the one has x for it's Absciss, and $\overline{ax^m + bx^n}^p$ for it's Ordinate; and the other y for it's Absciss, and $\frac{dy^{n-1}}{\sqrt{e + fy^n}}$ for it's Ordinate: for these two Areas would be equal, by Prop. 9. Wherefore, if these Areas be found, and equated, you'll thence obtain the Relation of x and y . Now the Area of the Curve whose Ordinate is $\frac{dy^{n-1}}{\sqrt{e + fy^n}}$ by Species 1. Order 4. Table 1, is $\frac{2d}{nf} \sqrt{e + fy^n}$: therefore if you put this equal to the Area of the Curve whose Absciss is x and Ordinate $\overline{ax^m + bx^n}^p$ (which may be found by the Quadrature of Curves) thence the

the Fluent y will be known, when x is given. Thus suppose $m = 2$, $n = 4$, $p = \frac{1}{2}$: then the Ordinate of the Curve having x for it's Absciss, is $\sqrt{ax^2 + bx^4} = x\sqrt{a + bx^2}$, which belongs to the Curves of the first Species of Form third, Table 1: according to which, it's Area would be $\frac{a + bx^2}{3b} \sqrt{a + bx^2}$. Wherefore $\frac{a + bx^2}{3b} \sqrt{a + bx^2} = \frac{2d}{3f} \sqrt{e + fy^n}$ gives the Relation of the Fluents sought.

444. And thus our Author has shewn in what Cases of fluxional Equations, the Fluents may be found, by means of the Quadrature of Curves: in all which, there are always the Fluxions of two flowing Quantities either expressed or understood. One of these flowing Quantities, which our Author in this Schol. calls x or z is supposed to flow after an equable and uniform manner, whose Fluxion is designed by Unity: the other flowing Quantity, he calls y or v , which is supposed to flow according to any Law whatsoever. The first of these Quantities he considers as a Standard, by which the other is to be measured or estimated; even as the Absciss is with respect to the Ordinate and Area of a Curve. And as in this Case, we reckon the Ordinate and Area known, when their Relation to the Absciss is expressed by any algebraical Equation: so in the Case of the Fluents x and y , y is considered as the Fluent sought, and therefore when it's Relation to x is given, it is said to be given. See Art. 75. and Art. 78, 79 and 80* at the End. But now, after the Fluent y is thus found by the Quadrature of Curves; it must be considered that this is what we formerly called the *pure* Fluent, since it has no Part but what flows: and therefore the Fluent so found, may be increased or diminished by any *constant* Quantity, and yet still be the Fluent, *i. e.* a flowing Quantity, which has the same Fluxion as formerly, since the constant Quantity added to, or subtracted from the pure Fluent, having itself no Fluxion, can make no change in the Fluxion of the former Fluent. And thus you see the Reason of what our Author advances upon this Head in Art. 81*. And therefore, as was observed before †, when the Fluents are found from the Fluxions, these Fluents, in the Application to any particular Case, must be so ordered and limited, as the Nature and Circumstances of the Case or Question do necessarily require. It would be the same thing to say, that, after an Equation is found expressing the Relation of the Fluents, all the Parts of which are flowing, you may add to either or both the Sides of it, what constant Quantities you please, and it shall yet be an Equation exhibiting the

Art 81.

of the
Quadrature of
Curves.

* Of the
Quadrature of
Curves.

* Of the
Quadrature of
Curves.

† Art.
338, 339.
of this Ex-
plication.

the

the Relation of the Fluents: for it produces the same Relation of Fluxions as before.

Art. 82.
of the
Quadrature
of
Curves.

445. Our Author tells us that where there arises any Doubt about the Truth of the Conclusion, in finding the Fluents from the Fluxions, you may easily ascertain it, by collecting the Fluxions of the Fluents so found, by Prop. 1: for if they be the same with the Fluxions that were at first proposed, the Operation is just, and the Fluents found are true Fluents, otherwise not. In which last Case they must be corrected in such manner, as to make their Fluxions agree with the Fluxions at first proposed. For, says he, a Fluent may be assumed at pleasure, and the Assumption corrected, by putting the Fluxion of the assumed Fluent equal to the propos'd Fluxion, and comparing the *homologous* Terms with one another. But after what manner the Fluent of any Fluxion is to be found by such an Assumption of the Fluent, and Comparison of the homologous Terms of the Fluxions: as likewise how this general Problem, including the whole of the *inverse* Method of Fluxions, *viz. An Equation being given involving any Number of Fluxions, to find the Fluents*, i. e. an Equation defining their Relation, which is the Converse of the first Proposition of this Treatise: I say, how these things are to be performed, as it don't properly belong to the Doctrine of Quadratures, I have no Design nor Inclination to shew in this Place: since it would open too large a Field, and carry us beyond the original Design of this Work, which was to explain and vindicate the first Principles upon which the Doctrine of Fluxions is founded, and to explain and illustrate this Treatise of the Quadrature of Curves.

446. Our celebrated Author crowns all with a Saying, that expresses the *Greatness* of his Modesty, and *Vastness* of his Understanding. What Man but Sir *Isaac Newton* himself, could call the *noble* and *sublime* Discoveries contained in this short Treatise, by the Name of *Principles* or *Beginnings* of Knowledge, the clear apprehending and full understanding of which, after they are discovered and laid before us, requires *Pains* and *Application* to Understandings of the *common* Size. Notwithstanding, it is true, what he affirms, that these pave the Way to innumerable still more valuable and sublime Discoveries in Mathematics and *Natural Philosophy*, as he himself hath partly shewn in his other Performances, and especially in that *invaluable* and *inimitable* one his mathematical Principles of natural Philosophy.

The End of the Quadrature of Curves.

O F

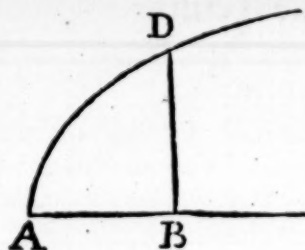
ANALYSIS

B Y

Equations of an infinite Number of Terms.

1. *THE General Method, which I had devised some considerable Time ago, for measuring the Quantity of Curves, by Means of Series, infinite in the Number of Terms, is rather shortly explained, than accurately demonstrated in what follows.*

2. Let the Base AB of any Curve AD have BD for it's perpendicular Ordinate; and call $AB=x$, $BD=y$, and let a , b , c , &c. be given Quantities, and m and n whole Numbers. Then



The Quadrature of Simple Curves,

R U L E I.

3. If $ax^{\frac{m}{n}}=y$; it shall be $\frac{an}{m+n} x^{\frac{m+n}{n}} = \text{Area ABD.}$

The thing will be evident by an Example.

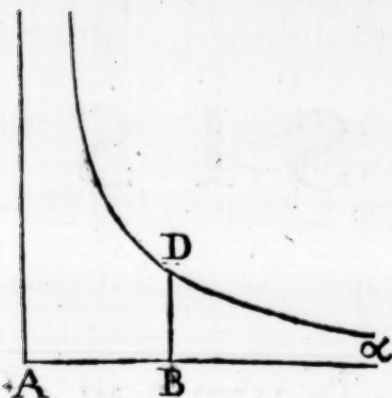
1. If $x^2 (=1x^{\frac{2}{1}})=y$, that is $a=1=n$, and $m=2$; it shall be $\frac{1}{3}x^3 = \text{ABD.}$

T t

2. Suppose

ANALYSIS by EQUATIONS

2. Suppose $4\sqrt{x} (=4x^{\frac{1}{2}}) = y$; it will be $\frac{8}{3}x^{\frac{3}{2}} (= \frac{8}{3}\sqrt{x^3}) = ABD$.
 3. If $\sqrt[3]{x^3} (=x) = y$; it will be $\frac{3}{4}x^{\frac{4}{3}} (= \frac{3}{4}\sqrt[3]{x^8}) = ABD$.
 4. If $\frac{1}{x^2} (=x^{-2}) = y$, that is if $a = 1 = n$, and $m = -2$;



It will be $\frac{1}{-1}x^{-1} (= -x^{-1} (= \frac{-1}{x})) = aBD$, infinitely extended towards α , which the Calculation places negative, because it lies upon the other side of the Line BD.*

5. If $\frac{1}{\sqrt{x^3}} (=x^{-\frac{3}{2}}) = y$; it will be $(\frac{2}{-1}x^{-\frac{1}{2}} = \frac{2}{-\sqrt{x}} = BD\alpha$.

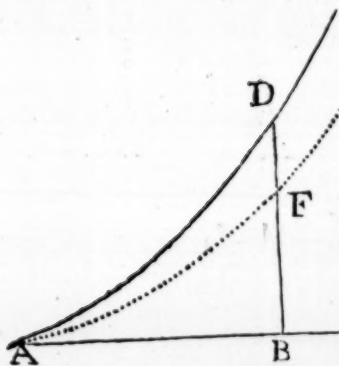
6. If $\frac{1}{x} (=x^{-1}) = y$; it will be $\frac{1}{0}x^0 = \frac{1}{0} \times 1 = \frac{1}{0} =$ an infinite Quantity; such as is the Area of the Hyperbola upon both Sides of the Line BD.

The Quadrature of Curves compounded of simple ones.

RULE II.

4. If the Value of y be made up of several such Terms, the Area likewise shall be made up of the Areas which result from every one of the Terms.

The first Examples.



5. If it be $x^2 + x^{\frac{1}{2}} = y$; it will be $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} = ABD$.

For if it be always $x^2 = BF$ and $x^{\frac{1}{2}} = FD$, you will have by the preceding Rule $\frac{1}{3}x^3 =$ Superficies AFB; described by the Line BF; and $\frac{2}{5}x^{\frac{5}{2}} = AFD$ described by DF; wherefore $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} =$ the whole Area ABD.

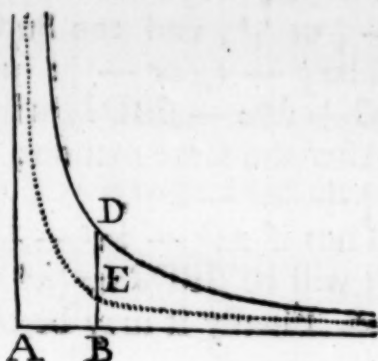
Thus if it be $x^2 - x^{\frac{1}{2}} = y$; it will be $\frac{1}{3}x^3 - \frac{2}{5}x^{\frac{5}{2}} = ABD$. And if it be $3x - 2x^2 + x^3 - 5x^4 = y$; it will be $\frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - x^5 = ABD$.

* Whatever is laid down by our Author with respect to the Position of Areas of Curves in this and the following Rules, is explained at full length in Sect. 5 of the preceding Treatise. Which see.

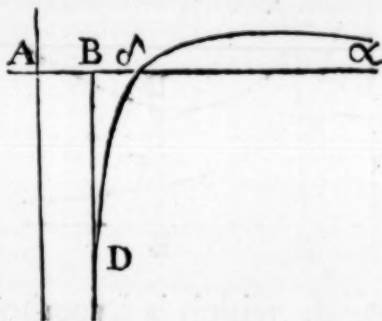
The second Examples.

6. If $x^{-2} + x^{-\frac{1}{2}} = y$; it will be $-x^{-1} - 2x^{-\frac{1}{2}} = \alpha BD$. Or if it be $x^{-2} - x^{-\frac{1}{2}} = y$; it will be $-x^{-1} + 2x^{-\frac{1}{2}} = \alpha BD$.

And if you change the Signs of the Quantities, you will have the affirmative Value ($x^{-1} + 2x^{-\frac{1}{2}}$, or $x^{-1} - 2x^{-\frac{1}{2}}$) of the Superficies αBD , provided the whole of it fall above the Base $AB\alpha$.



7. But if any Part fall below (which happens when the Curve decussates or crosses it's Base betwixt B and α , as you see here in δ) you are to subtract that Part from the Part above the Base; and so you shall have the Value of the Difference: but if you would have their Sum, seek both the Superficies's separately, and add them. And the same thing I would have observed in the other Examples belonging to this Rule.

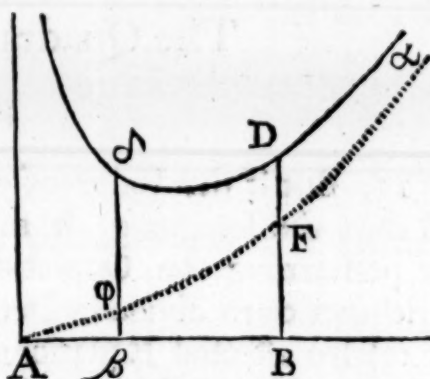


The third Examples.

8. If $x^2 + x^{-2} = y$; it will be $\frac{1}{3}x^3 - x^{-1} =$ the Superficies described.

But here it must be remarked that the Parts of the said Superficies so found, lye upon opposite Sides of the Line BD .

That is, putting $x^2 = BF$, and $x^{-2} = FD$; it shall be $\frac{1}{3}x^3 = ABF$ the Superficies described by BF , and $-x^{-1} = DF\alpha$ the Superficies described by DF .



9. And this always happens when the Indexes ($\frac{m+n}{n}$) of the Ratios of the Base x in the Value of the Superficies sought, are affected with different Signs. In such Cases any middle part $BD\delta\beta$ of the Superficies (which only can be given, when the Superficies is infinite upon both Sides) is thus found.

Subtract the Superficies belonging to the lesser Base $A\beta$ from the Superficies belonging to the greater Base AB , and you shall have $\beta BD\delta$ the Superficies insisting upon the difference of the Bases. Thus in this Example (see the preceding Fig.)

T t 2

If

If $AB = 2$, and $A\beta = 1$; it will be $\beta BD\delta = \frac{1}{6}$:

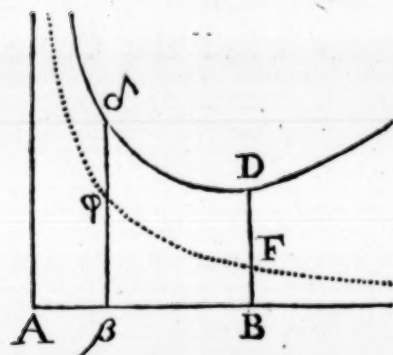
For the Superficies belonging to AB (*viz.* $ABF - DF\alpha$) will be $\frac{8}{3} - \frac{1}{2}$ or $\frac{13}{6}$; and the Superficies belonging to $A\beta$ (*viz.* $A\phi\beta - \delta\phi\alpha$) will be $\frac{1}{3} - 1$, or $-\frac{2}{3}$: and their Difference (*viz.* $ABF - DF\alpha - A\phi\beta + \delta\phi\alpha - \beta BD\delta$) will be $\frac{13}{6} + \frac{2}{3}$ or $\frac{17}{6}$.

After the same manner, if $A\beta = 1$, and $AB = x$; it will be $\beta BD\delta = \frac{2}{3} + \frac{1}{3}x^3 - x^{-1}$.

Thus if $2x^3 - 3x^5 - \frac{2}{3}x^{-4} + x^{-\frac{1}{2}} = y$, and $A\beta = 1$;

It will be $\beta BD\delta = \frac{1}{2}x^4 - \frac{1}{2}x^6 + \frac{2}{9}x^{-3} + \frac{5}{2}x^{\frac{3}{2}} - \frac{4}{18}$.

10. Finally it may be observed, that if the Quantity x^{-1} be found in the Value of y , that Term (since it generates an hyperbolical Surface) is to be considered apart from the rest.



As if it were $x^2 + x^{-3} + x^{-1} = y$: let it be $x^{-1} = BF$, and $x^2 + x^{-3} = FD$; and $A\beta = 1$; and it will be $\delta\phi FD = \frac{1}{6} + \frac{1}{3}x^3 - \frac{1}{2}x^{-2}$, as being that which is generated by the Terms $x^2 + x^{-3}$.

Wherefore if the remaining Superficies $\beta\phi FB$, which is hyperbolical, be given by any Method of Computation, the whole $\beta BD\delta$ will be given.

The Quadrature of all other Curves.

R U L E III.

11. But if the Value of y , or any of it's Terms be more compounded than the foregoing, it must be reduced into more simple Terms; by performing the Operation in Letters, after the same Manner as Arithmeticians divide in Decimal Numbers, extract the Square Root, or resolve affected Equations; and afterwards by the preceding Rules you will discover the Superficies of the Curve sought.

Examples, where you divide.

12. Let $\frac{aa}{b+x} = y$; *Viz.* where the Curve is an Hyperbola.

Now that that Equation may be freed from it's Denominator, I make the Division thus.

$b+x)$

$$b+x) aa + o \left(\frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4} \&c. \right.$$

$$\frac{aa + \frac{aax}{b}}{b+x}$$

$$o - \frac{aax}{b} + o$$

$$- \frac{aax}{b} - \frac{aax^2}{b^2}$$

$$o + \frac{aax^2}{b^2} + o$$

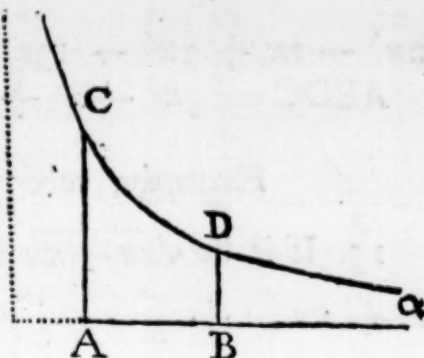
$$+ \frac{aax^2}{b^2} + \frac{aax^3}{b^3}$$

$$o - \frac{aax^3}{b^3} + o$$

$$- \frac{aax^3}{b^3} - \frac{aax^4}{b^4}$$

$$o + \frac{aax^4}{b^4}$$

&c.



And thus in Place of this $y = \frac{aa}{b+x}$, a new Equation arises, viz.
 $y = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} \&c.$ this Series being continued infinitely;
 and therefore (by the second Rule)

The Area sought ABDC will be equal to $\frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}$
 &c. an infinite Series likewise, but yet such, that a few of the initial
 Terms are exact enough for any Use, provided that b be equal to x re-
 peated some few times.

13. After the same Manner if it be $\frac{1}{1+xx} = y$, by dividing there
 arises

$$y = 1 - xx + x^4 - x^6 + x^8 \&c. \text{ Whence (by the second Rule)}$$

$$\text{You will have } ABDC = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 \&c.$$

Or if x^2 be made the first Term in the Divisor, viz. thus: $x^2 + 1$
 there will arise $x^{-2} - x^{-4} + x^{-6} - x^{-8} \&c.$ for the Value of y ;
 whence (by the second Rule)

It will be $BD\alpha = -x^{-1} + \frac{1}{3}x^{-3} - \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7} \&c.$ You must
 proceed in the first Way when x is small enough, but the second Way,
 when it is supposed great enough.

14. Finally,

14. Finally, if it be $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = y$; by dividing there arises
 $2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}} \&c.$ whence it will be
 $ABDC = \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{3}x^3 \&c.$

Examples, where the Square Root must be extracted.

15. If it be $\sqrt{aa + xx} = y$, I extract the Root thus:

$$aa + xx \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \&c. \right.$$

aa

$$\begin{array}{r} 0 + x^2 \\ \hline \end{array}$$

$$\begin{array}{r} x^2 + \frac{x^4}{4a^2} \\ \hline \end{array}$$

$$\begin{array}{r} 0 - \frac{x^4}{4a^2} \\ \hline \end{array}$$

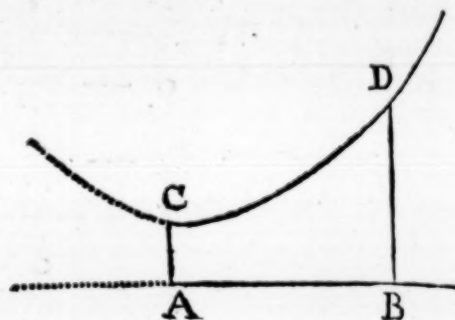
$$\begin{array}{r} - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\ \hline \end{array}$$

$$\begin{array}{r} 0 + \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\ \hline \end{array}$$

$$\begin{array}{r} + \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}} \\ \hline \end{array}$$

$$\begin{array}{r} 0 - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}} \\ \hline \end{array}$$

$\&c.$

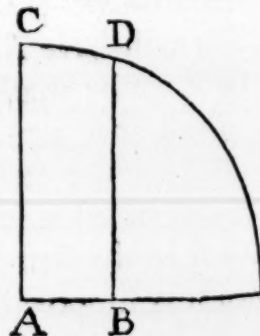


Whence for the Equation $\sqrt{aa + xx} = y$, a new one is produced,
viz. $y = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \&c.$ And (by the second Rule)

You will have the Area sought $ABDC = ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} \&c.$

And this is the Quadrature of the Hyperbola.

16. After the same Manner if it be $\sqrt{aa - xx} = y$, it's Root will be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \&c.$ and therefore the Area sought ABDC will be equal to $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} \&c.$ And this is the Quadrature of the Circle.

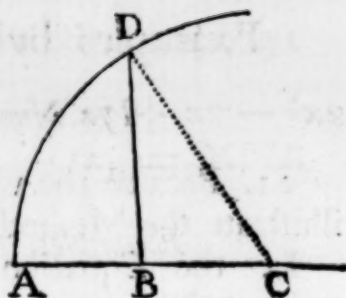


17. Or if you suppose $\sqrt{x - xx} = y$, the Root will be equal to this infinite Series $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} + \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} + \frac{5}{128}x^{\frac{9}{2}} \&c.$

And the Area fought ABD will be $\frac{2}{3}x^{\frac{3}{2}} - \frac{11}{5}x^{\frac{5}{2}} + \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{704}x^{\frac{9}{2}} \&c.$

Or $x^{\frac{1}{2}} \times \frac{2}{3}x - \frac{1}{5}x^2 - \frac{1}{128}x^3 - \frac{1}{72}x^4 - \frac{5}{704}x^5 \&c.$

And this is the Quadrature of the Circle's Area.



18. If $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}} = y$ (whose Quadrature gives the Length of the Curve of the Ellipse¹) by extracting both Roots, there arises

$$\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6 - \frac{5}{128}a^4x^8}{1 - \frac{1}{2}bx^2 + \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6 + \frac{5}{128}b^4x^8} \&c.$$

And by dividing as is done in the Case of Decimal Fractions, you will have

$$\begin{array}{r} 1 + \frac{1}{2}bx^2 + \frac{3}{8}b^2x^4 + \frac{5}{16}b^3x^6 + \frac{35}{128}b^4x^8 \&c. \\ + \frac{1}{2}a \quad + \frac{1}{4}ab \quad + \frac{3}{16}ab^2 \quad + \frac{5}{32}ab^3 \\ - \frac{1}{8}a^2 \quad - \frac{1}{16}a^2b \quad - \frac{3}{64}a^2b^2 \\ + \frac{1}{16}a^3 \quad + \frac{1}{32}a^3b \\ - \frac{5}{128}a^4 \end{array}$$

And therefore the Area required $x + \frac{1}{6}bx^3 + \frac{3}{40}b^2x^5 \&c.$
 $+ \frac{1}{6}a \quad + \frac{1}{20}ab$
 $- \frac{1}{40}a^2$

19. But it is to be observed that the Operation may be often abbreviated by a due Preparation of the Equation, as in the Example just now adduced $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}} = y$. If you multiply both Parts of the

Fraction by $\sqrt{1-bx^2}$, there will arise $\frac{\sqrt{1+ax^2-ahx^4}}{1-bx^2} = y$; and the rest of the work is performed by extracting the Root of the Numerator only, and then dividing by the Denominator.

20. From these Examples, I suppose, it will be sufficiently evident after what Manner any Value of y may be reduced (with whatever Roots or Denominators it may be involved, as you may see here

$x^3 + \frac{\sqrt{x - \sqrt{1-xx}}}{\sqrt{axx + x^3}} - \frac{\sqrt{x^3 + 2x^5 - x^{\frac{3}{2}}}}{\sqrt{x+x^2} - \sqrt{2x-x^{\frac{2}{3}}}} = y$) into infinite Series of simple Terms, from which, by the second Rule, the Superficies required may be known.

¹ For the Explication of this see Art. 341—343 of the preceding Treatise.

Examples

Examples by the Resolution of EQUATIONS.

The Numeral Resolution of Affected Equations.

21. Because the whole Difficulty lyes in the Resolution, I shall first illustrate the Method I use in a numeral Equation.

Let the Equation $y^3 - 2y - 5 = 0$ be proposed to be resolved: and let 2 be a number which differs from the Root sought, by less than a tenth Part of itself. Then I put $2 + p = y$, and I substitute this Value in Place of it in the Equation, and thence a new Equation arises, viz. $p^3 + 6p^2 + 10p - 1 = 0$, whose Root p is to be sought for, that it may be added to the Quotient: viz. thus (neglecting $p^3 + 6p^2$ upon the Account of their smallness) $10p - 1 = 0$, or $p = 0,1$ is near the Truth; therefore I write 0,1 in the Quotient, and then suppose $0,1 + q = p$, and this it's value I substitute, as formerly, whence results $q^3 + 6,3q^2 + 11,23q + 0,061 = 0$.

22. And since $11,23q + 0,061 = 0$ comes near to the Truth, or since q is almost equal to $-0,0054$ (viz. by dividing until as many Figures arise as there are places betwixt the first Figures of this and the principal Quotient) I write $-0,0054$ in the lower Part of the Quotient, since it is negative.

23. And then supposing $-0,0054 + r = q$, I substitute this as formerly, and thus the Operation is continued as far as you please. But if I desire to continue the Work only to twice as many Figures as there are in the Quotient except one, instead of q I substitute $-0,0054 + r$ into this $6,3q^3 + 11,23q + 0,061$, viz. neglecting it's first Term (q^3) upon the Account of it's Smallness, and there arises $6,3r^2 + 11,16196r + 0,000541708 = 0$ almost, or (rejecting $6,3r^2$) $r = \frac{-0,000541708}{11,16196} = -0,00004853$ almost, which I write in the negative Part of the Quotient. Finally, subducting the negative Part of the Quotient from the affirmative, I have 2,09455147 the Quotient sought.

$y^3 - 2y - 5 = 0$	$ \begin{array}{r} + 2,10000000 \\ - 0,00544853 \\ \hline + 2,09455147 = y \end{array} $	
$2 + p = y$	$ \begin{array}{r} + y^3 \\ + 2y \\ - 5 \\ \hline \text{Sum} \end{array} $	$ \begin{array}{r} + 8 + 12p + 6p^2 + p^3 \\ - 4 - 2p \\ - 5 \\ \hline - 1 + 10p + 6p^2 + p^3 \end{array} $
$0,1 + q = p$	$ \begin{array}{r} + p^3 \\ + 6p^2 \\ + 10p \\ - 1 \\ \hline \text{Sum} \end{array} $	$ \begin{array}{r} + 0,001 + 0,03q + 0,3q^2 + q^3 \\ + 0,06 + 1,2 + 6,0 \\ + 1, + 10 \\ - 1, \\ \hline + 0,061 + 11,23q + 6,3q^2 + q^3 \end{array} $
$- 0,0054 + r = q$	$ \begin{array}{r} + 6,3q^2 \\ + 11,23q \\ + 0,061 \\ \hline \text{Sum} \end{array} $	$ \begin{array}{r} + 0,000183708 - 0,06804r + 6,3r^2 \\ - 0,060642 + 11,23 \\ + 0,061 \\ \hline + 0,000541708 + 11,16196r + 6,3r^2 \end{array} $
$- 0,00004854 + s = r$		

24. Equations of more Dimensions are resolved the same way, and you'll save a good deal of Trouble, towards the End, as was done here, if you omit the first Terms gradually.

25. Moreover it is to be observed that in this Example, if I had any doubt whether $0,1 = p$ approached near enough to the Truth, instead of $10p - 1 = 0$, I had feigned $6p^2 + 10p - 1 = 0$, and writ the first Figure of it's Root in the Quotient; and indeed it's convenient to search out the second and third Figure of the Quotient after this Manner, when in the last resulting Equation, the Square of the Coefficient of the penult Term is not ten times greater than the Product of the last Term and Coefficient of the antepenult Term.

26. Yea you will for the most part save Labour, especially in Equations of many Dimensions, if you find out all the Figures to be adjoined to the Quotient by the same Means, *i. e.* by extracting the lesser of the Roots out of the three last Terms of the last resulting Equation: For in that way you will gain double as many Figures, at each Time, to be annexed to the Quotient.

U u

27. Whether

27. Whether this Method of resolving Equations be vulgarly practised I cannot tell, but surely to me it appears simple in Comparison of others, and more accommodated to Practice. The Demonstration of it appears from the very Manner of Operation, whence, as Occasion requires, you may easily call it into your Mind.

28. Equations wanting any of the Terms, or not, are managed almost with the same Ease: And an Equation is always left, whose Root, together with the Quotient already found, is equal to the Root of the Equation at first proposed. Whence you may examine or prove the Work here as well as in other Parts of Arithmetic, *viz.* by taking away the Quotient from the Root of the first Equation (as is known to Analysts) that so the last Equation; or two or three of it's last Terms may be thence produced. Any Labour there is in this Affair will be found to arise from the substituting the one Kind of Quantities for the other: Which you may do different Ways, but I think the following the most expeditious, especially when the numeral Coefficients consist of many Figures.

29. Let $p + 3$ be to be substituted for y in this Equation $y^4 - 4y^3 + 5y^2 - 12y + 17 = 0$: And since it may be resolved into this Form,

$y - 4 \times y + 5 \times y - 12 \times y + 17 = 0$, the new Equation will be generated thus; $p - 1 \times p + 3 = p^2 + 2p - 3$, and $p^2 + 2p + 2 \times p + 3 = p^3 + 5p^2 + 8p + 6$, and $p^3 + 5p^2 + 8p - 6 \times p + 3 = p^4 + 8p^3 + 23p^2 + 18p - 18$, and $p^4 + 8p^3 + 23p^2 + 18p - 1 = 0$, which was sought.

The literal Resolution of affected Equations.

30. These things being thus shewn in Numbers: Let $y^3 + a^2y - 2a^3 + axy - x^3 = 0$ be proposed to be resolved.

I first search for the Value of y when x is nothing, that is, I seek the Root of this Equation $y^3 + a^2y - 2a^3 = 0$, and I find it to be $+a$. Therefore I write $+a$ in the Quotient; and supposing $+a + p = y$, I substitute for y it's Value, and put the Terms thence resulting ($p^3 + 3ap^2 + 4a^2p$ &c.) in the Margin; out of which I assume $+4a^2p + a^2x$, *viz.* the Terms where p and x separately are of the least Dimensions, and I suppose them equal almost to nothing, or $p = -\frac{1}{4}x$ almost, or $p = -\frac{1}{4}x + q$. And writing $-\frac{1}{4}x$ in the Quotient, I substitute $-\frac{1}{4}x + q$ for p ; and the Terms thence resulting I write again in the Margin, as you may see in the Scheme annexed; and

and thence I assume the Quantities $4a^2e - \frac{1}{16}ax^2$, viz. those in which q and x separately are of the least Dimensions, and I feign $q = \frac{x^2}{64a}$ almost, or $q = \frac{x^2}{64a} + r$; and subjoining $+\frac{x^2}{64a}$ to the Quotient, I substitute $\frac{xx}{64a}$ for q ; and thus I proceed as far as you please.

$y^3 + a^2y - 2a^3 + axy - x^3 = 0$ $y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \&c.$		
$+a + p = y$	$+y^3$ $+a^2y$ $+axy$ $-2a^3$ $-x^3$	$+a^3 + 3a^2p + 3ap^2 + p^3$ $+a^3 + a^2p$ $+a^2x + axp$ $-2a^3$ $-x^3$
$-\frac{1}{4}x + q = p$	$+p^3$ $+3ap^2$ $+4a^2p$ $+axp$ $+a^2x$ $-x^3$	$-\frac{1}{64}x^3 + \frac{3}{16}x^2q - \frac{1}{4}xq^2 + q^3$ $+\frac{3}{16}ax^2 - \frac{3}{4}axq + 3aq^2$ $-a^2x + 4a^2q$ $-\frac{1}{4}ax^2 + axq$ $+a^2x$ $-x^3$
$+\frac{x^2}{64a} + r = q$	$+3aq^2$ $+4a^2q$ $-\frac{1}{2}axq$ $+\frac{3}{16}x^2q$ $-\frac{1}{16}ax^2$ $-\frac{6}{64}x^3$	$+\frac{3x^4}{4096a} + \frac{3}{32}x^2r + 3ar^2$ $+\frac{3}{16}ax^2 + 4a^2r$ $-\frac{1}{128}x^3 - \frac{1}{8}axr$ $+\frac{3x^4}{1024a} + \frac{3}{16}x^2r$ $-\frac{1}{16}ax^2$ $-\frac{6}{64}x^3$
$+4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2) + \frac{131}{512}x^3 - \frac{15x^4}{4096a} \left(+ \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \right.$		

31. But if I desire only double as many Terms save one to be further adjoined to the Quotient: I omit the first Term (q^3) of the last resulting Equation, and likewise that Part ($-\frac{1}{4}xq^2$) of the second Term, where x is of as many Dimensions as in the penult Term of the Quotient; and substitute $\frac{x^2}{64a} + r$ for q into the other Terms ($3aq^2 + 4a^2q \&c.$) placed in the Margin as you see; and then from the last two Terms

U u 2

$\left(\frac{15x^4}{4096a} \right.$

$(\frac{15x^4}{4096a} - \frac{131x^3}{128} + \frac{9x^2}{2} - \frac{1}{2}ax + 4a^2r)$ of the Equation thence resulting, by dividing thus:

$4a^2 - \frac{1}{2}ax + \frac{9}{2}x^2) + \frac{131x^3}{128} - \frac{15x^4}{4096a}$ (I bring out $+\frac{131x^3}{512a^2} + \frac{509x^4}{10384a^3}$ to be adjoined to the Quotient.

32. Finally that Quotient $(a - \frac{x}{4} + \frac{x^2}{64a} \&c.)$ by the second Rule, will give $ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \frac{509x^5}{81920a^3} \&c.$ for the Area sought, which approaches so much nearer to the Truth, the less that x is.

Another Way of resolving the same Equations.

33. But if the Value of the Area ought to approach the nearer to the Truth the greater that x is; of which let this be an Example $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. Therefore, being about to resolve this, I chuse out the Terms $y^3 + x^2y - 2x^3$, in which x and y either separately, or multiplied together are of the most and at the same Time of equal Dimensions every where; and from them as if equal to nothing I discover the Root. This I find to be x , which I write in the Quotient. Or, which comes to the same Purpose, I extract the Root out of $y^3 + y - 2 = 0$ (substituting Unity for x) which Root here comes out 1, and I multiply it by x , and write the Product (x) in the Quotient. Finally I put $x + p = y$, and so I proceed as in the former Example until I obtain the Quotient $x - \frac{a}{4} + \frac{a^2}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$

$\&c.$ and so the Area sought $\frac{x^2}{2} - \frac{ax}{4} + \left[\frac{a^2}{64x}\right]^1 - \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$; concerning which see the third Examples of the second Rule. For the sake of illustration I have given this Example the same in every Respect with the former, provided x and a be there substituted for one another, that there might not be any Necessity to adjoin here any other Example of this Resolution.

34. But the Area $\frac{x^2}{2} - \frac{ax}{4} + \left[\frac{a^2}{64x}\right] \&c.$ is terminated at a Curve which extends itself infinitely along some Assymptote; and the initial

¹ Sir Isaac uses this Notation to denote the curvilinear Area belonging to the Ordinate $\frac{a^2}{64x}$, which belongs to the Hyperbola. For understanding of which consult Art. 190—192 of the preceding Treatise.

Terms $x - \frac{1}{4}a$) of the Value of y extracted, always terminate in that Assymptote; whence you may easily find the Portion of the Assymptote. The same thing is always to be observed when the Area is denoted by Terms divided more and more by x continually, only that in place of a right-lined Assymptote, sometimes you will have the conical Parabola, or perhaps one more compounded.

35. But passing this Manner of Resolution, as being particular, because not applicable to Curves which return into themselves like Ellipses; with respect to the other Manner of Resolution, which was shewn above in the Example $y^3 + a^2y + axy - 2a^3 - x^3 = 0$ (*viz.* that Method in which the Dimensions of x perpetually increase in the Numerators of the Quotient) the following Things may be remarked.

1. If it happen at any Time that the Value of y , when x is supposed to be nothing, be a surd Quantity; or one entirely unknown, yet you may design it by some Letter. Thus in the Example $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, if the Root of this Equation $y^3 + a^2y - 2a^3 = 0$ had been a Surd, or unknown, I would have supposed any Quantity (b) put for it; and have performed the Operation as follows. Writing b in the Quotient, I suppose $b + p = y$, and substitute that for y , as you may see; whence a new Equation $p^3 + 3bp^2 \&c. = 0$ arises, in which the Terms $b^3 + a^2b - 2a^3$ are to be rejected, as being equal to nothing, because b is supposed the Root of this Equation $y^3 + a^2y - 2a^3 = 0$. Then the Terms $3b^2p + a^2p + abx$ give $-\frac{abx}{3b^2 + a^2}$ to be adjoined to the Quotient; and $-\frac{abx}{3b^2 + a^2} + q$ to be substituted for p , &c.

$$y^3 +$$

$y^3 + a^2y + axy - 2a^3 - x^3 = 0.$ Put $cc = 3b^2 + a^2.$ $y = b - \frac{abx}{cc} + \frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^4b^3x^3}{c^8} - \frac{a^5bx^3}{c^8} + \frac{a^5b^3x^3}{c^{10}} \&c.$		
$b + p = y$	y^3 $+ axy$ $+ aay$ $- x^3$ $- 2a^3$	$+ b^3 + 3b^2p + 3bp^2 + p^3$ $+ abx + axp$ $+ aab + a^2p$ $- x^3$ $- 2a^3$
$-\frac{abx}{cc} + q = p$	p^3 $+ 3bp^2$ $+ axp$ $+ ccp$ $- x^3$ $+ abx$	$- \frac{a^4b^3x^3}{c^6} \&c.$ $+ \frac{3a^2b^3x^2}{c^4} - \frac{6ab^3x}{cc} q \&c.$ $- \frac{a^2bx^2}{cc} + axq$ $- abx + ccq$ $- x^3$ $+ abx.$
$c^2 + ax - \frac{6ab^2x}{cc} + \frac{a^4bx^2}{c^4} + x^3 + \frac{a^4b^3x^3}{c^6} \left(\frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^4b^3x^3}{c^8} \right) \&c.$		

The Work being finished, I take any Number for a , and I resolve this Equation $y^3 + a^2y - 2a^3 = 0$, after the same Manner as was shewn above in the Case of Numeral Equations; and then I substitute it's Root in Place of b .

2°. If it should happen that the said Value be nothing, that is to say, if in the Equation to be resolved, there be no Term but what is multiplied by x or y , as in this $y^3 - axy + x^3 = 0$; then I select the Terms $(-axy + x^3)$ in which x separately, and also y separately, if that can be had, otherwise multiplied by x , is of the fewest Dimensions. And these Terms give $+\frac{xx}{a}$ for the first Term of the Quotient; and $\frac{xx}{a} + p$ to be substituted for y .

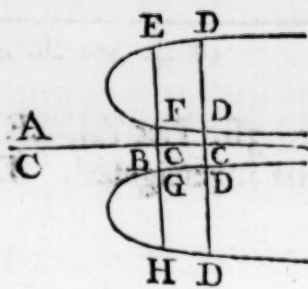
In this Equation $y^3 - a^2y + axy - x^3 = 0$, you may discover the first Term of the Quotient, either from $-a^2y - x^3$; or from $y^3 - a^2y$.

3°. If that Value of y be imaginary, as in this $y^4 + y^2 - 2y + 6 - x^2y^2 - 2x + x^2 + x^4 = 0$, I increase or diminish the Quantity x until the said Value become real.

Thus

Thus in the Figure annexed, when AC (x) is nothing, then CD (y) is imaginary.

But if AC be diminished by the given Line AB, that so BC may become x ; then supposing BC (x) to be nothing, CD (y) has a fourfold Value (CE, CF, CG or CH) each of them real; any of which Roots (CE, CF, CG or CH) may be the first Term of the Quotient, according as the Surface BEDC, BFDC, BGDC or BHDC is desired. In other Cases likewise, where you find any Stop, you may extricate yourself by the same Means.

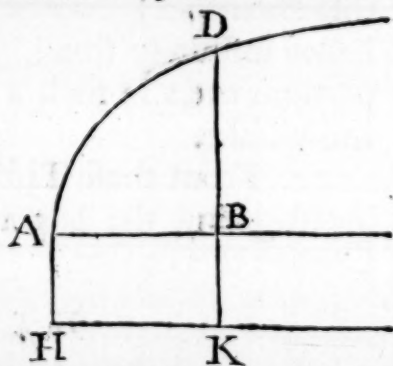


Finally if the Index of the Power of x or y be a Fraction, I reduce it to an Integer; as in this Example $y^3 - xy^{\frac{1}{2}} + x^{\frac{1}{2}} = 0$; putting $y^{\frac{1}{2}} = v$, and $x^{\frac{1}{2}} = z$, there arises $v^6 - z^3v + z^{\frac{1}{2}} = 0$, whose Root is $v = z + z^3 \&c.$ or (by restoring the Values) $y^{\frac{1}{2}} = x^{\frac{1}{2}} + x \&c.$ and by squaring, $y = x^{\frac{1}{2}} + 2x^{\frac{1}{2}} \&c.$

36. And this much shall suffice to have said touching the Investigation of the Areas of Curves. Yea since all Problems concerning the Length of Curves; the Quantity and Surface of Solids; and the Centers of Gravity may at length be brought to this, *viz.* that the Quantity of some Plane Surface bounded by a Curve Line is sought, there is no Necessity to say any thing about them here. However I shall shew very briefly after what Manner I perform the Operation in these Things.

The Application of what has been said to other Problems of that Kind.

37. Let ABD be any Curve, and AHKB a Rectangle, whose Side AH or BK is Unity: And imagine the Right Line DBK to move uniformly from AH, so as to describe the Areas ABD and AK; and that BK (t) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually increased; and that from the Moment BD continually given, you can, by Means of the preceding Rules, investigate the Area ABD described by it, or compare it with AK (x), which is described with the Moment t .

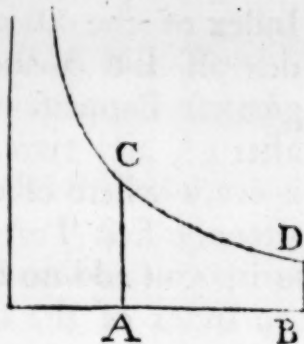


Now by the same Means that the Superficies ABD from it's Moment being at all Times given, is discovered by the foregoing Rules, by the like Means may any other Quantity be investigated from it's Moment given in like manner. The Thing will be clearer by an Example.

To

To find the Base from the Area given.

43. Thus if from the Area ABDC of the Hyperbola ($\frac{1}{1+x} = y$) given I wanted to investigate the Base AB, calling the Area z , I extract the Root of this Equation $z(ABCD) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglecting those Terms in which x is of more Dimensions than z is desired in the Quotient.



As if I would have z to rise to five Dimensions only in the Quotient, I neglect all the Terms $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.		
$z + p = x$	$+\frac{1}{5}x^5$ $-\frac{1}{4}x^4$ $+\frac{1}{3}x^3$ $-\frac{1}{2}x^2$ $+x$ $-z$	$+\frac{1}{5}z^5$ &c. $-\frac{1}{4}z^4 - z^3p$ &c. $+\frac{1}{3}z^3 + z^2p + zp^2$ &c. $-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$ $+z + p$ $-z$
$\frac{1}{2}z^2 + q = p$	$+zp^2$ $-\frac{1}{2}p^2$ $-z^3p$ $+z^2p$ $-zp$ $+p$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$	$+\frac{1}{4}z^5$ &c. $-\frac{1}{8}z^4 - \frac{1}{2}z^2q$ $-\frac{1}{2}z^5$ &c. $+\frac{1}{2}z^4 + z^2q$ $-\frac{1}{2}z^3 - zq$ $+\frac{1}{2}z^2 + q$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2 - \frac{1}{6}z^3 + \frac{1}{8}z^4 + \frac{1}{20}z^5 (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

44. I have laid the Steps of the Resolution before you, as you see, upon the Account of the two following Remarks.

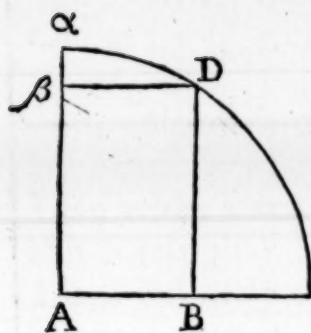
1. That in the Substitution, I always omit those Terms, which I foresee will be of no Use afterwards. Concerning which this is the Rule;

X x

Rule; That after the first Term resulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimension of that first Term wants Units of the Index of the greatest Dimension. As in this Example, where the greatest Dimension is 5, I neglect all the Terms after z^5 , I put one after z^4 , and two only after z^3 . When the Root (x) to be extracted, is every where of even or odd Dimensions, let this be the Rule: That after the first Term, resulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimension of that first Term, wants Pairs of Units of the Index of the highest Dimension; or no more than what it wants Ternaries of Units, when the Indexes of the Dimensions of x differ by three Units; and so in others.

2. When I see that p , q , or r , &c. in the last resulting Equation, is found of one Dimension only, I seek it's Value, that is to say the remaining Terms, which are still to be added to the Quotient, by means of Division; as you see done here.

To find the Base from the Length of the Curve given.



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, viz. $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being supposed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c.

46. And moreover if the Cosine $A\beta$ were required from that Arch given, make $A\beta (= \sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

Concerning the Continuation of the Series of the Progressions.

47. Let it be observed here, by the bye, that when 5 or 6 Terms of those Roots are known, they may be continued at Pleasure for most Part, by observing the Analogy of the Progression.

Thus you may continue this $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. by dividing the last Term by the following Numbers in Order, 2, 3, 4, 5, 6, 7, &c.

And this $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, &c. by dividing by these Numbers 2×3 , 4×5 , 6×7 , 8×9 , 10×11 , &c.

And

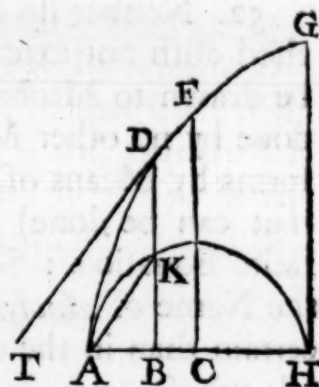
And this $x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$, &c. by these $1 \times 2, 3 \times 4, 5 \times 6, 7 \times 8, 9 \times 10$, &c.

And this $x = x + \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{112}x^7$, &c. by multiplying by these, viz. $\frac{1 \times 1}{2 \times 3}, \frac{3 \times 3}{4 \times 5}, \frac{5 \times 5}{6 \times 7}, \frac{7 \times 7}{8 \times 9}$, &c. And so in others.

The Application of what has been said to Mechanical Curves.

48. And what has been said shall suffice with Respect to Geometrical Curves. But now, although the Curve be Mechanical, yet this Method of ours may be applied to it.

For Example, let ADFG be the Trochoid whose Vertex is A, and Axis AH, and AKH the Wheel or Circle with which it is described. And the Superficies ABD is required. In Order to which, putting $AB = x$ and $BD = y$ as formerly; and $AH = 1$, I first seek the Length of BD. Now from the Nature of the Trochoid, you have $KD = \text{Arch AK}$: Wherefore the whole Line $BD = BD + \text{Arch AK}$. But $BK (=$



$\sqrt{x - xx}$) $= x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c. And (from what was formerly shewn) the Arch $AK = x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{1}{40}x^{\frac{5}{2}} + \frac{1}{112}x^{\frac{7}{2}}$, &c. Therefore the whole $BD = 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}}$, &c. And (by Rule the second) the Area $ABD = \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} + \frac{1}{252}x^{\frac{9}{2}}$, &c.

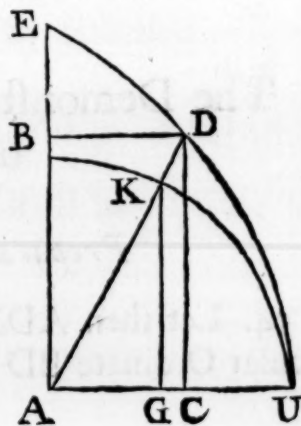
Or more shortly thus: Since the Right Line AK is parallel to the Tangent TD, it will be AB to BK, as the Moment of the Line AB to the Moment of the Line BD, that is, $x : \sqrt{x - xx} :: 1 : \frac{1}{x} \sqrt{x - x^2}$

$= x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}}$, &c. Wherefore (by Rule the second) $BD = 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} - \frac{5}{576}x^{\frac{9}{2}}$, &c. And the Superficies $ABD = \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} - \frac{5}{3168}x^{\frac{11}{2}}$, &c.

49. Much after the same Manner (if C be the Center of the Circle and $CB = x$) you may find the Area CBDF &c,

50. Let it be required to find the Area ABDU of the Quadrature UDE (whose Vertex is U, and A the Center of the interior Circle UK to which it is fitted.

Having drawn any Line as AKD, I let fall the Perpendiculars DB, DC, KG. And so it shall be $KG : AG :: AB (x) : BD (y)$,
X x 2 or



or $\frac{x \times AG}{KG} = y$. But from the Nature of the Quadratrix, you have BA (=DC) = Arch UK; or UK = x . Wherefore putting AU = 1, it will be GK = $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, &c. from what was shewn above, and GA = $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$ &c.

And therefore $y (= \frac{x \times AG}{KG}) = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \text{ \&c.}}{1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{30240}x^9 \text{ \&c.}}$, or, by actual Division $y = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6 \text{ \&c.}$ and (by Rule the second) the Area AUDB = $x - \frac{1}{9}x^3 - \frac{1}{225}x^5 - \frac{2}{6075}x^7 \text{ \&c.}$

51. Thus also the Length of the Quadratrix UD may be determined, although the Calculation be something more difficult.

52. Neither do I know any Thing of this Kind to which this Method doth not extend, and that in various Ways. Yea Tangents may be drawn to Mechanical Curves by it, when it happens that it can be done by no other Means. And whatever the common Analysis performs by Means of Equations of a finite Number of Terms (provided that can be done) this can always perform the same by Means of infinite Equations: So that I have not made any Question of giving this the Name of *Analysis* likewise. For the Reasonings in this are no less certain than in the other; nor the Equations less exact; albeit we Mortals whose reasoning Powers are confined within narrow Limits, can neither express, nor so conceive all the Terms of these Equations, as to know exactly from thence the Quantities we want: Even as the surd Roots of finite Equations can neither be so expressed by Numbers, nor any analytical Contrivance, that the Quantity of any one of them can be so distinguished from all the rest, as to be understood exactly.

53. To conclude, we may justly reckon that to belong to the *Analytic Art*, by the Help of which the Areas and Lengths &c. of Curves may be exactly and geometrically determined (when such a thing is possible). But this is not a Place for insisting upon these Things. There are two Things especially which an attentive Reader will see need to be demonstrated.

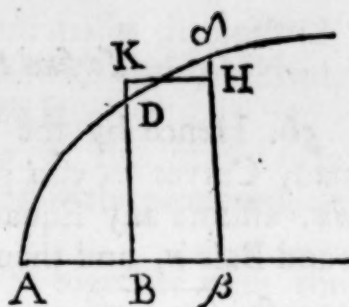
I. The Demonstration of the Quadrature of Simple Curves belonging to Rule the first.

Preparation for demonstrating the first Rule.

54. Let then AD δ be any Curve whose Base AB = x , the perpendicular Ordinate BD = y , and the Area ABD = z , as at the Beginning.

ning. Likewise put $B\beta = 0$, $BK = v$; and the Rectangle $B\beta HK$ (ov) equal to the Space $B\beta D$.

Therefore it is $A\beta = x + 0$, and $A\delta\beta = z + ov$: Which Things being premised, assume any Relation betwixt x and z that you please, and seek for y in the following Manner.



Take at Pleasure $\frac{2}{3}x^{\frac{1}{2}} = z$; or $\frac{4}{9}x^3 = z^2$. Then $x + 0$ ($A\beta$) being substituted for x , and $z + ov$ ($A\delta\beta$) for z , there arises $\frac{4}{9}$ into $x^3 + 3x0^2 + 3x0^2 + 0^3 =$ (from the Nature of the Curve) $z^2 + 2z0v + 0^2v^2$. And taking away Equals ($\frac{4}{9}x^3$ and z^2) and dividing the Remainders by 0 , there arises $\frac{4}{9}$ into $3x^2 + 3x0 + 00 = 2zv + ovv$. Now if we suppose $B\beta$ to be diminished infinitely and to vanish, or 0 to be nothing, v and y , in that Case will be equal, and the Terms which are multiplied by 0 will vanish: So that there will remain $\frac{4}{9} \times 3x^2 = 2zv$, or $\frac{2}{3}x^2 (=zy) = \frac{2}{3}x^{\frac{1}{2}}y$; or $x^{\frac{1}{2}} (= \frac{x^2}{x^{\frac{1}{2}}}) = y$. Wherefore conversely if it be $x^{\frac{1}{2}} = y$, it shall be $\frac{2}{3}x^{\frac{1}{2}} = z$.

The Demonstration.

55. Or universally, if $\frac{n}{m+n} \times ax^{\frac{m+n}{n}} = z$; or, putting $\frac{na}{m+n} = c$, and $m+n = p$, if $cx^{\frac{p}{n}} = z$; or $c^n x^p = z^n$: Then by substituting $x + 0$ for x , and $z + ov$ (or which is the same $z + oy$) for z , there arises c^n into $x^p + p0x^{p-1}$, &c. $= z^n + n0yz^{n-1}$ &c. the other Terms, which would at length vanish being neglected. Now taking away $c^n x^p$ and z^n which are equal, and dividing the Remainders by 0 , there remains $c^n p x^{p-1} = n y z^{n-1} (= \frac{nyz^n}{z}) = \frac{ny c^n x^p}{c x^{\frac{p}{n}}}$, or, by dividing by $c^n x^p$, it shall be $p x^{-1} = \frac{ny}{c x^{\frac{p}{n}}}$; or $p c x^{\frac{p-n}{n}} = ny$; or by restoring $\frac{na}{m+n}$ for c , and $m+n$ for p , that is m for $p-n$, and na for pc , it becomes $ax^{\frac{m}{n}} = y$. Wherefore conversely, if $ax^{\frac{m}{n}} = y$, it shall be $\frac{n}{m+n} ax^{\frac{m+n}{n}} = z$. Q. E. D.

To find those Curves which can be squared.

56. Hence by the Way you may observe after what Manner as many Curves as you please may be found, whose Areas are known; viz. assume any Equation you please for the Relation betwixt the Area x and Base x , and thence let the Ordinate y be sought. Thus if you suppose $\sqrt{aa+xx} = z$, by performing the Calculation you will find $\frac{x}{\sqrt{aa+xx}} = y$. And so in other Cases.

II. The Demonstration of the Resolution of Affected Equations.

57. The other Thing to be demonstrated, is the literal Resolution of affected Equations. Viz. That the Quotient, when x is sufficiently small, the further it is produced, approaches so much nearer to the Truth, so that the defect (p , q , or r &c.) by which it differs from the full Value of y , at Length becomes less than any given Quantity; and that Quotient being produced infinitely is exactly equal to y . Which will appear thus.

1°. Because that Quantity in which x is of the lowest Dimension (that is to say, more than the half of the last Term, provided you suppose x small enough) in every Operation is perpetually taken out of the last Term of the Equations, of which p , q , r , &c. are the Roots: Therefore that last Term (by 1. 10. Elem.) at length becomes less than any given Quantity; and would entirely vanish, if the Operation were infinitely continued.

Thus if it be $x = \frac{1}{2}$, you have x the half of all these $x + x^2 + x^3 + x^4$, &c. and x^2 the half of all these $x^2 + x^3 + x^4 + x^5$, &c. Therefore if $x > \frac{1}{2}$, x shall be greater than the half of all these $x + x^2 + x^3$, &c. and x^2 greater than the half of all these $x^2 + x^3 + x^4$, &c.



Thus if $\frac{x}{b} > \frac{1}{2}$, you shall have x more than the half of all these $x + \frac{x^2}{b} + \frac{x^3}{b^2}$, &c. And the same Way

of others. And as to the numeral Coefficients, for most part they perpetually decrease: Or if at any Time they increase, you need only suppose x some few Times less.

2°. If

2°. If the last Term of any Equation be continually diminished, till it at length vanish, one of it's Roots shall likewise be diminished, until the last Term vanishing, it vanish along with it.

3°. Wherefore one Value of the Quantities $p, q, r, \&c.$ continually decreases until at length, when the Operation is infinitely produced, it vanish entirely.

4°. But the Values of these Roots $p, q, r, \&c.$ together with the Quotient already extracted, are equal to the Roots of the proposed Equation (thus in the Resolution of the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, shewn above, you will observe that $y = a + p = a - \frac{1}{4}x + q = a - \frac{1}{4}x + \frac{xx}{64a} + r, \&c.$) Whence it is sufficiently evident that the Quotient infinitely produced, is one of the Values of y : Which was what I proposed to shew.

58. The same Thing will appear by substituting the Quotient instead of y in the Equation proposed. For you will perceive that those Terms perpetually destroy one another in which x is of the least Dimensions.



Sir

Sir ISAAC NEWTON'S

ANALYSIS

BY

Equations of an infinite Number of
Terms, *explained.*

SECT. I.

INTRODUCTION.

1. **H**AVING endeavoured, in what goes before, to explain and illustrate Sir *Isaac Newton's* Treatise of the Quadrature of Curves: In order to render this Doctrine, together with what depends upon it, as compleat as I can, for the Use of the young Geometrician, I now design to give him some Assistance, for his better understanding this other *twin* Performance of our celebrated Author, according as I promised. And therefore, as this Analysis treats of the Doctrine of Infinite Series and Equations, chiefly with a View to the Quadrature of Curves, and other Parts of the Geometry of Curve-Lines, I suppose it will neither be unsuitable to the Design, nor unacceptable to the Reader, to lay before him a brief Account of the Rise and Progress of the Contemplation of Infinites and the Quadrature of Curves, in Mathematics, from it's Infancy in the Method of *Exhaustions*, used by the Ancients, to it's present State, as it has been perfected by Sir *Isaac Newton* into the Method of *Infinite Series* and *Fluxions*.

2. When

2. When the Ancients began to indulge themselves in the Study and Contemplation of the Properties and Relations of Magnitudes, they very soon found it necessary to consider and investigate the fundamental Principles of the Doctrine of Proportion: And accordingly we find it delivered by *Euclid* in the fifth Book of his Elements. In the third Definition, he defines Ratio or Proportion to be ¹ *a certain Habitude of two Magnitudes of the SAME KIND to each other according to Quantity:* And in the next Definition, he defines Magnitudes capable of bearing Ratio or Proportion, to be such, *as being multiplied or repeated some certain Number of Times, may EXCEED one another.* From whence it appears that Quantities that are of different Kinds, such as Lines and Surfaces, Surfaces and Bodies; any of these and Time; Time and Velocity, &c. cannot be said to have Ratio or Proportion to one another: Both because they are of different Kinds; and likewise (which indeed follows from the other) because no Multiplication or Repetition of the one can ever make it exceed the other; or even constitute any Quantity of that Kind: For no Multiplication of a Line can ever make any Surface; nor any Portion of Time, however multiplied, make any Degree of Velocity, &c. But it is evident, by the second of the preceding Definitions, that *Euclid* designed by it to limit Magnitudes that bear Proportion to each other, to some other Idea than that of Homogeneity: Otherwise his fourth Definition had been superfluous. And by what comes afterwards, in the succeeding Parts of these Elements, it appears, that he meant to lay a Foundation for the Doctrine of *Exhaustions*, as it is wont to be called; as well as that of Incommensurables. For being about to treat of the Doctrine of Incommensurables in the tenth Book, he lays down and demonstrates this Proposition by way of a Lemma, *viz. Two unequal Magnitudes being proposed, if from the greater you take away more than the half, and from the Remainder more than it's half; and so on continually, there will remain a Magnitude at length, by this continual Subduction, which is less than the least of the two Magnitudes proposed.* In demonstrating which he assumes this Principle or Postulate, that *the lesser of the two Magnitudes being multiplied, will at length exceed the greater:* For if not, they can bear no Ratio to one another by *Def. 3. B. 5.* which yet they are supposed to do, that they may be capable of any Comparison or Relation to one another.

3. Now this Proposition or Lemma is the Foundation upon which the Method of *Exhaustions* is built. Which Method is made use of by *Euclid* in demonstrating some Propositions, as the second, tenth,

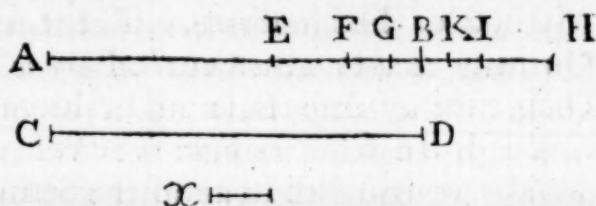
¹ Δύο ἐστὶ δύο μεγεθῶν ὁμογενῶν ἢ κατὰ πληρότητα πρὸς ἀλλήλα ποία σχέσις.

eleventh, twelfth, and eighteenth, B. 12. El. in which he proves that Circles are to one another as the Squares of their Diameters; that a Cone is a third part of a Cylinder of the same Base and Altitude; that Cones and Cylinders of the same Altitude are to one another as their Bases; that similar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Bases; and that Spheres are to one another in a triplicate Proportion of their Diameters. And *Archimedes* makes constant Use of it throughout all his Writings. For he, beginning where *Euclid* and other Geometricians before himself left off, found that this, or some Principle equivalent to it, was absolutely necessary to make any Progress in the Contemplation of Plain Curvilinear Figures; and Solid Figures generated by their Rotation; which were not capable of being divided into Parts bounded by Right Lines, or Plain Surfaces, and therefore we find him, in several different Places, assuming this Postulate, that of two unequal Magnitudes of the same Kind, the Excess of the greater above the lesser cannot be so small, but that it may be so multiplied as to exceed either of them; or any other Magnitude of the same Kind. Accordingly, by the Help of the Principle, he has demonstrated, by the Method of Exhaustion or continual Approach, the Quadrature of the Parabola; having shewn that any Segment of a Parabola cut off by a Right Line is to the inscribed Triangle of the same Base and Altitude as 4 to 3; that a Circle is equal to a right-angled Triangle whereof the two Sides including the Right-angle, are equal the one to the Semi-diameter, the other to the Periphery of the Circle; and has shewn how to approach to the Quadrature of the Circle and Ellipse. He has demonstrated the Ratio of the Cylinder, Hemisphere and Cone having the same Base and Height, both with respect to their solid Contents, and Surfaces; and shewn in general how to compare Spheres, Spheroids, Parabolical, and Hyperbolical Conoids, or any Portions of them, with Cones or Cylinders: And with Respect to the Surface of the Sphere, he has demonstrated that it is equal to 4 times the Area of one of it's great Circles. And besides many noble and useful Discoveries with Respect to these Figures mentioned, he wrote a most elaborate Treatise upon the Spiral Line, which commonly goes under the Name of *Archimedes's* Spiral, in which he demonstrates the Proportion of the Spiral Spaces to the Areas of the circumscribing Circles; and that of the Subtangents in the Spiral, to the Perimeters of the Circles. All these Things are demonstrated mostly by the Method of Exhaustions. Which may be considered as the first Step towards the general Method of Quadratures, and infinite converging Series lately introduced. This hath got the Name of the Method

Method of *Exhaustions*, because, in comparing two unequal Quantities with one another, by taking more than half the Difference, and adding it to the least, or subducting it from the greatest; and again taking more than the half of the remaining Difference, and adding it to the least, or subducting it from the greatest; and so on continually, the Difference betwixt the two Quantities is gradually *exhausted*, so as to become less than any given Quantity, in virtue of what is demonstrated by *Euclid* in that Lemma, which hath been mentioned: And it would be the same thing, if, instead of taking more than half the Difference, you take half the Difference, or even any given Part of the Difference, and add it continually to the least; or subduct it continually from the greatest of the two Quantities: For this likewise will gradually *exhaust* the Difference, so as to make it less than any given Quantity.

4. That the Way of proceeding in this Method may be the better understood, I shall shew how *Archimedes* demonstrates by it, that a Circle is equal to a right-angled Triangle, whereof one of the Sides including the Right Angle, is equal to the Radius of the Circle, and the other to it's Circumference. In Order to which, let the Circle

and Triangle be represented by the two Right Lines AB and CD, as Symbols. Moreover let AE, AF, AG, &c. represent a Square, an eight sided, a sixteen sided Figure, &c. regularly inscribed in the Circle: The



Number of Sides being always doubled. Again let AH, AI, AK, &c. represent a Square, a regular eight sided, sixteen sided, &c. Figure, circumscribed about the Circle. Then I say $AB = CD$. For if not, then AB is either greater than CD; or less. If AB be greater than CD, let the Excess be x : Then since EB, FB, GB, &c. are the Differences betwixt the inscribed Figures and the Circle, EF is greater than the half of EB; FG greater than the half of FB, &c. (as may easily be demonstrated) consequently (by Prop. 1. Elem. 10.) you must come at length to some Figure inscribed as AG, whose Difference from the Circle AB, viz. GB is less than x : Wherefore AG must be greater than the Triangle CD, since AB was equal to $CD + x$: But AG, being an inscribed Figure, is only equal to a right-angled Triangle having one of the sides equal to the Perpendicular drawn from the Center upon one of the Sides of the Polygon; and the other to the Sum of the Sides of the Polygon; the first of which is less than the Radius of the Circle; and the other less than the Circumference;

to which the Sides of the Triangle CD are equal: Therefore AG is less than CD, contrary to what has been demonstrated. Wherefore it is not true that AB is greater than CD. I say neither is AB less than CD. For if AB be less than CD; let the Defect be x , so that it be $AB + x = CD$; then it is easy to shew that HI is greater than $\frac{1}{2}HB$; that IK is greater than $\frac{1}{2}IB$, &c. and that therefore you must (by the Proposition already mentioned) at length arrive at some Difference as KB less than x : Wherefore AK is less than CD, since $AB + x = CD$. But AK, being a regular Polygon circumscribed about the Circle, is equal to a right-angled Triangle having one of it's Sides equal to the Radius, and the other equal to the Sum of all the Sides of the Polygon, *i. e.* greater than the Circumference of the Circle: Which Triangle therefore is greater than the Triangle CD: Wherefore AK is also greater than CD; contrary to what has been demonstrated: For it was shewn to be less. Therefore the Circle AB is not less than the Triangle CD; and it was demonstrated that it is not greater. Therefore they are equal. Q. E. D.

5. By this example it may appear in what Manner the ancient Geometricians proceeded, when they demonstrated any Proposition by the Method of Exhaustions. The *Characteristic* of which is, that one Quantity makes a *continual Approach* to another, so as to *exhaust* the Difference by degrees, until it become less than any given Quantity. And it is the same in Ratio's: When a Ratio, which is variable, continually approaches to another constant Ratio, so as to differ from it at length by less than any given Ratio.

Where they could discover any two Quantities, or two Ratios, which were both Limits to which the same variable Quantity, or Ratio, continually approached, so as at length to come nearer them than by any given Difference, but never to go beyond them, they concluded these Quantities, or Ratios, which were the Limits, to be equal. And this they demonstrated by that kind of Demonstration which is called *ducens ad absurdum*: Which Demonstrations were sometimes long; but always strict and exact. To avoid the Tedioufness of such Demonstrations, it was that our Author invented his Method of *prime and ultimate Ratios*¹. Which is founded upon a like Principle, as the Method of Exhaustions; and the same Objections lye against the one as against the other, as we have shewn towards the Beginning of our Notes upon the Quadrature of Curves.

6. As, by what has been said, it appears in what Manner the Ancients entered into the Contemplation of Curvilinear Areas, and Solids

¹ Phil. Nat. Prin. Sect. 1. Schol.

generated by their Revolution; so likewise we may observe the Doctrine of *infinite converging Series*, beginning to appear in Embryo, as it were, in their continual Approaches, by exhausting the Difference. For the Steps of the Approach, may be supposed to be continued indefinitely: For although a finite number, or even a very few of them, is sufficient to make out their Demonstrations; yet it is evident that no finite Number of them can ever *entirely* exhaust the Difference. Thus in the preceding Example taken from *Archimedes*, $AE + EF + FG, \&c.$ in *inf.* represents an infinite Series of simple Terms, according to Sir *Isaac Newton's* Method; which is that generally used; where the Aggregate of all the Terms, infinite in Number, is equal to AB ; and the greater the Number of them, taken from the Beginning, is, the nearer doth the Aggregate of them approach to AB : So that a finite Number of Terms differs from AB by less than any given Difference. Again $AH. AE, AI. AF, AK. AG, \&c.$ represents Mr. *James Gregory's* Method of converging Series in his excellent Treatise *Vera Circuli & Hyperbolæ Quadratura*. In which he supposes the Terms of the Series to proceed by Pairs, as A and B , C and D , E and F , $\&c.$ such, that the Difference betwixt A and B is greater than the Difference betwixt C and D ; and that after the same Manner that C and D are formed by an analytical Operation from A and B ; after the like Manner, or by the like analytical Operation, E and F are formed from C and D ; and so on continually: Every new Pair of Terms being produced always in the same Manner from the Pair of Terms immediately preceding. By which the Difference continually lessening, becomes less than any given Quantity; and the Series being supposed to be continued *in infinitum*, that Difference quite vanishes, and the two Terms become equal; either of which is the Quantity sought; and the other Terms approach so much nearer to it, in order, the further they are distant from the Beginning of the Series. He mentions another Form of converging Series consisting of simple Terms, to which the foregoing may be reduced; which he defines thus: Let $A, B, C, D, E, \&c.$ be a Series of simple Terms, and let them be of such a Nature that the third Term C is compounded after the same Manner of A and B , the first and second, that the fourth D is of B and C ; and the fifth E is of C and D , and so on *in infinitum*: And let the Difference of the antecedent Terms A and B , be always greater than the Difference of the immediately subsequent B and C . This Series being supposed to be continued infinitely, each new Term makes a nearer Approach to the Quantity required, and you may continue it so far as to arrive at a Term, which differs from it by less than any given Difference: So that
what

what he calls the last Term (for want of another Word to express it) is the very Quantity required. Such Series appears in Embryo, likewise, in the preceding Proposition from *Archimedes*; where AB being the Quantity sought, AE, AF, AG, &c. may represent the Terms of one such Series; and AH, AI, AK, &c. the Terms of another.

7. But notwithstanding there is some Analogy in all the three Methods, viz. that of Exhaustions; that of Mr. Gregory's Forms of Series; and that of Sir Isaac's; as they represent a continual Approach to some determined Limit; yet there is a manifest *Gradation* as to their *Extent* and *Universality*. For the Approach used by the Ancients proceeded by taking away or adding the half Difference, or more than the half Difference, continually. In Mr. Gregory's Form of converging Series, it is sufficient if the Terms by which the continual Approach is made be so related, that the subsequent ones be always compounded of the preceding ones after the same Manner, by any analytical Operation whatsoever, including Addition, Subtraction, Multiplication, Division, and Extraction of Roots. Sir Isaac Newton's Form of Infinite Series doth not even require this; it being sufficient that the Sum of the Terms from the Beginning, continually approach to the Quantity sought; so as a finite Number of them may differ from the thing sought by less than any given Difference, without any further Limitation; although the Law according to which the Approximation is made, may often, if not always, be discovered. The Method of investigating these Series's being much more general, as well as the Relation of the Terms of which they are composed, Mr. Gregory afterwards fell into the same Method; when he saw it to have so much Advantage over the Method he had formerly used in his Quadrature of the Circle, Ellipse and Hyperbola; as we shall see afterwards.

8. After *Archimedes* there was little or nothing done in the Geometry of Curve Lines, till towards the Middle of the last Century. In the Year 1635 *Bonaventura Cavallerius* introduced his Method of *Indivisibles*: By which he endeavoured to render the Method of *Exhaustions* of the Ancients, somewhat easier and shorter. In this Method Figures are conceived as made up of infinitely small or *indivisible* Parts or Elements; a Line of an infinite Number of Points; a Surface of innumerable Lines, right or curve, as is most convenient; or perhaps of other Elements, as infinitely small Triangles, Sectors, &c. a Solid, of an infinite Number of Surfaces, plain or curve; or perhaps of other Elements; as a Sphere of innumerable small Pyramids, all terminating in the Center, and constituting the spherical Surface with their Bases, &c. In which Method, Points, Lines, and Surfaces, considered

sidered as the Elements of which Magnitudes are composed, are not to be conceived as *geometrical* Points, Lines and Surfaces; but as geometrical Lines, Surfaces and Solids, of infinitely small Length, Breadth and Thickness. By this Method many useful Truths were discovered; and demonstrated in a much more concise Way than by the Method of Exhaustions: Yet one had need proceed with great Caution, otherwise one may be very readily led, by it, into Errors, and false Conclusions. It was found of very great Use for investigating those Properties and Relations of Magnitudes; which could not be easily discovered by the former Methods: Which being once discovered might always be ascertained, by reducing the Demonstrations to the Apagogical Form of the Ancients; where there was any Scruple left as to their Truth; which indeed was frequently the Case. For the Foundation, upon which the Geometry of Indivisibles is built, seems not to be quite so satisfying, or strictly *geometrical*; in Regard the Supposition of finite Magnitudes being made up of infinitely small and indivisible Parts, is a Notion hard to digest; and, to say the best of it, is not capable of being altogether freed from some *Confusion*, which still remains in the Mind, after all the Pains we can be at to comprehend it.

9. After *Cavallerius* and his Scholar *Torricellius*, there arose a great Number of famous Men in the Mathematical Way, who have considerably enlarged and promoted this Science, in it's higher Parts; such were M. *Des Cartes*, *Fermat*, *Huddenius*, *Gregory* of *St. Vincent*, Mr. *Huygens*, Dr. *Barrow*, and many others: But the most considerable Step by far that was made in the Geometry of Curve Lines, and the Doctrine of Quadratures, was by Dr. *Wallis* in his *Arithmetic of Infinites*, published in the Year 1656. The Doctrine of infinite Series had been very little considered before this Time: But he perceiving the Analogy betwixt the Terms of certain Series's and the Ordinates of certain Curves, sought out Rules for finding the amount of all the Terms of these Series's: By Means of which he was capable of squaring all those Curves, whose Ordinates were proportional to the Terms of such Series's. He began by considering an arithmetical Progression: From that he proceeded to Progressions of Powers, that is whose Terms were as the Squares, Cubes, Biquadrates, &c. of the Terms of an Arithmetical Progression as 0, 1, 2, 3, 4, &c. then to Progressions of Roots, that is, whose Terms are as the Square Roots, Cube Roots, Biquadratic Roots, &c. of the Terms of an arithmetical Progression. Afterwards he was led to consider Progressions whose Terms are any Dimension whatsoever of the Terms of the Arithmetical Progression, *i. e.* the Indexes or Exponents of whose Dimensions are any Numbers integral, fractional,

fractional, or surd, whether positive, or negative¹. He considered these Progressions as consisting of an infinite Number of Terms, the last Term, which represented the lowest Ordinate of the Curve, being still finite, and the intermediate Terms from 0 to the last, being infinite in Number, represented Ordinates applied to the Axis at infinitely small and equal Distances, betwixt the Vertex and lowest Ordinate: Or perhaps these Terms represented any other Lines Right or Curve; or any Plain or Curve Surfaces, in the Case of Solids, which were proportional to them. And he found, in a Way of Induction, this most general and comprehensive Proposition, *That the Sum of all the Terms of any such Series is to so many times the greatest, as Unity is to the Index or Exponent of the Power, Root, or Dimension whatsoever, encreased by 1*. Which amounts to this: Supposing 0, 1, 2, 3, 4, &c. x to be an Arithmetical Progression consisting of an infinite Number of Terms, in the natural Order of the Numbers, having the last Term x ; and let $0^m, 1^m, 2^m, 3^m, \&c. x^m$ be a Progression of Terms, which are any the same Power, Root, or Dimension whatsoever, of the former Terms, whose Exponent is denoted by m , then shall the Sum of this last Series or Progression be equal to $\frac{x^{m+1}}{m+1}$. And this is in Substance the same with

the first Rule of this Analysis of Sir *Isaac Newton's*, which was discovered by an Induction of Particulars by Dr. *Wallis*; but is demonstrated universally, by Means of an indefinite Index, by Sir *Isaac*.

10. From thence Dr. *Wallis* proceeded to the Consideration of such Progressions as have their Terms made up of Combinations of the simple Terms of the forementioned Series's, by Addition and Subtraction; and likewise of the Powers of such Sums and Differences: And thence he investigated what is in Substance the same with our Author's second Rule of his Analysis, for the Quadrature of compound Curves. By this Means he was enabled to give the Quadrature of all sorts of Parabolas, of spiral spaces, and of many plain Figures which extend themselves infinitely along an assymptote: Likewise the Contents of such solids as are generated by the Revolution of these Parabolical and Hyperbolical Curves; and thence discovered other Things, which depended upon these Quadratures, such as the rectifying of several Curve Lines, and plaining of Curve Surfaces; as he has shewn in his Arithmetic of Infinites, and other mathematical Works.

11. But he found his Method fail him in the Case of those Progressions, whose Terms were the Roots of the Sums or Differences of simple

¹ Dr. *Wallis* is reckoned the first Person who brought in the Use of fractional and negative Indexes; and likewise of indefinite and surd ones.

Terms, called Roots universal, such as $\sqrt{r^2 - 0a^2}$, $\sqrt{r^2 - 1a^2}$, $\sqrt{r^2 - 4a^2}$, $\sqrt{r^2 - 9a^2}$, $\sqrt{r^2 - 16a^2}$, &c. $\sqrt{r^2 - r^2}$: Which he calls a Series of Terms in the subduplicate Ratio of a Series of Equals lessened by a Series of Secundans or Squares. Where if r stand for the Radius of a Circle; and a for any of the infinitely small and equal Distances of the Ordinates in the Quadrant of the Circle, beginning at it's Center, such Quadrant is equal to the Sum of all the Terms of this Progression; as he shews in Prop. 121. Arith. Inf. The same Series with the Sine of the second Term under the Vinculum, changed into it's Opposite, that is $\sqrt{r^2 + 0a^2}$, $\sqrt{r^2 + 1a^2}$, $\sqrt{r^2 + 4a^2}$, &c. $\sqrt{r^2 + r^2}$, being summed up, would give the Quadrature of the equilateral Hyperbola. He likewise shews other two Series's, by the summing up of whose Terms, the Quadrature of the Circle and Hyperbola would be found, but labouring under the same Inconveniency. Here Dr. Wallis stuck: But this Difficulty was removed afterwards by our incomparable Author, by his third Rule in this Analysis, as we shall see anon. However Dr. Wallis being very solicitous to do something towards the Quadrature of the Circle; which was his principal View when he engaged in the Prosecution of these Enquiries (as he tells us in the Preface to that Work) he thought upon another Method, which he calls *Interpolation* of Series. By which he means a Method of discovering certain intermediate Terms of a regular Series or Progression, by considering the Properties of the Progression, and the Relations of the Terms to each other. Of this he gives several Instances for finding the Area of the Circle. Of which this is one: In the Progression 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. whose Terms are produced by the continual Multiplication of $1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8}$, &c. to find the intermediate Term betwixt 1 and $\frac{1}{2}$. Such intermediate Term he expresses by this Symbol $\square$, even as the Square Root of 2 is expressed thus, $\sqrt{2}$; and he shews that the Circle is to the Square of it's Diameter as 1 to $\square$. But the Result of his Enquiry was, that the Value of $\square$ cannot be adequately expressed by any received Way of Notation; which is nothing more strange than that $\sqrt{2}$ or any other surd Root is not capable of being adequately expressed, in that Way: But whereas the Value of $\sqrt{2}$, or any other Surd, may be expressed approximately by the common Notation, so likewise he found that the approximate Value of $\square$, was $1 \times \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \text{ \&c. in inf.}}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \text{ \&c. in inf.}}$ or $1 + \frac{1}{8}A + \frac{1}{24}B + \frac{1}{48}C + \frac{1}{80}D \text{ \&c. in inf.}$ the Capitals A, B, C, &c. denoting the first, second, third, &c. Terms.

12. When these Discoveries, which were indeed very noble and useful, and in Point of Generality and Extent, far exceeding any thing

that had been formerly done by others in the Geometry of Curve Lines, were made public, it was objected to him by Monsieur *Fermat*¹, and others, that however valuable his Discoveries were, and true in themselves, yet the Demonstrations of them in the Way of Induction by Dr. *Wallis*, did not come up to that Accuracy which a Geometrical Subject required, and the ancient Geometricians had all along observed in all their Performances. To which Dr. *Wallis* made Answer (even as he had remarked in the Arithmetic of Infinites itself) that he did not so much design to demonstrate his Discoveries, as to lay open to others the Method he used in making them; which the Ancients purposely concealed; that, notwithstanding, he thought the Proof by way of Induction was satisfying and convincing; that it would be an easy Matter for any Person moderately skilled in Geometry to demonstrate these Things with all the Pomp and Apparatus made Use of by the Ancients: But that was a Labour he never designed to undertake. However, to give some Satisfaction in this Matter, he shews by some Examples in the 78th Chap. of his Algebra, how the Propositions he had discovered, might be demonstrated after the Manner of the Ancients, in imitation of what had been done by *Archimedes* in the 10th and 11th Propositions of his Treatise of Spiral Lines: In which *Archimedes* demonstrates, what is the Sum of all the Terms of a Progression of Squares, whose Sides constitute an arithmetical Progression of Lines, having the common Difference equal to the first Term; when compared with so many times the greatest Square: And the Limits betwixt which the Sum of the Terms of such a Progression is contained, although the common Difference of their Sides be not the same with the first of them. Which he applies to the finding the Relation of the Spiral Spaces to the Circular Sectors; even as Dr. *Wallis*, by prosecuting this Affair to a much greater Length, shews how to find not only the Sum of all the Squares; but the Sum of any Powers, or Roots whatsoever, of an arithmetical Progression; and thereby to compare infinite Numbers of curvilinear Areas with right-lined Figures, and with one another. And truly, when one attentively considers this elaborate Treatise of *Archimedes*, and the other Works of that subtle and penetrating Genius, one cannot help seeing Dr. *Wallis*'s Arithmetic of Infinites, Mr. *Gregory*'s Method of Inscription and Circumscription of Polygons; and even our second *Archimedes*'s Method of prime and ultimate Ratios, beginning to discover themselves, as it were in Embryo, in order to be brought forth afterwards to open Light.

¹ See Dr. *Wallis*'s Algebra, Chap. 79.

13. And now we are arrived at the last Step of this Progress, as it was perfected into the Method of infinite Series and Fluxions by our celebrated Author. By what Means this was done, I cannot better represent to the Reader than in the Account he gives of it himself in his Letter to Mr. Oldenburg, Octob. 24, 1676.²

14. " About the Beginning of my mathematical Studies, as soon
 " as the Works of our celebrated Countryman Dr. Wallis fell into my
 " Hands, by considering the Series's, by the Intercalation of which,
 " he exhibits the Area of the Circle and Hyperbola; viz. that in a
 " Series of Curves, whose common Base or Axis is x , and the Ordinates
 " $\sqrt{1-xx}$, $\sqrt{1-xx^2}$, $\sqrt{1-xx^3}$, $\sqrt{1-xx^4}$, $\sqrt{1-xx^5}$, &c.
 " if the Areas of every other of them, that is x , $x - \frac{1}{3}x^3$, $x - \frac{2}{5}x^5$
 " $+ \frac{1}{5}x^5$, $x - \frac{3}{5}x^5 + \frac{3}{5}x^5 - \frac{1}{7}x^7$, &c. could be interpolated, we
 " would have the Areas of the intermediate ones, of which the first
 " $\sqrt{1-xx}$ is the Circle: To interpolate these, I observed that the
 " first Term in all of them was x , and that the second Terms $\frac{1}{3}x^3$,
 " $\frac{1}{5}x^5$, $\frac{2}{5}x^5$, $\frac{3}{5}x^5$, &c. were in arithmetical Progression; and that there-
 " fore the two first Terms of the Series's to be intercalated, should be
 " $x - \frac{1}{2}x^3$, $x - \frac{3}{5}x^5$, $x - \frac{5}{7}x^7$, &c.

" For intercalating the rest, I considered that the Denominators 1,
 " 3, 5, 7, &c. were in arithmetical Progression, and that nothing
 " remained but to investigate the numeral Coefficients of the Numera-
 " tors. Now these I saw, in the alternate Areas that were known,
 " to be the Figures which express the Powers of the Number 11, viz.
 " 11^0 , 11^1 , 11^2 , 11^3 , 11^4 , &c. that is the first was 1; the second
 " 1, 1; the third 1, 2, 1; the fourth 1, 3, 3, 1; the fifth 1, 4, 6,
 " 4, 1, &c.

" Wherefore I enquired how, from the two first Figures given in
 " these Series's the rest might be derived; and I found that if the
 " second Figure be called m , the rest might be produced by a conti-
 " nual Multiplication of the Terms of this Series $\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times$
 " $\frac{m-3}{4} \times \frac{m-4}{5}$, &c.

" For Example, let the second Term $m=4$, then $4 \times \frac{m-1}{2}$, that
 " is 6, will be the third Term; and $6 \times \frac{m-2}{3}$, that is 4, the fourth

² See this Letter, which contains many valuable Discoveries of Sir Isaac in the Com. Epist. No 55.

“ Term; and $4 \times \frac{m-3}{4}$, that is 1, the fifth; and $1 \times \frac{m-4}{5}$, that is 0,
 “ the sixth Term: With which the Series, in this Case, terminates.

“ Therefore I applied this Rule for intercalating the Series's. And
 “ since, for the Circle, the second Term was $\frac{\frac{1}{2}x^3}{3}$, I put $m = \frac{1}{2}$, and
 “ the Terms arising were $\frac{1}{2} \times \frac{\frac{1}{2}-1}{2}$, or $-\frac{1}{8}$; $-\frac{1}{8} \times \frac{\frac{1}{2}-2}{3}$, or $+\frac{1}{16}$;
 “ $+\frac{1}{16} \times \frac{\frac{1}{2}-3}{4}$, or $-\frac{5}{128}$; and so on *in inf.* Whence I came to
 “ understand that the Area of the circular Segment, which I wanted,
 “ was $x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9}$, &c.

“ And the like Method may be made use of to interpolate other
 “ Series's, and that although they be deficient by the Intervals of two
 “ or more Terms together. Such was my first Entry upon these Studies;
 “ which had certainly escaped me, had I not accidentally cast my Eye
 “ upon some loose Papers that came in my Way a few Weeks ago.
 “ But having arrived this Length, I immediately began to consider that
 “ the Terms $\sqrt{1-xx}$, $\sqrt[3]{1-xx}$, $\sqrt[4]{1-xx}$, $\sqrt[5]{1-xx}$, &c. that is
 “ to say 1, $1-x^2$, $1-2x^2+x^4$, $1-3x^2+3x^4-x^6$, &c. might
 “ be intercalated after the same Manner as the Areas generated by
 “ them; and that nothing was required for this Purpose but to omit
 “ the Denominators 1, 3, 5, 7, &c. which are in the Terms expressing
 “ the Areas: That is to say that the Coefficients of the Terms of the
 “ Quantity to be intercalated $\sqrt{1-xx}$, or $\sqrt[3]{1-xx}$; or in general of
 “ $\sqrt[m]{1-xx}$, arise by the continual Multiplication of the Terms of
 “ this Series $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c. And thus I perceived e. g.
 “ that $\sqrt{1-xx}$ was equivalent to $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$; and $\sqrt[3]{1-xx}$,
 “ to $1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6$, &c. and $\sqrt[4]{1-xx}$, to $1 - \frac{1}{3}x^2 - \frac{1}{9}x^4$
 “ $-\frac{5}{81}x^6$, &c.

“ And thus I was possessed of a general Method of reducing Radicals
 “ to infinite Series, by the Rule I mentioned towards the Beginning of
 “ my former Letter¹, and that before I was acquainted with the
 “ Extractions

¹ This refers to his Letter wrote to Mr. Oldenburg 13 June 1676, at the Beginning of

which he had set down his famous Binomial Theorem thus, $\sqrt[m]{P+PQ} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ$, &c. which is what he understands by the Rule he mentioned. See
 Com.

“ Extractions of Roots. But once this was known, I could not remain
 “ long ignorant of that other. For in order to prove these Operations,
 “ I multiplied $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$, &c. into itself, and I found
 “ the Product $1 - xx$, the rest of the Terms *in inf.* vanishing, by the
 “ Continuation of the Series. And thus likewise the Series $1 - \frac{1}{3}x^2 -$
 “ $\frac{1}{9}x^4 - \frac{5}{81}x^6$, &c. multiplied twice into itself produced $1 - x^2$.
 “ Which, as it afforded a certain Proof of the Truth of these Con-
 “ clusions; so it led me to try whether, conversely, these Series’s,
 “ which thus appeared to be the Roots of the Quantity $1 - x^2$,
 “ might not in an arithmetical Manner, be extracted out of it.
 “ The Matter succeeded to my Mind. The Form of the Operation
 “ in the Extraction was this. (*See it under the Examples for extracting*
 “ *Roots in this Analysis. Art. 15. and following.*)

“ Having discovered this, I entirely neglected the Interpolation of
 “ Series’s, and made use of these Operations only, as a more genuine
 “ Foundation. And as to Reduction into infinite Series by Division,
 “ it did not lye long concealed, being a much easier Affair. Yea I
 “ immediately attempted the Resolution of affected Equations, and
 “ obtained it likewise. Whence the Ordinates, the Segments of the
 “ Axes, and any other Right Lines, were at once known, from the
 “ Areas or Arches of the Curves being given.

“ About that Time the Plague breaking out², obliged me to get
 “ hence (from *Cambridge*) and think of something else. Notwith-
 “ standing, I added to my other Discoveries, at that Time, a certain
 “ Construction of Logarithms deduced from the Area of the Hyperbola.”
 And after shewing how this Construction was affected, he proceeds
 thus :

“ I am ashamed to tell to how many Places of Figures I carried
 “ these Computations, having no other Business at that Time: For I
 “ pleased myself too much with these Discoveries. But as soon as
 “ that ingenious Performance of *Nicolaus Mercator* appeared, I mean
 “ his *Logarithmotechnia* (who I doubt not, first discovered the Things
 “ he published) I began to cool a little upon these Things, suspecting
 “ that either he understood the Extraction of Roots; as well as Division
 “ of Fractions; or at least that others upon the Discovery of Division,
 “ would find out the rest, before I could be of a sufficient Age for
 “ writing.

Com. Epist. N^o 48. These two Letters of Sir *Isaac Newton* to Mr. *Oldenburg* give an Account
 of many of his admirable Discoveries in the Business of the Quadrature of Curves, and infinite
 Series; and upon that Account deserve the mathematical Reader’s careful Perusal.

² This happened in the Years 1665 and 1666.

“ Yet about the same Time that Performance appeared, I commu-
 “ nicated to Mr. *Collins* by the Intervention of our Friend Dr. *Barrow*
 “ (then Professor of Mathematics at *Cambridge*) a certain Compend
 “ of these Series's³. In which I had signified that the Areas and
 “ Lengths of all Curves; and the Surfaces and Contents of Solids,
 “ might be determined from Right Lines given; and conversely
 “ from these given, the others might be found: And I illustrated
 “ the Method I pointed out, by several Series's.

“ After this a literary Correspondence having arisen betwixt us,
 “ Mr. *Collins*, a Person born for the Advancement of mathematical
 “ Learning, insisted strongly upon my publishing these other Things.
 “ Accordingly five Years ago (1671) when at the Persuasion of my
 “ Friends, I had resolved to publish a Treatise of the Refraction of
 “ Light; and of Colours, which I had then ready by me, I began
 “ to turn my Thoughts again to these Series's, and wrote likewise
 “ a Treatise upon them¹, that I might publish both together.

“ But having wrote you a Letter upon the Subject of the reflecting
 “ Telescope, in which I explained briefly my Notions of the Nature
 “ of Light, something unexpected fell out, which made me imagine
 “ it concerned me to write to you speedily about printing that Letter.
 “ Immediately upon the back of this I met with frequent Interrup-
 “ tions, from several Persons letters, taken up with Objections, and
 “ other Things, which frightened me entirely from executing my De-
 “ sign; and made me find Fault with myself for my Imprudence,
 “ that by catching at a Shadow, I had been so far deprived of the Peace
 “ and Quiet of my own Mind, which is a thing of more substantial
 “ Worth.

“ About that Time, *James Gregory*, from one certain Series of
 “ mine, which Mr. *Collins* had sent him, after much Thought and Ap-
 “ plication (as he himself wrote back to *Collins*) lighted upon the
 “ same Method, and hath left a Treatise upon it, which we hope
 “ will be published by his Friends. Since, considering his extraordina-
 “ ry Genius, he must have added many new Discoveries of his own,
 “ which would be a great Loss to mathematical Learning, should they
 “ be lost². “ But

³ By which he means this Analysis by infinite Equations.

¹ This Treatise is mentioned by Mr. *Collins* in two Letters published in the Com. Epist. N^o 22, 23; and it appears, by the Account given of it by him, and by what follows in this Letter of the Author, to be the same with that published by the ingenious Mr. *John Colson*, now Professor of Mathematics in the University of *Cambridge*, A^o 1736, under the Title of *The Method of Fluxions and infinite Series; with it's Application to the Geometry of Curve Lines*.

² Here we have Sir *Isaac Newton*'s own Testimony that Mr. *James Gregory* discovered a General Method of Quadratures and Series's, of the same Sort with his own. This is sufficient-ly

“ But as for my Treatise I mentioned, I had not quite finished it,
 “ when I gave over my Design: And I have had no Inclination to
 “ this very Day, to add the other Things to it. For it wanted that
 “ Part, in which I had designed to explain the Method of solving
 “ such Problems as cannot be reduced to Quadratures: Although I had
 “ laid down something of the Foundation of this also. However the
 “ Doctrine of Infinite Series made but a small Part of that Treatise.

“ I put a good many Things together, among which was the Me-
 “ thod of drawing Tangents; which the sagacious *Slusius* communi-
 “ cated to you some two or three Years ago; concerning which (at

ly confirmed by Mr. Gregory's Letters to Mr. Collins, published in the Com. Epist. N^o 16, 17, 18, 20, wrote in the Years 1670 and 1671; by Mr. Oldenburg's Letter to Mr. Leibnitz, Ap. 15. A^o 1675. N^o 36. and by Mr. Collins's Letter to David Gregory, Brother to James, Aug. 11. A^o 1676. N^o 47; which was the Year after James Gregory died. But although this be evident from these and other incontestable Proofs, yet by any thing that ever I could learn, Mr. Gregory left no such Treatise behind him, at his Death, as Sir Isaac Newton speaks of in this Place: And had there been any such, it is highly probable it would have been made public by his Nephew, Dr. David Gregory, late Professor of Astronomy at Oxford, with whom all his Uncle's Papers were left. These Papers, which are now in the Custody of Dr. David Gregory, Canon of Christ's Church in that University, according to his known Humanity, were allowed me and another Gentleman to peruse in the Year 1736, with the Hopes of finding some such Treatise as that mentioned by Sir Isaac Newton; but we could not discover any such Thing among these Papers, although we saw several curious ones upon particular Subjects, which are not in Print. On the contrary, by some Letters we saw, it appears that James Gregory had not compiled any Treatise containing the Foundations of this General Method, a very short Time before his Death. So that all that can be known about his Method, can only be collected from his Letters published in the Com. Epist. and the short History of his Mathematical Discoveries compiled by Mr. Collins in twelve Sheets after his Death, at the Desire of the Gentlemen of the Royal Academy at Paris: Which was repositied among the Archives of the Royal Society at London. See Com. Epist. N^o 47. The Principles of his Method seem to have been few and simple; which he could easily retain in his Mind, and apply as Occasion offered. And I am much mistaken, if they have not been the very same, or nearly the same, with those of Sir Isaac Newton's Method; as may be collected from his Letters to Mr. Collins, mentioned already. In one of which he says, *I had not observed that Mr. Newton's Series for the Zone of a Circle (which you did me the Pleasure to transmit some Time ago) together with an infinite Number of such Series's, are easy Consequences from that one which I sent you about Logarithms; viz. a Logarithm being given, to find it's Number; or to change the Root of any pure Power into an infinite Series.* And in another he says: *As to Mr. Newton's General Method, I am possessed of it in some Measure, I think, as well with respect to mechanical as geometrical Curves.* And, after giving him a considerable Number of different Series's for the Circle, Ellipse, and Hyperbola, he adds: *I would not incline to give you so much Trouble; nor have I a Mind to print any thing, unless it be to give a new Edition of my Quadrature of the Circle, with some trifling Additions.* As to my Method of finding the Roots of all Equations; one Series discovers one Root only; but there are infinite such for every Root: Although it requires some Pains to begin the Series right, and to distinguish the Root to which it belongs. But I shall perhaps write you more of this afterwards. You need be in no Strait to communicate whatever I send you, to any Person: For I am not at all anxious, whether it be published under my Name, or that of any other. This last Sentence shews a worthy Disposition of Mind, which regarded the Discovery of Truth only, without affecting the Honour of being the Discoverer. From these and some other Passages in other Letters in the Com. Epist. it appears that Mr. Gregory's Method he made use of after the Year 1670, and Sir Isaac Newton's were in Effect the same.

“ the

“ the Desire of Mr. *Collins*) you writ back that I was in the Knowledge of the same. We fell upon it, it is true, in different Ways. When one knows the Foundation I proceed upon, he cannot draw Tangents, after any other Manner, unless he designedly go out of the right Way.

“ Moreover there is no Stop, whatever Way the Equations be affected with Radicals, affecting one or both the indefinite Quantities; but without any Reduction of such Equations (which would render the Work frequently insupportable) the Tangent is directly drawn. And the Thing is much the same in the Questions *de Maximis & Minimis*; and some others, which I shall not here mention.

“ The Foundation of these Operations, which is obvious enough in itself (because I can't here pursue the Explication of it) I chuse rather to conceal in Cypher thus *6accdaē13eff7i3l9n4o4qrr4s9t12vx*¹.

“ Upon this Foundation I have endeavoured likewise to render the Speculation concerning the Quadrature of Curves, more simple; and have arrived at some more general Theorems.”

Here he gives an Account of these Theorems, and some Examples of their Application: By which it appears, that, by applying the Foundation just now mentioned to render the Speculations about the Quadrature of Curves more simple, he means the Substance of the Doctrine contained in his Treatise of the Quadrature of Curves; which likewise appears from that Part of the *Method of Fluxions and Infinite Series* itself, published by Mr. *Colson*, which treats of this Affair. (See *Prob. 9. throughout, especially Art. 63.*) And after he has given some Account of these Theorems; and shewn how, at length, they become very complex, when the Progressions of the Curves are continued; he adds, “ So that I am of Opinion that they could scarce be found out by the Transmutation of Figures, which *James Gregory*² and some others have made use of, without some other Foundation.

“ For my own Part, I could not arrive at any thing that was general in these Matters, until I abstracted from the Contemplation of Figures, and reduced the whole Affair to the simple Consideration of the Ordinates only.

¹ This being decyphered, is, *Datā Equations quocunque Fluentes Quantitates involvente, Fluxiones invenire; & vice versa. i. e.* An Equation being given, involving any Number of Flowing Quantities, to find the Fluxions; and contrarily.

² This is the Method of Quadratures which Mr. *James Gregory* had formerly made use of in his Book published at *Padua* in 1668, under the Title of *Geometriæ pars universalis inferens Quantitatum Curvarum Transmutationi & Mensuræ*.

15. And thus I have given some Account ³ of the Rise and Progress of the Contemplation of Curves and Infinite Series: And therefore come now, as I propos'd, to give some Account of the Contents of this Treatise of *Analysis by Equations of an infinite Number of Terms*; and to shew wherein it agrees with the other Treatise of the Quadrature of Curves; and wherein they differ.

16. As to the Analysis by Equations of an infinite Number of Terms; we may take it up as containing the following Things.

1°. The Quadrature of simple Curves; where the Ordinate is equal to one Term only, including some Power of the Base or Absciss x , any Way involved with known Quantities; thus generally expressed,

$$y = ax^{\frac{m}{n}}.$$

2°. The Quadrature of compound Curves: *i. e.* where the Ordinate is equal to an Aggregate of such simple Terms.

3°. The Quadrature of all other Curves, where the Value of the Ordinate y , or some Part of it, is more complex than in the two preceding Cases. And here, such complex Value, or Part of the Value of y , may be of three Kinds. 1°. A Fraction, which, after it is reduced as far as may be, by the common Rules of Algebra, still retains the Denominator a compound Quantity, including the Absciss x in one or more of it's Terms, as $\frac{aa}{b+x}$. 2°. Some complex Radical, involving the Absciss x in one or more Terms below the Vinculum, as $y = \sqrt{aa - xx}$. 3°. The Root of an affected Equation, as $y^3 + a^2y + axy - 2a^3 - x^3 = 0$. In all these Cases, we are here taught how to reduce the Value of y into an infinite Series of simple Terms. For doing of which we have three different Methods, corresponding to these three different Cases; which are Division, Extraction of the Roots of compound Quantities, and Extraction of the Roots of affected Equations; all explained here in Order. And under the Head of the Extraction of Roots of affected Equations, our Author shews the Method of doing it in *numeral* Equations, as well as *specious*.

³ In this short Account of the Doctrine of Curves and Infinite Series, I have omitted making Mention of a great Number of ingenious and learned Men, both of our own Nation and Foreigners, who by their Industry and Labours, have deserved well of Mathematicks in these Particulars. For, an exact Account of their Discoveries and Improvements, both before and since our Author's Time, would take up a large Volume of itself; which I think might be a very useful Work. Neither have I entered into the Dispute betwixt the *English* and Foreigners about the first Inventor of the Method of Fluxions. With respect to which I am of Opinion, that no disinterested Person, who will take the Trouble to read the *Commercium Epistolicum*, published by Order of the Royal Society, can be in any Doubt.

A a a

4°. He

And

4°. He likewise shews how the foregoing Doctrine of the Quadrature of Curves, may be applied to other Subjects; and gives an Example in finding the Lengths of Curves.

5°. He lets us see how we may return again from the Area, or Length, &c. given, to the Base or Absciss of the Curve, by Means of the Resolution of affected Equations.

6°. He shews how this Doctrine may be applied to mechanical Curves.

7°. He gives a Demonstration of the Quadrature of simple Curves; which in Effect contains a Demonstration of the Quadrature of all others. And here, by the Way, he shews what Way you may find as many Curves as you please, that can be squared.

Finally, you have a Demonstration of the preceding Method of resolving affected Equations.

17. Wherefore by what is delivered in this short Compend of Sir Isaac Newton's first Discoveries, we are taught the Method of finding the Areas of all Sorts of Curves, either exactly or approximately. But this wonderful Person, not content with this noble Discovery, which infinitely surpassed all that had been done before in this Affair, still proceeded further, and made considerable Improvements in it; which are delivered in the other twin Performance, his Treatise of the *Quadrature of Curves*. For it often happens that those Curves, whose Areas can only be exhibited by *infinite* Equations, according to the Methods laid down in this Analysis, are capable of being expressed by finite Equations, and so may be geometrically squared; or they may be compared with other more simple Curves, whose Areas may in some Respect be considered as known; such as the Conic Sections. Accordingly he has taught us the Way of doing this in his *Quadrature of Curves*: and thereby rendered this Speculation more simple; and advanced that Doctrine to a greater Degree of Perfection. For as the Solution of a Problem by an Equation of an higher Order, when it may be solved by one of a lower, is reckoned ungeometrical, because not so simple as the Nature of the Problem will admit; so, much rather a Solution by means of an infinite Equation is not to be admitted, when the same Thing can be done by means of a finite Equation. For if this can be done, the Curve is geometrically squared, *i. e.* compared with rectilinear Figures. And when this cannot be done, yet it is of considerable Use to reduce the Areas of Curves, to the most simple Curves, with which they admit of a geometrical Comparison. The fifth and sixth Propositions of the *Quadrature of Curves* present us with two General Theorems for the Areas of all Curves which are thus

thus generally expressed $y = z^{\theta-1} \times \sqrt[\lambda-1]{e + fz^{\eta} + gx^{2\eta} + bz^{3\eta} \&c.}$

$\times a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} \&c. \text{ or } y = z^{\theta-1} \times \sqrt[\lambda-1]{e + fz^{\eta} + gx^{2\eta} \&c.}$

$\times k + lz^{\eta} + mz^{2\eta} \&c. \text{ }^{\mu-1} \times a + bz^{\eta} + cz^{2\eta} \&c. : y \text{ being the Ordinate; } z \text{ the Absciss; } \theta, \eta, \lambda, \mu, \text{ any Numbers whatsoever, integral or fractional, positive or negative; the other Letters representing any known Quantities. Where the Series's expressing the Areas often terminate; and thereby shew the Curve to be geometrically quadrable; when at the same Time the Area exhibited by the Method in the Analysis, can be exprest no other Way but by an infinite Series.}$

For understanding of which the Reader may consult these two Propositions, and our Explication of them.

18. There are other Cases of Curves likewise, which may be geometrically squared by the ninth and tenth Propositions: whose Areas can only be exprest by infinite Equations, according to the Method of the Analysis. And by the tenth Prop. of the Treatise of Quadratures, you can always discover what are the most simple Figures, whether rectilinear, or curvilinear, with which any Curve can be geometrically compared, whose Ordinate y is determined from the Absciss z given, by an Equation not affected. By Cor. 1. of that Prop. you can find the most simple Figures with which any Curve can be compared, whose Ordinate y is the Root of an affected quadratic Equation: such as $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$. And therefore also, every Curve, whose Ordinate is defined by any affected Equation, which can be made to pass into another Curve whose Ordinate is defined by a quadratic affected Equation, by the Method taught in Cor. 8. Prop. 9. may be geometrically squared; or compared with the most simple Figures with which it admits of a geometrical Comparison. Moreover by Cor. 2. of that Prop. every Curve whose Ordinate is defined by such an affected Equation, as may be made to pass into an Equation not affected, by the Method taught in Cor. 7. Prop. 9. can be geometrically squared; or compared with other the most simple Curves, it admits of being compared with. Under this last head are included all such Curves, whose Equation consists of three Terms, whatever Way affected: thus generally exprest $ax^l + bx^my^r + cy^s = 0$.

19. These are the Cases of Curves whose Areas may be found by what is taught in our Author's Quadrature of Curves. By which also we find when they admit of a geometrical Quadrature, or Comparison with right-lined Figures: which often happens, when their Areas can only be exprest by an infinite Series, by the Methods taught

in this Analysis. For when the Ordinate, or any Part of it, is express'd by a complex Fraction or Radical, the Area which is exhibited by the Analysis is always express'd by an infinite Series: and it very seldom happens to be otherwise, when the Ordinate is the Root of an affected Equation. Wherefore it hence appears that our Author's Treatise of the Quadrature of Curves is a considerable Improvement of that Subject, and what is connected with it. Besides what has been said, it contains the fundamental Principles of the Method of Fluxions; many useful Things respecting the Comparison and Transmutation of Curves; and shews in the last Scholium, how far the Quadrature of Curves serves for finding the Fluents from the Fluxions given. Finally it presents us with two most useful Tables of Quadratures: the first of which exhibits the Areas of the more simple Kinds of Curves that can be squared; the other contains the more simple Kinds of Curves that may be compared with the Conic Sections. So that the Areas in both Cases may be found by simple Inspection.

20. But in regard there are Curves, whose Areas cannot be found at all by the Methods taught in the Quadrature of Curves, *viz.* those that are defined by Equations that come not under any of the Heads mentioned in Art. 17. and 18. above: therefore, in these Cases, we must have Recourse to the Method taught in this Analysis, by the Resolution of affected Equations into infinite Series. Which is one of the chief Reasons of annexing this Analysis with the Explication of it, to the foregoing Treatise of the Quadrature of Curves.

S E C T. II.

Explication of the first Part of the Author's third Rule; which shews the Method of finding the Areas of those Curves, which require the Reduction of complicate Fractions and Radicals: contained in Art. 12 — 20.

21. **A**S to the two first Rules for the Quadrature of *simple* and *compound* Curves, I think I have so fully explained the Doctrine contained in them, in the Notes upon the preceding Treatise, especially in Sect. 5. and the Explication of Prop. 5. that it would be a needless Repetition to say any thing upon that Head, in this Place.

22. I told

22. I told already that there are three different Ways of reducing Quantities into infinite Series. 1°. By *Division*. 2°. By extracting the Roots of *surd Quantities*. 3°. By extracting the Roots of *affected Equations*. The first two are to be explained in this Section.

23. Any Fraction, whose Denominator consists of more Terms than one after the Fraction is brought to it's most simple Expression, may be reduced to an infinite Series of simple Terms (*i. e.* which either have no Denominators; or else only simple ones) by dividing continually; as is done in the Case of Division in Decimal Arithmetic, where there is a Remainder: in which Case, as is commonly known, the Quotient may be continued in decimal Fractions: which very often run out infinitely. The Conveniency of thus reducing Fractions having compound Denominators; and the Use to be made of it in the Quadrature of Curves, appears from what our Author shews of it's Application to the equilateral Hyperbola; and some other Curves.

24. In the Application of this Operation to the squaring of Curves, where the Ordinate y , or any Part of it, is exprest by a Fraction; it is only then that the Denominator of the Fraction is reckoned compound, and the Fraction to be turned into an infinite Series, when, after it is reduced by the common Rules for reducing of Fractions in Algebra, the Denominator is still made up of several Terms, of which one or more include the Base or Absciss x : and no otherwise. Thus

$\frac{aa}{b+x}$, $\frac{4a-4x}{2a^2-4ax+2x^2}$, are such complex Fractions; but $\frac{a^2+ax}{a-b}$,

$\frac{a^2+2ax+x^2}{3a^2+3ax-ab-bx}$, are not reckoned such: since in the first of them,

the Absciss x does not enter the Denominator; and the other, when reduced to it's most simple Expression, becomes $\frac{a+x}{3a-b}$, where the De-

nominator doth not include x neither. But in the other two Examples, the first, which is already in it's lowest Terms, includes x in one Term of it's Denominator; and the other, after Reduction, becomes

$\frac{2}{a-x}$, which still includes x in one Part of it's Denominator.

25. The Author's first Example is for finding the Area of the external rectangular Hyperbola $ABDC$ (*See the Fig. in the Analysis Art. 12.*) The Manner of Operation in dividing is clearly laid before us, and contains no Difficulty in it. In this Example, $AB=x$ the Absciss, which commences at the given Point A in one of the Asymptotes; $BD=y$ the perpendicular Ordinate; the Distance betwixt

twixt the Center and the Point A is denoted by b ; and a denotes the Side of a Square inscribed betwixt the Hyperbola and it's Asymptotes; and AC is parallel to BD. We spoke of this Case at the End of Sect. 5. of the preceding Treatise: which the Reader may consult. To make the Series, which arises by this Operation, for the Value of the Area ABDC, to converge, it is necessary that b be greater than x . We may also suppose x to be negative, so that, the Point A remaining, the Point B fall betwixt it and the Center: then you have $y = (BD =) \frac{a^2}{b-x}$: and if you actually divide aa by $b-x$, the Quo-

tient is $\frac{a^2}{b} + \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} + \frac{a^2x^3}{b^4}$, &c: therefore the Hyperbolic Area adjacent to AB, in this Case, by the Author's second Rule, is $\frac{a^2x}{b} + \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} + \frac{a^2x^4}{4b^4}$ &c: which differs from the other in no other Respect, but that all the Terms are positive, which in the other are alternately positive and negative: so that the other Series, by supposing x negative, becomes the same with this. Therefore if the Area adjacent to any Part of the Asymptote be required, let such Part be bisected: and calling b the Distance from the Center to the Point of Bisection (which is considered as a constant Quantity) and the variable Quantity x standing for the half of the foresaid Part of the Asymptote; and a being the same as before: the Area adjacent to the Part $2x$ will be $\frac{2a^2x}{b} + \frac{2a^2x^3}{3b^3} + \frac{2a^2x^5}{5b^5} + \frac{2a^2x^7}{7b^7}$ &c. made by adding together the Values of the two Areas lying on each Side of the Ordinate which bisects the Part of the Asymptote $2x$. Which Series will always converge, whatever be the Value of x , since, by the Hypothesis it is less than b . And the Convergency is much quicker than in the preceding Series's.

26. If we attempt the Division of aa by $x+b$, putting x first in the Divisor, it will give $y = \frac{a^2}{x} - \frac{a^2b}{x^2} + \frac{a^2b^2}{x^3} - \frac{a^2b^3}{x^4}$ &c. $= a^2x^{-1} - a^2bx^{-2} + a^2b^2x^{-3} - a^2b^3x^{-4}$: therefore, by the second Rule, the Area is $\frac{a^2x^0}{0} + a^2bx^{-1} - \frac{a^2b^2x^{-2}}{2} + \frac{a^2b^3x^{-3}}{3}$ &c. or $\frac{a^2}{0} + \frac{a^2b}{x} - \frac{a^2b^2}{2x^2} + \frac{a^2b^3}{3x^3}$ &c. which denotes the Area BD α upon the other Side of the Ordinate BD opposite to A, (See Sect. 5. of the preceding Treatise at the beginning) and infinitely extended along the Absciss produced. And this cannot be found, in regard the first Term $\frac{a^2}{0}$ is infinitely great.

27. But

27. But in our Author's second Example, viz. $y = \frac{1}{1+x^2}$, (Art. 13.) which belongs to an Hyperbola of a higher Order, if you divide 1 by $x^2 + 1$, putting x^2 foremost, the Quote is $x^{-2} - x^{-4} + x^{-6} - x^{-8} \&c.$ $= y$: and therefore, by the second Rule, the Area is $-x^{-1} + \frac{x^{-3}}{3} - \frac{x^{-5}}{5} + \frac{x^{-7}}{7} \&c.$ Which is the Value of the Area BD α , extending itself along the Absciss infinitely produced beyond the Ordinate BD, the Signs of all the Terms being changed into their Opposites. Which Value is finite, although the Area be infinitely extended: and the Series converges, if x be greater than 1. Whereas the other Series, viz. $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \&c.$ denotes the Area ABDC adjacent to the Absciss AB: which converges when x is less than 1. These two Expressions of the Area are exactly the same which the fifth Prop. of the Treatise of the Quadrature of Curves would be found to give, by putting $y = \frac{1}{1+x^2}$; or $x^{-2} \times \frac{1}{1+x^2}$, and comparing them with the general Form in that Prop. viz. $y = z^{q-1} \times \frac{e + f z^r}{a + b z^n} \times$ $a + b z^n \&c.$

28. The Division in the Author's last Example (Art. 14.) viz. $\frac{2x^{\frac{1}{2}} - x^{\frac{1}{2}}}{1+x^{\frac{1}{2}} - 3x} = y$, is performed in the following Manner.

$$\begin{array}{r} 1 + x^{\frac{1}{2}} - 3x \quad 2x^{\frac{1}{2}} - x^{\frac{1}{2}} \quad (2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 \&c. = y \\ \underline{2x^{\frac{1}{2}} + 2x - 6x^{\frac{3}{2}}} \end{array}$$

$$\begin{array}{r} - 2x + 5x^{\frac{3}{2}} \\ - 2x - 2x^{\frac{1}{2}} + 6x^2 \end{array}$$

$$\begin{array}{r} + 7x^{\frac{3}{2}} - 6x^2 \\ + 7x^{\frac{1}{2}} + 7x^2 - 21x^{\frac{3}{2}} \end{array}$$

$$\begin{array}{r} - 13x^2 + 21x^{\frac{3}{2}} \\ \&c. \end{array}$$

And

And if you invert the Order of the Terms of the Divisor and Dividend, the Operation is as follows.

$$\begin{array}{r}
 -3x + x^{\frac{1}{2}} + 1 \Big) -x^{\frac{3}{2}} + 2x^{\frac{1}{2}} \left(\frac{x^{\frac{1}{2}}}{3} + \frac{1}{9} - \frac{14}{27x^{\frac{1}{2}}} - \frac{11}{81x} - \frac{53}{243x^{\frac{3}{2}}} \&c. = y \right. \\
 \underline{-x^{\frac{3}{2}} + \frac{1}{3}x + \frac{1}{3}x^{\frac{1}{2}}} \\
 \qquad -\frac{1}{3}x + \frac{5x^{\frac{1}{2}}}{3} \\
 \qquad \underline{-\frac{1}{3}x + \frac{1}{9}x^{\frac{1}{2}} + \frac{1}{9}} \\
 \qquad \qquad + \frac{14x^{\frac{1}{2}}}{9} - \frac{1}{9} \\
 \qquad \qquad \underline{+ \frac{14x^{\frac{1}{2}}}{9} - \frac{14}{27} - \frac{14}{27x^{\frac{1}{2}}}} \\
 \qquad \qquad \qquad + \frac{11}{27} + \frac{14}{27x^{\frac{1}{2}}} \\
 \qquad \qquad \qquad \underline{+ \frac{11}{27} - \frac{11}{81x^{\frac{1}{2}}} - \frac{11}{81x}} \\
 \qquad \qquad \qquad \qquad + \frac{53}{81x^{\frac{3}{2}}} + \frac{11}{81x} \\
 \qquad \qquad \qquad \qquad \qquad \&c.
 \end{array}$$

Therefore the Area in the former Case, which is adjacent to the Absciss, is $\frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{9}x^{\frac{1}{2}} - \frac{11}{3}x^{\frac{1}{2}} \&c.$ But in the other Value of y , there is included the Term $-\frac{11}{81x}$, where x is of one Dimension in the Denominator: which therefore denotes the Ordinate applied to the Asymptote in the Conical Hyperbola: the Quadrature of which Term produces an Area infinitely great. By such a Series you can neither find the Area adjacent to the whole Absciss x ; nor adjacent to the Absciss produced infinitely beyond the Ordinate; but only the Area adjacent to some intermediate Part of the Absciss: which would be the Case likewise here, although there were no such Hyperbolical Term, in regard the Powers of x would be found in the Numerators of some of the Terms; and in the Denominators of other of the Terms of the Value of the Curvilinear Area. For understanding of this consult Sect. 5. of the former Treatise.

From

From these Examples of the Quadrature of Curves, which require Reduction of their Ordinates into infinite Series by Means of Division, you may understand how you are to proceed in any similar Case of Curves, where such Operation is required¹.

29. In the whole Business of Infinite Series, whether arising by Division; or otherwise, the Operation resembles those Operations in *Decimal Arithmetic*, where the Quotient runs out into a Number, which approximates to the Value of the Thing sought continually, but so as that it never terminates. Such is that Operation by which Vulgar Fractions are reduced to Decimal Fractions, when the Fractions cannot be wholly express'd in Decimals; and that whereby the Roots of such Numbers are extracted, which are not true Powers of their Kind. In which Cases you approximate to the Quotient, or Root, in Decimals; and may approach nearer to it than by any given Difference; but can never arrive at the full Value. The chief Difference betwixt these infinite Series's in *Decimal Arithmetic*, and those in the *Literal or Specious*, is, that, in the former, there is only one Scale or Progression of Terms, which decreases in a decuple Proportion; and the Coefficients

¹ Mr. James Gregory, in his *Vera Circuli & Hyperbolæ Quadratura*, published in the Year 1667, gave a converging Series; or at least shewed the Constitution of a converging Series, which proceeds by Pairs of Terms, of that Sort I have mentioned above, Art. 7. by which he shews how to approach to the Area of the Hyperbola, as well as that of the Circle, as near as you please. In April next Year, Lord Viscount Brouncker published an infinite Series (of that kind, which I have called above, Art. 6. the *Newtonian*) for the same Purpose: which Series

he expresses thus $\frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \frac{1}{7 \times 8}$, &c. in *inf.* That same Year Mr. Nicholas

Mercator, an *Holsatian* by Birth, but who had spent a great Part of his Life in *England*, published his *Logarithmotechnia*, in which he shewed how Lord Viscount Brouncker's Series might be found, by reducing a complex Fraction to an infinite Series of simple Terms by Division. Which was but a small Improvement upon what Dr. Wallis had shewn in his *Math. Univ.* Chap. 33. with respect to Division (although the Doctor had not given any explicate Example, in which the Number of Terms in the Quotient was infinite, as some alledge) Mercator having reduced the complex Fraction to an infinite Series of simple Terms, had no more ado after that, but apply the same Person's Method of Quadratures in his *Arith. Inf.* for squaring the several Terms. Upon the Publication of the *Logarithmotechnia*, Dr. Wallis illustrated the Discovery; and gave another infinite Series for the same Purpose, in the *Philosoph. Trans.* for Aug. 1668. Mercator and Wallis's Series's were in effect the same with those mentioned above,

Art. 25. viz. $\frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}$, &c. and $\frac{a^2x}{b} + \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} + \frac{a^2x^4}{4b^4}$, &c.

the former being Mercator's; the latter Dr. Wallis's. Towards the End of the same Year 1668, Mr. James Gregory published his *Exercitationes Geometricæ*: in which he promoted and enlarged Mercator's Discovery, and gave a geometrical Demonstration of it, by means of summing up the Secants of a circular Arch. However it appears from what we have taken out of Sir Isaac Newton's Letter to Mr. Oldenburg, recorded above, and other undoubted Evidences, that Sir Isaac was posses'd of his General Method of reducing Complicate Fractions, Radicals, and affected Equations, into infinite Series's, above two Years before the *Logarithmotechnia* was published.

are all positive Integers below 10; whereas in the latter, which is of a more general and indefinite Nature, the Scales or Progressions may be infinitely varied, so as to decrease in a decuple, or any other Proportion whatsoever; and the numeral Coefficients may be any Numbers, integral or fractional; positive or negative. Thus we may conceive any integral Number as a Series of Powers with their proper Coefficients, *i. e.* the Number 86027 may be conceived as expressed thus; $8 \times 10^4 + 6 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$: and if the Number be continued into Places below the Place of Units, as thus 86027.4638, then it is equivalent to this $8 \times 10^4 + 6 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 7 \times 10^0 + 4 \times 10^{-1} + 6 \times 10^{-2} + 3 \times 10^{-3} + 8 \times 10^{-4}$: the fractional Part of which may be otherwise denoted thus $\frac{4}{10^1} + \frac{6}{10^2}$

$+ \frac{3}{10^3} + \frac{8}{10^4}$. We may likewise suppose 0.1 to be the Root instead of 10, and then the preceding Number 86027.4638 will stand thus $\frac{8}{0.1^4} + \frac{6}{0.1^3} + \frac{0}{0.1^2} + \frac{2}{0.1^1} + \frac{7}{0.1^0} + 4 \times 0.1^1 + 6 \times 0.1^2 + 3 \times 0.1^3 + 8 \times 0.1^4$: the fractional Part of which, *viz.* .4638 may likewise stand thus $\frac{4}{0.1^{-1}} + \frac{6}{0.1^{-2}} + \frac{3}{0.1^{-3}} + \frac{8}{0.1^{-4}}$.

All which Series's would be infinite if the Decimal Fraction were infinite. And after the same Manner a great Variety of other Scales, or Forms of Progression, might be devised, with suitable Roots for different Kinds of Arithmetic: such as the sexagenary used in Astronomy; in which the Root is 60 or it's Reciprocal. Further it may be observed, that the Operations in Decimal Arithmetic, by which the Quotient, or Quantity sought, is discovered, correspond to the Operations relating to infinite Series in specious Arithmetic, in this Respect that every new Step of the Operation, in both Cases, by which the constituent Parts of the Quotient are found, makes as great an Advance towards the Supply of what the Quotient is yet deficient, as the Nature of the Progression will admit.

30. In those Series's which belong to the literal Arithmetic, the Indexes or Exponents of the Powers or Dimensions, may be fractional as well as integral; and are sometimes left indefinite: but they are always supposed to proceed by equal Differences. The numeral Coefficients

cients are likewise sometimes left indefinite. And then these indefinite Exponents and Coefficients are exprest by Letters. Such a general and indefinite converging Series may be represented thus, $Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n}$, &c. where $x^{\pm n}$ denotes the Root of the Scale or Progression; A, B, C, D, &c. any Coefficients, positive or negative; m and n any positive Numbers whatsoever: m may be nothing; and so may one or more of the Quantities, A, B, C, &c.

31. But every Series in specious Arithmetic, when it is to be applied in Practice, must have it's several Species and indefinite Characters, reduced to determinate and known Values, that so the Value of the several Terms, and consequently of the whole, may be collected. Thus suppose we are going to make use of the Series $\frac{2a^2x}{b} + \frac{2a^2x^3}{3b^3} + \frac{2a^2x^5}{5b^5} + \frac{2a^2x^7}{7b^7}$ &c. (See above Art. 25.) for finding the hyperbolical Area ABDC (See the Fig. referred to in that Place) where $AB = 2x : a$, and b , the same as there mentioned: suppose $a = 1 = b$; and $x = 0.1$ or $AB = 0.2$, then you'll have ABDC ($= \frac{2a^2x}{b} + \frac{2a^2x^3}{3b^3} + \frac{2a^2x^5}{5b^5} + \frac{2a^2x^7}{7b^7}$ &c.) $= 0.2 + \frac{0.002}{3} + \frac{0.00002}{5} + \frac{0.0000002}{7}$ &c. or $0.2 + 0.0006666$ &c. $+ 0.000004 + 0.000000028$ &c. $= 0.20067069$ &c.

I now proceed to explain the Reduction of complex Radicals into infinite Series: and the Quadratures depending thereon: of which our Author treats in Art. 15—20.

32. In the Account I have given above of the Rise and Progress of the Doctrine of Quadratures and Series's, I have shewn after what Manner, and by what Steps our Author discovered the Reduction of Radicals into infinite Series. As for the Manner of Operation, it resembles the like Operation in Decimal Arithmetic for extracting the Roots of Numbers which are not true Powers of their Kind: In which Case, the Root or Quotient runs out into an infinite Decimal. Wherefore this being understood; and the Reader being supposed to be acquainted with the Operations of literal Arithmetic; and the Foundation upon which the Extraction of Roots is built, I shall not detain him in explaining this Operation, when applied to the Reduction of complex Surds into infinite Series, further than to represent the Parts of the Operation in the Example Art. 15. a little more fully.

33. Let it be proposed then to extract the Square Root of $aa + xx$; or to find the Value of y in an infinite Series, supposing that $y = \sqrt{aa + xx}$. The Manner of proceeding is as follows.

B b b 2

$aa + xx$

$$\begin{array}{r}
 aa + xx \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9} \text{ &c. in inf.} = \right. \\
 \hline
 2a + \frac{x^2}{2a} \left. \right)^* + xx \\
 \times \frac{x^2}{2a} \quad + xx + \frac{x^4}{4a^2} \\
 \hline
 2a + \frac{x^2}{a} - \frac{x^4}{8a^3} \left. \right)^* - \frac{x^4}{4a^2} \\
 \times - \frac{x^4}{8a^3} \quad - \frac{x^4}{4a^2} - \frac{x^6}{8a^5} + \frac{x^8}{64a^6} \\
 \hline
 2a + \frac{x^2}{a} - \frac{x^4}{4a^3} \left. \right)^* + \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
 \times + \frac{x^6}{16a^5} \quad + \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} \\
 \hline
 2a + \frac{x^2}{a} \left. \right)^* * * * \quad * - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} \\
 \times - \frac{5x^8}{128a^7} \quad - \frac{5x^8}{64a^6} - \frac{5x^{10}}{128a^8} \\
 \hline
 * + \frac{7x^{10}}{128a^8} \text{ &c.} \\
 \text{&c.}
 \end{array}$$

Where, having set down the Quantity $aa + xx$, whose Square Root is to be extracted, you take the Square Root of a^2 the first Term, viz. a , which you place in the Quotient: and then subtracting a^2 from the proposed Quantity, the Remainder $+ x^2$ is divided by $2a$, which gives $+\frac{x^2}{2a}$ to be subjoined to a in the Quotient: You annex it likewise to the Divisor $2a$, making $2a + \frac{x^2}{2a}$, and then multiplying this augmented Divisor by it, place the Product $+ xx + \frac{x^4}{4a^2}$ under the Resolvend or Dividend $+ xx$: subtract it from it, and the Remainder $-\frac{x^4}{4a^2}$ gives a new Resolvend or Dividend. For forming the Divisor to which, take double the Part of the Root or Quotient already found, viz. $2a + \frac{x^2}{a}$: which place for the Divisor as you see, leaving Room for another Term to be subjoined. Divide $-\frac{x^4}{4a^2}$ by $2a$, the first Term of this Divisor; which gives $-\frac{x^4}{8a^3}$ to be subjoined to the preceding Part

Part of the Quotient; and likewise to the Divisor: then multiply the augmented Divisor $2a + \frac{x^2}{a} - \frac{x^4}{8a^3}$ by it, forming thereby the Ablativum $-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$, to be subtracted from the Resolvend $-\frac{x^4}{4a^2}$: and so you proceed, repeating always the same Operation, until you obtain as many Terms in the Root as you desire. Where you may observe that those Terms may be neglected in the Operation, which have no Influence upon the Quotient within the proposed Limits. Thus you'll observe that if the Quotient is to be continued to the 10th Power of x only, those Terms are neglected, which can be of no Use within these Bounds: which I have accordingly marked with Asterisks.

34. The Operation being finished as far as you intend, the Truth of it may be proved by multiplying the Root into itself: for that ought to produce $a^2 + x^2$, the subsequent Terms *in inf.* destroying one another: which you will find to be so here, as far as the Product can be carried by the defective Root found, *viz.* to as many Terms as have been found in the Quotient. And after the same Manner you are to proceed in other Examples.

35. And thus having found $y = (\sqrt{a^2 + x^2}) = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9} \&c.$ the Area of the Curve whose Property is defined by $y^2 = a^2 + x^2$, or $y = \sqrt{a^2 + x^2}$, is (by the second Rule) $ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} + \frac{7x^{11}}{2816a^9} \&c.$ Now if CD (See the Fig. Art. 15.) be a rectangular Hyperbola, A it's Center, AC (=a) the half of the determinate Axis; and consequently of the Latus Rectum, AB (=x) any Part of the second Axis, and BD (=y) it's Ordinate: from the Property of the Hyperbola, you have $\overline{BD} + AC \times \overline{BD} - AC = \overline{AB}q$, or $y + a \times y - a = x^2$, that is $y^2 - a^2 = x^2$ or $y^2 = a^2 + x^2$: which is the Property required. Therefore $ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} \&c. =$ ABDC the Area required: for the Area begins at A.C. since it vanishes, when $x = AB$ vanishes. And to make this Series converge, x should be less than a : and the less x is, a remaining, the quicker is the Convergency. The same hyperbolical Area would be found by Prop. 5. of the Treatise of Quadratures, to be $aa + xx +$

$$x \frac{x}{a^2} - \frac{4x^3}{3a^4} + \frac{8x^5}{5a^6} - \frac{64x^7}{35a^8} + \frac{128x^9}{63a^{10}} \&c. \text{ or } y \times x - \frac{x^3}{3a^2} + \frac{4x^5}{15a^4} - \frac{8x^7}{35a^6} + \frac{64x^9}{315a^8} \&c.$$

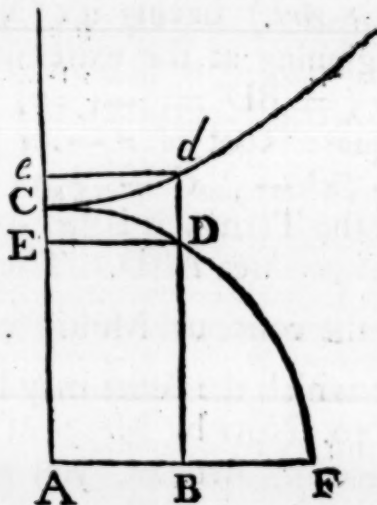
36. If

36. If it were $y = \sqrt{aa - x^2}$, the Equation would be to the Circle (See the Fig. Art. 16.) ABCD: where the Radius $AC = a$, $AB = x$, $BD = y$: for $BD^2 = AC^2 - AB^2$, that is $y^2 = a^2 - x^2$. Now if the Square Root be extracted out of $a^2 - x^2$, after the Manner of the preceding Example, it will be found to be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9} \&c.$ which differs from the Series for the Ordinate to the second Axis of the Hyperbola, as above, in no other Respect but that all the Terms after the first are negative; which in the Hyperbola are positive and negative alternately. Whence the circular Area ABDC = $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} - \frac{7x^{11}}{2816a^9} \&c.$ differing from the Series for the Hyperbolical Area in the same Respect those for the Ordinates differ. The double of this Series will give the Area of the Circular Zone adjacent to the Diameter, and having it's Right Sine or Altitude equal to x , viz. $2ax - \frac{x^3}{3a} - \frac{x^5}{20a^3} - \frac{x^7}{56a^5} - \frac{5x^9}{576a^7} \&c.$ which by substituting R for a the Radius, and B for x the Sine, is $2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5} \&c.$ the same with that which was sent by Mr. Collins in one of his Letters to Mr. James Gregory, as a Specimen of Sir Isaac's Method of Series. See Com. Epist. N^o. 18. The same circular Area ABDC would be found by Prop. 5. of the Quadratures to be $y \times x + \frac{1}{3} \frac{x^3}{a^2} + \frac{1}{15} \frac{x^5}{a^4} + \frac{8}{35} \frac{x^7}{a^6} + \frac{64}{315} \frac{x^9}{a^8} \&c.$ the same with that for the hyperbolical Area found by the same Prop. except only in the Signs. The former Series $ax \pm \frac{x^3}{6a} - \frac{x^5}{40a^3} \pm \frac{x^7}{112a^5} \&c.$ and this one $y \times x + \frac{1}{3} \frac{x^3}{a^2} + \frac{1}{15} \frac{x^5}{a^4} \&c.$ will be found the same, if for y you insert it's Value, viz. $a \pm \frac{x^2}{2a} - \frac{x^4}{8a^3} \pm \frac{x^6}{16a^5} \&c.$ and then multiply it by $x \mp \frac{x^3}{3a^2} + \frac{4x^5}{15a^4} \mp \frac{8x^7}{35a^6} \&c.$

37. When a few of the initial Terms of a Series are once found, it is of considerable Use to know the Law of the Progression by which the Series proceeds, that thereby it may be continued at Pleasure. This Law is found either by observing the Steps of the general Theorem from whence the Series is deduced; or by considering the Relations of those Terms which are already found in the beginning of the Series. Thus

Thus the Series $ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} + \frac{7x^{11}}{2816a^9} \&c.$ for the hyperbolical; or circular Area: found as above by the Method of infinite Series, may be continued at Pleasure, by observing that the numeral Coefficients $\frac{1}{6} \cdot \frac{1}{40} \cdot \frac{1}{112} \cdot \frac{5}{1152} \cdot \&c.$ are formed by multiplying continually by $\frac{1 \times 1}{2 \times 3} \cdot \frac{1 \times 3}{4 \times 5} \cdot \frac{3 \times 5}{6 \times 7} \cdot \frac{5 \times 7}{8 \times 9} \cdot \frac{7 \times 9}{10 \times 11} \cdot \&c.$ due Regard being had to the Signs. So likewise the other Series for the same Areas, derived from Prop. 5. of the Quadratures, *viz.* $y \times x + \frac{x^3}{3a^2} + \frac{4x^5}{15a^4} + \frac{8x^7}{35a^6} + \frac{64x^9}{315a^8} \&c.$ may be continued, by observing that the Numeral Coefficients $\frac{1}{3} \cdot \frac{4}{15} \cdot \frac{8}{35} \cdot \frac{64}{315} \cdot \&c.$ are formed by the continual Multiplication of the following Fractions $\frac{1}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{11} \cdot \&c.$

38. Hence if CD, Cd, be a Circle, and rectangular Hyberbola, described with the Center A, and Transverse Axis AC; and AB = x the Base or Absciss, as before, upon the conjugate Axis; BD or Bd = y the Ordinate: and the Ordinates DE, de, to the Transverse Axis be drawn: then



from $ABDC = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} - \frac{7x^{11}}{2816a^9} \&c.$ subtracting the Rectangle $ABDE = AB \times BD = x \times a - \frac{x^4}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9} =$
 $ax - \frac{x^3}{2a} - \frac{x^5}{8a^3} - \frac{x^7}{16a^5} - \frac{5x^9}{128a^7} - \frac{7x^{11}}{256a^9} \&c.$ there remains DCE
 $= \frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5} + \frac{5x^9}{144a^7} + \frac{35x^{11}}{1408a^9} \&c.$ which serves for finding the Area or half Segment DCE from the Right Sine AB or DE given. And after the like manner, from AB or $de = x$ given in the Hyperbola, you'll find the Hyperbolical half Segment dCe equal to $\frac{x^3}{3a} - \frac{x^5}{10a^3} + \frac{3x^7}{56a^5} - \frac{5x^9}{144a^7} + \frac{35x^{11}}{1408a^9} \&c.$ which two Series may be continued at Pleasure by forming the numeral Coefficients of each Term by the continual Multiplication of so many Terms of the following Series, as the Place of the Term denotes, *viz.* $\frac{1 \times 1}{1 \times 3} \cdot \frac{1 \times 3}{2 \times 5} \cdot \frac{3 \times 5}{4 \times 7} \cdot \frac{5 \times 7}{6 \times 9} \cdot \frac{7 \times 9}{8 \times 11} \cdot \&c.$ By adding the two Series $\frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5} \&c.$ and $\frac{x^3}{3a} - \frac{x^5}{10a^3} +$

+ $\frac{3x^7}{56a^5}$ &c. together, you have $\frac{2x^3}{3a} + \frac{3x^7}{28a^5} + \frac{35x^{11}}{704a^9}$ &c. = DCE + dCe: and by subtracting the latter from the former you have $\frac{x^3}{5a^3} + \frac{5x^9}{72a^7}$ &c. = DCE - dCe: the first is the Sum of the two half Segments; the last their Difference: both which converge very fast; so doth the Series $\frac{x^3}{3a} + \frac{x^7}{56a^5} + \frac{7x^{11}}{1408a^9}$ &c. = the intermediate Area DCd the Difference betwixt ABDC and ABdC: and also $2ax - \frac{x^3}{20a^3} - \frac{5x^9}{576a^7}$ &c. equal to their Sum.

39. Our Author's next Example (Art. 17.) is $y = \sqrt{x - xx}$ or $\sqrt{1 \times x - xx}$: which is to the Circle ABD (See the Fig. belonging to that Art.) having it's Diameter equal to 1; and the Absciss AB = x, beginning at the extremity of the Diameter; in which Case you have $yy (= BD^2 = 1 - x \times x) = x - x^2$; or $y = \sqrt{x - x^2}$. Now if the Square Root of $x - xx$ be extracted, you'll find it to be $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}} - \frac{7}{256}x^{\frac{11}{2}}$ &c. = y: and therefore by squaring all the Terms, it is $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{8}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}} - \frac{7}{16768}x^{\frac{13}{2}}$ &c. = Area ABD. The numeral Coefficients of which are produced by the continual Multiplication of $\frac{2}{3} \cdot \frac{-1 \times 3}{2 \times 5} \cdot \frac{1 \times 5}{4 \times 7} \cdot \frac{3 \times 7}{6 \times 9} \cdot \frac{5 \times 9}{8 \times 11} \cdot \frac{7 \times 11}{10 \times 13}$ &c. by which the Series may be continued as far as you please.

40. Now by Means of the foregoing Series's contained in the four preceding Articles, you may easily find any hyperbolical, or circular Area, in any Case proposed. For the Satisfaction of the young Geometrician, I shall lay before him the Computation of the circular Area from the last Series $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{8}x^{\frac{7}{2}}$ &c. = ABD. Here the Diameter is 1: let us suppose AB (= x) to be any Part of the Diameter, the less Part it is taken, the Series will converge the more quickly: but because it is most convenient for the computing the Area of the whole Circle, let it be $x = \frac{1}{4}$ that is the versed Sine of 60 Degrees. By what was shewn in the last Article, the first Term being $\frac{2}{3}x^{\frac{3}{2}}$, the second Term will be found by multiplying the first by $-\frac{1 \times 3}{2 \times 5}x = -$

$\frac{1}{10}x$; the third, by multiplying the second, viz. $-\frac{1}{5}x^{\frac{5}{2}}$ by $\frac{1 \times 5}{4 \times 7}x$ or $-\frac{1}{28}x$: and so on. But in such numeral Calculations, you'll do best to reduce all to Decimals: and set down each Term, as you see in the Margin, after it is found. Now since $x = \frac{1}{4}$, it appears that $\frac{2}{3}x^{\frac{3}{2}} = \frac{2}{3} \times \frac{1}{8} = \frac{1}{12} = 0.083333$ &c. Which I set down in the Margin with it's

it's proper Sign. This multiplied by $-\frac{3}{10}x = -\frac{3}{40}$, makes -0.00625 . Which being placed duly under the former with it's proper Sign, you multiply it by $\frac{5}{8}x = \frac{5}{112}$, and it produces $-0.0002790178571429 \text{ \&c.}$ And so on for the rest; which will be such as you see expressed in the Margin. Where the first Line only is positive; and all the rest negative. Wherefore subtract the Sum of all the Negatives, viz. $0.0065602271702860 \text{ \&c.}$ from the first Term $0.0833333333333333 \text{ \&c.}$ and there remains $0.0767731061630473 \text{ \&c.}$ = the half Segment ABD. The Radius CD being drawn, the Triangle $BCD = \frac{BC \times CD}{2} = \frac{\frac{1}{4} \times \sqrt{\frac{3}{8}}}{2} = \frac{\sqrt{3}}{32} = 0.0541265877365275 \text{ \&c.}$ which being added to ABD, makes $0.13089969389957 \text{ \&c.}$ = Sector ACD a sixth Part of the whole Circle: So that multiplying it by 6, you have $0.78539816339744 \text{ \&c.}$ for the Area of a Circle whose Diameter is 1: true to 14 Places of Decimals. And therefore the Diameter of any Circle being given, multiply this Number by the Square of that Diameter, and you have it's Area.

Moreover the Area being found divide by $\frac{1}{2}$ Radius, and you'll have the Circumference: that is in the present Case, divide the preceding Number by $\frac{1}{4}$ or multiply it by 4, and it gives $3.14159265358979 \text{ \&c.}$ for the Circumference of a Circle whose Diameter is 1.

41. After the like Manner you may find the Area of the Hyperbola from the Equation $y = \sqrt{x+xx}$: and both that of the Circle and that of the Hyperbola, by Means of the other Series's which have been mentioned. And it appears with what Ease and Expedition this is performed: So that for Ease and Expedition it vastly surpasses any former Method for approaching the Area of the Circle, made use of before our Author's Time: not excepting Mr. James Gregory's, in his *Vera Circuli & Hyperbolæ Quadratura* (which was the best) as will easily appear by considering the Operation made use of for this Purpose in Prop. 29; and comparing it with the Method here shewn: which yet is capable of great Improvement.

$$\begin{array}{r}
 + 0.0833333333333333 \\
 - 0.00625 \\
 - 0.0002790178571429 \\
 - 0.0000271267361111 \\
 - 0.0000034679066051 \\
 - 0.0000005135169396 \\
 - 0.0000000834465027 \\
 - 0.0000000144628917 \\
 - 0.0000000026285354 \\
 - 0.0000000004954581 \\
 - 0.0000000000961296 \\
 - 0.0000000000190948 \\
 - 0.0000000000038676 \\
 - 0.0000000000007963 \\
 - 0.0000000000001663 \\
 - 0.0000000000000352 \\
 - 0.0000000000000075 \\
 - 0.0000000000000016 \\
 - 0.0000000000000004 \\
 - 0.0000000000000001 \\
 \hline
 - 0.0065602271702860 \\
 + 0.0767731061630473 \\
 + 0.0541265877365275 \\
 \hline
 0.13089969389957
 \end{array}$$

42. Our Author's next Example (Art. 18.) is $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}} = y$: the Quadrature of which gives the Length of an Elliptical Arch. For suppose the Figure ACF, belonging to Art. 38 above, to be an Ellipse, having AF the half Transverse and AC the half conjugate Axis; and DB and DE Ordinates to them respectively: call $AB = x$, $BD = v$, $AF = t$, $AC = c$, and the Arch $CD = z$: then by what was demonstrated in the preceding Treatise (See Notes upon Art. 5. of the Treatise of Quadratures) it is $z = \sqrt{x^2 + v^2}$: that is (by inserting $\frac{t^2 x^2 x^2}{c^4 - c^2 x^2}$ for v^2 ; which is equal to it from the Property of the Curve, and reducing)

$$z = x \frac{\sqrt{c^4 + tt - cc x^2}}{\sqrt{c^4 - cc x^2}}.$$

Wherefore a Curve which has x for it's Absciss and $\frac{\sqrt{c^4 + tt - cc x^2}}{\sqrt{c^4 - cc x^2}}$ for it's Ordinate, flows with a Fluxion or Velocity analogous to that with which the Elliptical Arch CD flows: consequently the Area generated by that Ordinate is analogous to the Elliptical Arch CD: So that the same Algebraical Expression shall denote both agreeably to Art. 37. of this Analysis; and what was shewn with Respect to the Rectification of Curves in the preceding Treatise. Therefore if it be

$y = \frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}$, the Quadrature of it will give the Elliptical Arch, supposing $c = 1$, $tt - cc = a$, and $cc = b$.

43. For this Purpose you may extract the Square Roots of $1 + ax^2$ and $1 - bx^2$, and divide the former by the latter: but, as our Author observes, you may abridge the Operation by multiplying the Numerator and Denominator by $\sqrt{1 - bx^2}$: by which it becomes

$\frac{\sqrt{1 + ax^2 - abx^4}}{1 - bx^2}$: and so you need only extract the Square Root of the Numerator, and divide by the Denominator, in order to have the Value of y . The Operation follows.

$$\frac{1 + ax^2}{1 - bx^2}$$

$$\begin{array}{r}
 1 + \frac{a}{b}x^2 - abx^4 \left(1 + \frac{1}{2}\frac{ax^2}{b} - \frac{1}{8}\frac{a^2x^4}{b^2} + \frac{1}{16}\frac{a^3x^6}{b^3} \text{ & c.} \right) = \sqrt{1 + \frac{a}{b}x^2 - abx^4} \\
 \hline
 2 + \frac{1}{2}\frac{ax^2}{b} \left(1 + \frac{ax^2}{b} - \frac{1}{2}\frac{a^2x^4}{b^2} \right) \\
 \hline
 + \frac{ax^2}{b} + \frac{1}{2}\frac{a^2x^4}{b^2} + \frac{1}{4}\frac{a^3x^6}{b^3} \\
 \hline
 2 + \frac{ax^2}{b} - \frac{1}{2}\frac{a^2x^4}{b^2} \left(1 + \frac{ax^2}{b} - \frac{1}{2}\frac{a^2x^4}{b^2} \right) \\
 \hline
 - \frac{1}{2}\frac{a^2x^4}{b^2} - \frac{1}{2}\frac{a^3x^6}{b^3} * \\
 \hline
 - \frac{1}{2}\frac{a^2x^4}{b^2} - \frac{1}{2}\frac{a^3x^6}{b^3} * \\
 \hline
 * + \frac{1}{8}\frac{a^3x^6}{b^3} * \\
 \hline
 + \frac{1}{8}\frac{a^3x^6}{b^3} * \\
 \hline
 - \frac{1}{8}\frac{a^3x^6}{b^3} * \\
 \hline
 \text{& c.}
 \end{array}$$

Which Square Root, now reduced into a Series of simple Terms, may be easily divided, by $1 - bx^2$ after the Manner shewn above: by doing of which, you'll obtain this Quotient

$$1 + \frac{1}{2}\frac{ax^2}{b} - \frac{1}{8}\frac{a^2x^4}{b^2} + \frac{1}{16}\frac{a^3x^6}{b^3} \text{ & c.}$$

And therefore, by Rule the second, the Terms being squared, will give

$$x + \frac{1}{6}\frac{ax^3}{b} - \frac{1}{20}\frac{a^2x^5}{b^2} + \frac{1}{112}\frac{a^3x^7}{b^3} \text{ & c.} = \text{Elliptical Arch CD}$$

for the initial Limit of the Arch is at the Point C: since the Series becomes nothing when $x = AB$ vanishes: that is when CD vanishes.

44. I should next proceed to shew how the Cube Root; and other higher Roots, of complex Radicals may be extracted, and thereby reduced to infinite Series's: which Operations are performed after the same Manner as they are in specious Arithmetic, by considering the Genesis of the several Powers of a Binomial; and performing the Extraction or Resolution in every Case, as the Composition of the Power, whose Root is extracted, may require; and continuing the Operation, after the Manner you do in common Arithmetic, in Decimal Fractions: as may be easily understood by the Method of resolving quadratic Radicals shewn above. But I think it needless to detain the Reader with

C c c 2

this:

this : since our Author's famous *Binomial Theorem*, mentioned already in the History of his Discoveries, serves more readily and generally, not only for extracting Roots properly so called ; but for expressing by a Series of simple Terms, any Root, Power, or Dimension whatsoever of a compound Quantity, having any Index positive ; negative ; integral or fractional. Which, without all Peradventure, is one of the most general and useful Theorems, that ever were discovered.

45. Sir *Isaac* exhibits the Theorem thus: $\overline{P+PQ}^{\frac{m}{n}} + P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$ Where $P + PQ$ represents any compound Quantity, whose Root or Dimension whatsoever is to be investigated: P , the first Term of that Quantity; Q , the rest of the Terms divided by the first; and consequently, PQ , all the rest of the Terms: $\frac{m}{n}$ the numeral Index of the Root, Power, or Dimension of $P + PQ$; whether that numeral Index be integral, or fractional; positive, or negative. Moreover the Capitals $A, B, C, D, \&c.$ stand for the first, second, third, fourth, &c. Terms:

i. e. $A = P^{\frac{m}{n}}$; $B = \frac{m}{n}AQ$; $C = \frac{m-n}{2n}BQ$; &c. The same Theo-

rem may stand otherwise thus: $\overline{P+PQ}^{\frac{m}{n}} = (P^{\frac{m}{n}} \times \overline{1+Q}^{\frac{m}{n}}) = P^{\frac{m}{n}} \times 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-n}{2n}Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}Q^4 + \&c.$ Which is the same with the former, only differently exprest: sometimes the one Expression being more commodious; sometimes the other.

46. This celebrated Theorem was discovered at first by the Method of Trial and Induction, as may be seen in the foregoing Account of our Author's Discoveries, Art. 14. Since that Time several Demonstrations of it have been given; or alledged to be given. The Method of Fluxions is most frequently made use of for this Purpose: viz.

this Prop. that the Fluxion of $ax^{\frac{m}{n}}$ is $\frac{m}{n}ax^{\frac{m}{n}-1}\dot{x}$. Which therefore ought to be proved independently of the Binomial Theorem: otherwise the Proof goes in a Circle. But we shall supersede the Demonstration of it, in this Place, reserving it till we have explained the

¹ See his Letter to Mr. *Oldenburg* June 13, 1676. Com. Epist. N^o 48.

² See the Account of our Author's Discoveries above, in Art. 14.

Resolution of affected Equations: and only shew, by some Examples, in what Manner it is applied.

47. Ex. 1. Let it be required to express $\sqrt{cc - xx}$ in an infinite Series; or to find the Value of y in such a Series, supposing $y = \sqrt{cc - xx}$.

By comparing this Example with the General Theorem $P + PQ^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ \&c.$ You have $P = cc$, $Q = -\frac{x^2}{c^2}$, $m = 1$, $n = 2$: therefore also $P^{\frac{m}{n}} = cc^{\frac{1}{2}} = c = A$, $\frac{m}{n}AQ = \frac{1}{2}c \times -\frac{x^2}{c^2} = -\frac{x^2}{2c} = B$, $\frac{m-n}{2n}BQ = \frac{-1}{4} \times -\frac{x^2}{2c} \times -\frac{x^2}{c^2} = -\frac{x^4}{8c^3} = C$, $\frac{m-2n}{3n}CQ = \frac{-3}{6} \times -\frac{x^4}{8c^3} \times -\frac{x^2}{c^2} = -\frac{x^6}{16c^5} = D$, $\frac{m-3n}{4n}DQ = \frac{-5}{8} \times -\frac{x^6}{16c^5} \times -\frac{x^2}{c^2} = -\frac{5x^8}{128c^7} = E$, $\&c.$ Therefore it is $y = (\sqrt{cc - xx}) = c - \frac{x^2}{2c} - \frac{x^4}{8c^3} - \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} \&c.$ the same as above, Art. 36, being the Sine of an Arch in a Circle, whose Radius is c , the Cosine being x .

48. Ex. 2. Let it be $y = \sqrt[3]{ax^2 + x^3}$, y is sought. Here $P = ax^2$,

$Q = \frac{x^3}{a}$, $m = 1$, $n = 3$: therefore also it is $P^{\frac{m}{n}} = a^{\frac{1}{3}}x^{\frac{2}{3}} = A$, $\frac{m}{n}AQ = \frac{1}{3}a^{\frac{1}{3}}x^{\frac{2}{3}} \times \frac{x^3}{a} = \frac{x^{\frac{5}{3}}}{3a^{\frac{2}{3}}} = B$, $\frac{m-n}{2n}BQ = \frac{-2}{6} \times \frac{x^{\frac{5}{3}}}{3a^{\frac{2}{3}}} \times \frac{x^3}{a} = -\frac{x^{\frac{8}{3}}}{9a^{\frac{5}{3}}} = C$, $\frac{m-2n}{3n}CQ = \frac{-5}{9} \times -\frac{x^{\frac{8}{3}}}{9a^{\frac{5}{3}}} \times \frac{x^3}{a} = \frac{5x^{\frac{11}{3}}}{81a^{\frac{8}{3}}} = D$, $\frac{m-3n}{4n}DQ = \frac{-8}{12} \times \frac{5x^{\frac{11}{3}}}{81a^{\frac{8}{3}}} \times \frac{x^3}{a} = -\frac{10x^{\frac{14}{3}}}{243a^{\frac{11}{3}}} = E$, $\&c.$ Wherefore it is $y = (ax^2 + x^3)^{\frac{1}{3}} =$

$a^{\frac{1}{3}}x^{\frac{2}{3}} + \frac{x^{\frac{5}{3}}}{3a^{\frac{2}{3}}} - \frac{x^{\frac{8}{3}}}{9a^{\frac{5}{3}}} + \frac{5x^{\frac{11}{3}}}{81a^{\frac{8}{3}}} - \frac{10x^{\frac{14}{3}}}{243a^{\frac{11}{3}}} \&c.$ In which Example you might have inverted the Order of the Terms of the Binomial, so as to make it stand thus $x^3 + ax^2$: and in that Case it would have been

$P = x^3$, and $Q = \frac{a}{x}$: and therefore $P^{\frac{m}{n}} = x = A$, $\frac{m}{n}AQ = \frac{a}{3} = B$, $\frac{m-n}{2n}BQ = -\frac{a^2}{9x} = C$, $\&c.$ and consequently $y = x^3 + ax^2$.

+

$+\frac{x}{3}-\frac{a^2}{9x}+\frac{5a^3}{81x^2}-\frac{110a^4}{243x^3}$ &c. The former is to be chosen when x is small; the latter, when it is great in Comparison of a .

49. Ex. 3. Let it be $y = \sqrt[m]{c^5 + c^4x - x^5}$: and suppose we take the second Form of the Theorem, viz. $\overline{P+PQ^{\frac{m}{n}}} = (\overline{P^n \times 1 + Q^n})^{\frac{1}{n}}$ $\Rightarrow$ $\overline{P^n \times 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-n}{2n}Q^2}$ &c. Here we have $P = c^5$,

$$Q = \frac{c^4x - x^5}{c^5}, m=1, n=5; P^n = c: \text{ and therefore it is } y = c \times$$

$$1 + \frac{1}{5} \times \frac{c^4x - x^5}{c^5} + \frac{1}{5} \times \frac{1}{5} \times \frac{(c^4x - x^5)^2}{c^5} + \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{(c^4x - x^5)^3}{c^5} + \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{(c^4x - x^5)^4}{c^5} \text{ &c.}$$

Here it may be observed, when you make use of this Form of the General Theorem, the Series may be easily produced, by considering that the Factors both in the Numerators and Denominators of the Numeral Coefficients go on in Arithmetical Progressions. This Series by involving the Quantity $\frac{c^4x - x^5}{c^5}$ to the several Powers, and multiplying

$$\text{by } c, \text{ is } y = c + \frac{c^4x - x^5}{5c^4} - \frac{2c^8x^2 - 4c^4x^6 + 2cx^{10}}{25c^9} + \frac{6c^{12}x^3 - 18c^8x^7 + 18c^4x^{11} - 6x^{15}}{125c^{14}} \text{ &c.}$$

In this Example you might have inverted the Order of the Terms, making it stand thus $\sqrt[5]{-x^5 + c^4x + c^5}$: and then it would have been

$P = -x^5, Q = \frac{c^4x + c^5}{-x^5}; P^n = -x, \text{ &c. and so you would have had } y = -x + \frac{c^4x + c^5}{5x^4} + \frac{2c^8x^2 + 4c^4x^6 + 2c^{10}}{25x^9} \text{ &c. The former Series converges when } x \text{ is small; this when } x \text{ is great.}$

50. Ex. 4. Suppose $y = \sqrt[3]{a^2 + x^2}$. Here you have $P = a^2$, $Q = \frac{x^2}{a^2}, m=4, n=3; P^n = a^{\frac{8}{3}}, \text{ &c. Wherefore, by the second Form of the Theorem it is } y = (\overline{aa + xx})^{\frac{1}{3}} =) a^{\frac{2}{3}} \times$

$$1 + \frac{1}{3} \frac{x^2}{a^2} + \frac{4 \times 1 \times x^4}{3 \times 6 \times a^4} + \frac{4 \times 1 \times -2 \times x^6}{3 \times 6 \times 9 \times a^6} + \frac{4 \times 1 \times -2 \times -5 \times x^8}{3 \times 6 \times 9 \times 12 \times a^8} + \frac{4 \times 1 \times -2 \times -5 \times -8 \times x^{10}}{3 \times 6 \times 9 \times 12 \times 15 \times a^{10}} \text{ &c.}$$

$$= a^{\frac{2}{3}} + \frac{4a^{\frac{2}{3}}x^2}{3} + \frac{2x^4}{9a^{\frac{4}{3}}} - \frac{4x^6}{81a^{\frac{10}{3}}} + \frac{5x^8}{243a^{\frac{16}{3}}} \text{ &c.}$$

You might also have made x^2 the first Member under the Vinculum of the Root: and the Series would come out the same, only placing a for x , and x for a in the several Terms.

51. Ex. 5. Let $y = \frac{\sqrt{c^4 - cc + rr x^2}}{\sqrt{c^4 - cc x^2}}$ be proposed: and it is required to find the Value of y in an infinite Series, and then divide the one by the other; or you might first multiply both by the Denominator, by way of Preparation, as was shewn above, Art. 43: or lastly you may bring the Quantity $\sqrt{c^4 - cc x^2}$ from the Denominator to the Numerator, by changing the Sign of the Index $\frac{1}{2}$, into it's opposite, so that the Expression stand thus $\sqrt{c^4 - cc + rr x^2}^{\frac{1}{2}} \times \sqrt{c^4 - cc x^2}^{-\frac{1}{2}}$, and then reduce the two Factors into infinite Series by the Binomial Theorem; and afterwards multiply these two Series into one another. Suppose we take this last Method: then for Brevity sake, substitute a for cc and b for $-cc + rr$, so that it be $y = \sqrt{a^2 + bx^2}^{\frac{1}{2}} \times \sqrt{a^2 - ax^2}^{-\frac{1}{2}}$. By comparing the first of these with the general Theorem you have $P = aa$, $Q = \frac{bx^2}{a}$, $m = 1$, $n = 2$: by comparing the second, $P = aa$, $Q = \frac{-x^2}{a}$, $m = -1$, $n = 2$. And according to the first Form of that

Theorem, you'll have for the first Series $P^{\frac{m}{n}} = a = A$, $\frac{m}{n}AQ = \frac{bx^2}{2a} = B$, $\frac{m-n}{2n}BQ = -\frac{b^2x^4}{8a^3} = C$, $\frac{m-2n}{3n}CQ = \frac{b^3x^6}{16a^5} = D$, $\frac{m-3n}{4n}DQ = -\frac{5b^4x^8}{128a^7} \&c.$ and according to the second Form of the Theorem,

you'll have for the second Series $P^{\frac{m}{n}} = aa^{-\frac{1}{2}} = \frac{1}{a}$: and therefore $\sqrt{a^2 - ax^2}^{-\frac{1}{2}} = \frac{1}{a} \times 1 + \frac{x^2}{2a} + \frac{-1 \times -3x^4}{2 \times 4 a^2} - \frac{-1 \times -3 \times -5x^6}{2 \times 4 \times 6 a^3} + \frac{-1 \times -3 \times -5 \times -7x^8}{2 \times 4 \times 6 \times 8 a^4} \&c.$ So that $\sqrt{a^2 + bx^2}^{\frac{1}{2}} = a + \frac{bx^2}{2a} - \frac{b^2x^4}{8a^3} + \frac{b^3x^6}{16a^5} - \frac{5b^4x^8}{128a^7} \&c.$ and $\sqrt{a^2 - ax^2}^{-\frac{1}{2}} = \frac{1}{a} + \frac{x^2}{2a^2} + \frac{3x^4}{8a^3} + \frac{5x^6}{16a^4} + \frac{35x^8}{128a^5} \&c.$ Multiply them as follows,

$$\begin{array}{r}
 a + \frac{bx^2}{2a} - \frac{b^2x^4}{8a^3} + \frac{b^3x^6}{16a^5} - \frac{5b^4x^8}{128a^7} \&c. \\
 \frac{1}{a} + \frac{x^2}{2a^2} + \frac{3x^4}{8a^3} + \frac{5x^6}{16a^4} + \frac{35x^8}{128a^5} \&c. \\
 \hline
 1 + \frac{bx^2}{2a^2} - \frac{b^2x^4}{8a^4} + \frac{b^3x^6}{16a^6} - \frac{5b^4x^8}{128a^8} \&c. \\
 + \frac{x^2}{2a} + \frac{bx^4}{4a^3} - \frac{b^2x^6}{16a^5} + \frac{b^3x^8}{32a^7} \&c. \\
 + \frac{3x^4}{8a^2} + \frac{3bx^6}{16a^4} - \frac{3b^2x^8}{64a^6} \&c. \\
 + \frac{5x^6}{16a^3} + \frac{5bx^8}{32a^5} \&c. \\
 + \frac{35x^8}{128a^4} \&c.
 \end{array}$$

and

and you obtain this Product $1 + \frac{a+b}{2a^2}x^2 + \frac{3a^2+2ab-b^2}{8a^4}x^4 + \frac{5a^3+3a^2b-ab^2+b^3}{16a^6}x^6 + \frac{35a^4+20a^3b-6a^2b^2+4ab^3-5b^4}{128a^8}x^8$ &c. Which therefore is the Value of

$$\frac{\sqrt{a^2+bx^2}}{\sqrt{a^2-ax^2}}.$$

52. Hence it follows, by our Author's second Rule, that a Curve having such an Ordinate, has it's Area equal to $x + \frac{a+b}{6a^2}x^3 + \frac{3a^2+2ab-b^2}{40a^4}x^5 + \frac{5a^3+3a^2b-ab^2+b^3}{112a^6}x^7 + \frac{35a^4+20a^3b-6a^2b^2+4ab^3-5b^4}{1152a^8}x^9$, &c. Which by replacing c^2 for a , and $rr - cc$ for b , makes $x + \frac{r^2}{6c^4}x^3 + \frac{4r^2c^2-r^4}{40c^8}x^5 + \frac{8c^4r^2-4c^2r^4+r^6}{112c^{12}}x^7 + \frac{64c^6r^2-48c^4r^4+24c^2r^6-5r^8}{1152c^{16}}x^9$ &c. Which expresses the Length of an Elliptical Arch, beginning at the Extremity of either of the Axes whose Halves are r and c ; and having x for it's right Sine (as appears from Art. 42.) And this is the Series which Mr. James Gregory sent to Mr. Collins. See Com. Epist. N^o 20. And the same Series will serve for the Length of an Arch of the Hyperbola, if all the particular Parts of each Term be made positive; and the whole of the 3d, 5th, 7th, &c. Terms be made negative: that is, if it stand thus $x + \frac{r^2}{6c^4}x^3 - \frac{4r^2c^2-r^4}{40c^8}x^5 + \frac{8c^4r^2+4c^2r^4+r^6}{112c^{12}}x^7 - \frac{64c^6r^2+48c^4r^4+24c^2r^6+5r^8}{1152c^{16}}x^9$ &c.

And if you suppose p to be inserted for $\frac{c^2}{r}$, that is half the Parameter of the Axis $2r$; or, which is the same, insert pr for c^2 every where: the Series for the Arch of the Ellipse will appear in this Form:

$$x + \frac{1}{6p^2}x^3 + \frac{1}{10rp^3}x^5 + \frac{1}{14r^2p^4}x^7 + \frac{1}{18r^3p^5}x^9 \text{ \&c.}$$

$$- \frac{1}{40p^4} - \frac{1}{28rp^5} - \frac{1}{24r^2p^6}$$

$$+ \frac{1}{112p^6} + \frac{1}{48rp^7}$$

$$- \frac{1}{1152p^8}$$

Which is the Series given by Sir Isaac Newton for the Elliptical Arch, in his first Letter to Mr. Oldenburg, Com. Epist. N^o 48. And this Series may be continued by observing that the Numeral Coefficients of the uppermost Line, viz. $\frac{1}{6} \cdot \frac{1}{10} \cdot \frac{1}{14}$ &c. are Fractions whose Denominators go on by the common Difference 4, having Unity their common Numerator: and as to the Numeral Coefficients of all the inferior Terms

Terms in every Column, that they arise by multiplying the numeral Coefficient of the uppermost Term continually, by the Terms of this Progression $\frac{1}{2}n-1 \cdot \frac{1}{4}n-3 \cdot \frac{1}{6}n-5 \cdot \frac{1}{8}n-7 \cdot \&c.$ the Letter n denoting the Index of the Power of p , in the uppermost Term of the Column.

53. If in the preceding *Gregorian* Series, you suppose $r=c$; or in the *Newtonian* $r=p$; in which Case the Ellipse becomes a Circle: the Series becomes $x + \frac{x^3}{6r^2} + \frac{3x^5}{40r^4} + \frac{5x^7}{112r^6} + \frac{35x^9}{1152r^8} \&c.$ Which therefore denotes the Arch of a Circle whose Right Sine is x , the Radius being r . Which Series may be produced to any Length by multiplying continually by the Numbers $\frac{1 \times 1}{2 \times 3} \cdot \frac{3 \times 3}{4 \times 5} \cdot \frac{5 \times 5}{6 \times 7} \cdot \frac{7 \times 7}{8 \times 9} \&c.$ in forming the numeral Coefficients of the Terms. Which is the Series given by Sir *Iaac Newton* for this Purpose at Art. 38 of this Analysis, supposing $r=1$.

54. 'Tis evident from what has been said, that this Binomial Theorem will equally serve for reducing complicate Fractions to infinite Series; and also for involving any compound Quantity to any perfect Power: as it doth for reducing complicate Radicals.

Ex. 6. Let it be required to find the Value of y in simple Terms, supposing $y = \sqrt[5]{a+x}$. Here $P=a$, $Q=\frac{x}{a}$, $m=5$, $n=1$: conse-

quently $P^m = a^5 = A$, $\frac{m}{n}AQ = 5a^4x = B$, $\frac{m-n}{2n}BQ = 10a^3x^2 = C$, $\frac{m-2n}{3n}CQ = 10a^2x^3 = D$, $\frac{m-3n}{4n}DQ = 5ax^4 = E$, $\frac{m-4n}{5n}EQ = x^5 = F$, $\frac{m-5n}{6n}EQ = 0 = G$: and so the Series terminates, since $G (=0)$ enters into the Composition of every subsequent Term. So that you have $y = \sqrt[5]{a+x} = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$. And it's evident that in this Case of perfect Powers, the Series must always terminate; and that when the Number of Terms is one more than the Exponent of the Power.

55. And in order to reduce complicate Fractions, you have no more to do but bring the Denominator into the Numerator, with the Sign of the Index changed into it's opposite: and then apply the Theorem.

Ex. 7. Suppose $y = \frac{1}{a-x}$: I express it thus $y = \sqrt[1]{a-x}^{-1}$. By comparing which with the second Form of the general Theorem, you have $P=a$, $Q=-\frac{x}{a}$, $m=-1$, $n=1$. Therefore it is $y =$

D d d

($\frac{1}{a}$)

$$\left(\frac{1}{a} \times 1 - \frac{1}{1} \times \frac{x}{a} - \frac{x^2}{1 \times 2} \times \frac{1}{a^2} - \frac{1 \times 2 \times 3}{1 \times 2 \times 3} \times \frac{x^3}{a^3} - \frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3 \times 4} \times \frac{x^4}{a^4} \&c.\right)$$

$$=) \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \frac{x^3}{a^4} + \frac{x^4}{a^5} \&c.$$

Ex. 8. Let it be $y = \frac{a-x}{a^2+x^2}^{-3}$: that is $y = \overline{a-x} \times \overline{a^2+x^2}^{-3}$.

By comparing $\overline{a^2+x^2}^{-3}$ with the second Form of the Theorem, you have $P = a^2$, $Q = \frac{x^2}{a^2}$, $m = -3$, $n = 1$. Therefore by a due Substitution

$$\text{it is } \overline{a^2+x^2}^{-3} = (a^{-6} \times 1 - \frac{3x^2}{a^2} - \frac{3 \times 4 \times x^4}{1 \times 2 \times a^4} - \frac{3 \times 4 \times 5 \times x^6}{1 \times 2 \times 3 \times a^6} - \frac{3 \times 4 \times 5 \times 6 \times x^8}{1 \times 2 \times 3 \times 4 \times a^8} \&c.)$$

$$=) \frac{1}{a^6} - \frac{3x^2}{a^8} + \frac{6x^4}{a^{10}} - \frac{10x^6}{a^{12}} + \frac{15x^8}{a^{14}} \&c. \text{ Which being multiplied}$$

by $a-x$, produces $\frac{1}{a^5} - \frac{x}{a^6} - \frac{3x^2}{a^7} + \frac{3x^3}{a^8} + \frac{6x^4}{a^9} - \frac{6x^5}{a^{10}} \&c. = y$. And so the Area's of the Curves, having their Ordinates thus reduced into infinite Series, are easily found by the Author's second Rule.

56. And thus far I have shewn how the Binomial Theorem is applied, in a variety of Examples. But it may be proper in this Place to observe that, notwithstanding the Series for the Area; or Fluent, found by the Help of this Theorem, always becomes infinite, when the Index of the Power, Root, or Dimension of the Quantity expressing the Ordinate of the Curve; or Fluxion, is not a positive Integer: and consequently exhibits the Area or Fluent, only by an Approximation: yet it often happens that the Curve is *geometrically quadrable*; and so the Area or Fluent may be fully assigned. Which is done by Means of Propositions 5 and 6 of the Treatise of Quadratures. Which see, with our Notes upon them.

57. The same Thing may be found likewise by Means of the *Transmutation* of Curves, founded upon Prop. 9. of the Treatise of Quadratures. Thus let $y = dx^k \times \overline{e + fz^n}^l$ define the Relation betwixt the Absciss z and Ordinate y of a Curve: and suppose that the Index l is such, that the Quantity $\overline{e + fz^n}^l$, being reduced, runs out into an infinite Series: and you would know if the Curve be *geometrically*

quadrable. Put $e + fz^n = x$: hence $z = \sqrt[n]{\frac{x-e}{f}}$, $z^k = \sqrt[n]{\frac{x-e}{f}}^k$, and $z^n = \frac{x-e}{f}$. Therefore, by taking the Fluxions of $e + fz^n = x$, you have $nfz^{n-1} \dot{z} = \dot{x}$, that is $\dot{x} : \dot{z} :: nfz^{n-1} : 1 :: nfz^n : z$. Now, suppose v to be the Ordinate of a Curve, whose Absciss is x , and it's Area

Area equal to the Area of the Curve proposed, having z for it's Abfcifs, and $y = dz^k \times \overline{e + fz^n}^l$ for it's Ordinate: then, by that Prop. $y (= dz^k \times \overline{e + fz^n}^l) : v :: (\dot{x} : \dot{z} ::) n f z^n : z$. Whence $v = \frac{d}{nf} z^{k-n+1} \times \overline{e + fz^n}^l$

$\overline{e + fz^n}^l =$ (by substituting the preceding Values) $\frac{dx^l}{nf} \times \left[\frac{-e+x}{f} \right]^{\frac{k+1}{n}-1}$.

Therefore, if $\frac{k+1}{n} - 1$ be a positive Integer, the Quantity $\left[\frac{-e+x}{f} \right]^{\frac{k+1}{n}-1}$ is a perfect Power, which may be exprest by a finite Number of simple Terms (See Art. 54.) Let it be done, and each Term multiplied by $\frac{dx^l}{nf}$, and then squared by Rule 1st: So you shall have the Area required: unless perhaps the Values of k , l and n be such, as to give some Term with the Root x in it's Denominator: which would make the Area infinite (See Art. 28.) Therefore, when, in the Curve proposed $y = dz^k \times \overline{e + fz^n}^l$, the Expression $\frac{k+1}{n}$, is an Integer, the Curve is quadrable, with the Limitation mentioned.

58. Moreover, since the Ordinate or Expression $dz^k \times \overline{e + fz^n}^l$ may be converted into this other Form $dz^{k+ln} \times \overline{f + ez^{-n}}^l$ (by Article 28. Treatise of Quad.) if you put $x = f + ez^{-n}$; and proceed as in the former Article, you'll find that the Curve whose Abfcifs is z , and Ordinate $y = dz^k \times \overline{e + fz^n}^l$, is equal to another Curve, having the

Abfcifs x , and Ordinate $-\frac{dx^l}{ne} \times \left[-\frac{f}{e} + \frac{x}{e} \right]^{\frac{k+ln+1}{-n}-1}$. Therefore

if $\frac{k+ln+1}{-n}$ be a positive Integer, the Curve proposed will be quadrable, with the like Limitation as to an *hyperbolical* Term, as in the preceding Article: and the Area found the same Way. But these two Areas, found by this and the preceding Article, lye upon different Sides of the Ordinate (See Sect. 5 of the preceding Treatise.) And if they can both be exactly found, the Curve is of that Sort, which we call doubly Quadrable.

59. The 9th and 10th Corollaries of Prop. 9. of the Quadratures serve for the same Purpose. Thus, if in Cor. 9. you put $v = 0 = \theta$, $\pi = 1 = \tau = \sigma$, consequently $\theta = \lambda - 1$; and suppose it be $a = -\frac{e}{f}$, $b = \frac{1}{f}$, $g = 0$ &c. then in the Curve whose Abfcifs is z and Ordinate

D d d 2

Ordinate $\eta f z^{\omega+\eta-1} \times \overline{e+fz^\eta}^{\lambda-1}$ passes into another which is equal to it, whose Ordinate is $x^{\lambda-1} \times \overline{-\frac{e}{f} + \frac{x}{f}}^\omega$. Or (by multiplying both by the given Quantity $\frac{d}{\eta f}$) the Curve whose Ordinate is $dz^{\omega+\eta-1} \times \overline{e+fz^\eta}^{\lambda-1}$, passes into another equal Curve, whose Ordinate is $\frac{dx}{f} x^{\lambda-1} \times \overline{-\frac{e}{f} + \frac{x}{f}}^\omega$. Compare the Ordinate $dz^k \times \overline{e+fz^\eta}^l$, in Art. 57, with the former of these two, and you have $\eta = n$; $\omega\eta + \eta - 1 = k$, or $\omega = \frac{k-n+1}{n}$; $\lambda - 1 = l$: the rest the same. Therefore, by a due Substitution, the given Curve, whose Ordinate is $dz^k \times \overline{e+fz^\eta}^l$, passes into another of an equal Area, whose Ordinate is $\frac{dx}{\eta f} \times \overline{-\frac{e}{f} + \frac{x}{f}}^{\frac{k+1}{n}-1}$. Whence, if $\frac{k+1}{n}$ be a positive Integer, the Curve is geometrically quadrable, as before. And the same way, it may be shewn that the Curve proposed is likewise quadrable, if $\frac{k+ln+1}{-n}$ be a positive Integer.

But I treated more fully of this Affair; and in a different Manner, in the Explication of Prop. 5. Quad. Which the Reader may consult.

S E C T. III.

Explication of the Numeral Resolution of affected Equations, contain'd in Art. 21——29.

60. **I**N order to understand our Author's Method of extracting the Roots of affected Numeral Equations, with the Abbreviations made use of therein, it will be proper to explain some Things which respect Decimal Fractions, and the Operations about them.

61. 1°. In Multiplication of Decimals, the Number of Places of Decimal Fractions in the Product, must be made equal to the Sum of the Places of Decimal Fractions in both Factors. And in Division, the Number of such Places in the Quotient, must be made equal to the Excess of those in the Dividend above those in the Divisor. And

if

if the Number of Figures in the Product be not so great as the for-
said sum, you must prefix Decimal Cyphers upon the left hand of
the Product, until the Number of Places altogether be as great as
that Sum. And in Division if there be a Defect in the Number of
Places in the Quotient, it must be supplied the same way. But if
the Places of Decimal Fractions in the Dividend, be fewer than those
in the Divisor, you must add Decimal Cyphers upon the right hand
of the Dividend, untill the Places of Decimal Fractions be as many as
in the Divisor. Which Operations are commonly known; and the
Reasons of them understood.

2°. When any two Decimal Fractions are multiplied, whereof
one, or both, have any Number of Cyphers prefixt, the Number of
Cyphers prefixt in the Product, must at least be as great as the Sum
of those prefixt in both Factors. For, by the last, the Number of
Decimal Places in the Product, is equal to the Sum of those in the
two Factors: but the Number of significant Figures in the Product,
towards the right hand, can never be greater than the Number of
significant Figures in both Factors (which is evident from the Nature
of Multiplication): therefore the Number of Cyphers prefixt in the
Product is at least equal to those prefixt in both Factors. And often
times they are more. Thus $0.0024 \times 0.03 = 0.000072$.

3°. Hence it follows that if r denote any Decimal Fraction with
 n Number of Cyphers prefixt, then the Number of Cyphers prefixt
in its several Powers, viz. r^2, r^3, r^4, r^5 , &c. must be at least $2n, 3n,$
 $4n, 5n$, &c. viz. so many times n as the Exponent of the Power
denotes.

4°. If two Numbers are proposed to be multiplied, whereof the one
is either a Decimal Fraction or mixt Number; and the other a Decimal
Fraction, with Cyphers prefixt; and the Product is only required to a
certain Number of Decimal Places, you may cut off from the right
hand of the first as many Decimal Places, as will make those which
remain, together with the prefixed Cyphers in the other, equal to the
Number of Decimal Places required in the Product: and neglect those
which are cut off, in the Multiplication. Thus if it were required to
find the Product of 123,4567 and 0.000089, only to 6 Places of De-
cimals, you may neglect 67 in the Multiplication: if the Product were
required only to 5 Places, you may neglect 567: so as to multiply
0.000089 by 123.4 only: and if the Product be required only to 4
Places of Decimals, you need only multiply 0.000089 by the integral
Part 123. But if the Number of Decimal Places, to which the Pro-
duct is required, be less than the Number of Cyphers prefixt in the
Decimal

Decimal Fraction, which is one of the Factors, *e.g.* 0.000089, then, you may neglect, not only all the Decimal Places of the other Factor 123,4567; but as many of the integral Places as the Decimal Places required in the Product, want of the Cyphers prefixt in the first Factor. Thus if the Product be required only to 3, 2, 1, Places of Decimals, you need only multiply 0.000089 by 12, by 1, by 0, respectively.

For since 0.000089 is less than 0.0001; and all after 123,45 less than 0.01: which two multiplied by one another produce only 0.000001 by Obs. 1. Therefore it is evident that 0.000089 multiplied by what follows 123,45, must produce something less than 0.000001: and therefore cannot affect the Product within the 6th Place of Decimals. And the reasoning is the same in every other Case, by considering the Decimal Progression of the Places.

62. These things with respect to Decimals being premised, our Author's Method of extracting the Roots of affected Numeral Equations, with the Abbreviations he makes use of therein, will be the better understood.

Let $ar + br^2 + cr^3 + dr^4 \&c. = A$, represent any Numeral Equation, in which $a, b, c, d, \&c.$ and A are given Numbers, positive or negative, and r the Root whose Value is sought. Then by dividing by a , and transposing, you have $r = \frac{A}{a} - \frac{br^2}{a} - \frac{cr^3}{a} - \frac{dr^4}{a} \&c.$ Now when one of the Values of r is small, so as it may be denoted by a Decimal Fraction with Cyphers prefixed, you may easily find the Limits within which the supplementary Parts $\frac{br^2}{a}, \frac{cr^3}{a}, \frac{dr^4}{a}, \&c.$ are contained: and consequently the Limits within which the Difference betwixt the true Value of r , and the Approximate Value of it, *viz.* $r = \frac{A}{a}$, must be contained. For since $\frac{br^2}{a}$ is an Expression, in which r^2 is the Square of r a Decimal Fraction having Cyphers prefixt; and therefore r^2 twice as many at least (by Obs. 3.) you may always find how many Cyphers will be prefixt in the Expression $\frac{br^2}{a}$, from the Nature of Decimal Arithmetic. And after the same Manner, how many such Cyphers will be prefixt in the Values of $\frac{cr^3}{a}, \frac{dr^4}{a}, \&c.$ For if $\frac{b}{a}$ be less than 1 (in which Case it's Value in Decimal Arithmetick is entirely below the Place of Unites) there must be as many Cyphers,

at

at least, prefixt in $\frac{br^2}{a}$, as in r^2 (by Obs. 2.) But if $\frac{b}{a}$ contain any Number of Integral Places, then the Cyphers prefixt in $\frac{br^2}{a}$ will, at least, be equal to the Excess of the Cyphers prefixt in r^2 , above the Integral Places in $\frac{b}{a}$. Thus, if r had 4 Decimal Cyphers prefixt in it's Value; and suppose $a=11, 16196$; $b=6,3$; then, since $\frac{b}{a} (= \frac{6.3}{11.16196})$ is less than an Integer, and r^2 has 8 decimal Cyphers prefixt, $\frac{br^2}{a}$ must have as many prefixt at least. After the same Manner $\frac{br^3}{a}$ would have 12 at least: and universally $\frac{br^n}{a}$, as many, at least, as is denoted by $4n$. But if, the Value of r remaining, it were $a=0.243$, $b=6.3$; then since $\frac{b}{a} (= \frac{6.3}{0.243})$ is more than 1, and contains two integral Places, therefore $\frac{br^2}{a}$ may have only 6 ($= 8 - 2$) Decimal Cyphers prefixt: although in some Cases, when the Figures are small, it may have more. And universally $\frac{br^n}{a}$ has $4n - 2$ at least. And so in other Cases. Which Things easily follow from what was shewn in the preceding Observations. And therefore you may proceed thus: 1°. Consider how many integral Places would be contained in $\frac{a}{b}$, (which may be easily found by Inspection) cut off as many of the Cyphers prefixt to the significant Figures in r^2 , the remaining Cyphers are the fewest that can be prefixt to the significant Figures in $\frac{br^2}{a}$. 2°. If the Value of $\frac{b}{a}$ have no integral Places, but only decimal Places, then consider how many Decimal Cyphers would be prefixt to the significant Figures in the Value of $\frac{b}{a}$ (which you may easily discover by Inspection) and then the decimal Cyphers prefixt to the significant Figures in the Value of $\frac{br^2}{a}$ will be equal to the Sum of these prefixt in $\frac{b}{a}$ and r^2 , both together.

63. From what has been said it appears, that, if the Number of Decimal Cyphers prefixt in the Value of r , be known, in the preceding Equation $r = \frac{A}{a} - \frac{br^2}{a} - \frac{cr^3}{a}$ &c. in which $\frac{A}{a}$ is an approximate Value

Value of r , you may easily know to what Number of Decimal Places the approximate Value of r , viz. $\frac{A}{a}$, may be continued, before any of the supplementary Parts $\frac{br^2}{a}$, $\frac{cr^3}{a}$, &c. can affect the Root. Thus suppose you had this Equation $0.000541550536 + 11.16204748r + 6.2838r^2 + r^3 = 0$: Which is the third supplementary Equation in our Author's Example, supposing it to be fully drawn out: that is $r = -\frac{0.000541550536}{11.16204748} - \frac{6.2838r^2}{11.16204748} - \frac{r^3}{11.16204748}$, and it is known that the Value of r is a decimal Fraction having 4 Cyphers prefixt. Then because the Term $\frac{r^3}{11.16204748}$ is equal to $\frac{1}{11.162} \times r^3$, of which the first Part is a Decimal Fraction with one Cypher prefixt; and the other a Decimal Fraction with 12 Cyphers prefixt, therefore the first significant Figure of it's Value will fall upon the 14th Place of Decimals (by Art. 62.) so that the Value of the Root r cannot be affected by it within the Limits of 13 Places. After the same Manner the Term $\frac{6.2838r^2}{11.16204748}$ will not affect the Root within the Limits of 8 Places of Decimals. Whence you conclude that the Value of r , deduced from supposing $r = -\frac{0.000541550536}{11.16204748}$ will be true to 8 Decimal Places, that is the Division of 0.000541550536 by 11.16204748 may be carried forward to so many Decimal Places: all which will be true unless perhaps the last Figure, which may sometimes be increased or diminished by 1, by the Addition or Subtraction of the supplementary Part. And since r^2 will, at least, have twice as many Decimal Cyphers prefixed in it's Value as r has, therefore, generally speaking, the Number of Decimal Places or Figures, to which the Quotient denoted by $\frac{A}{a}$, may be carried, will be twice as many as those to which the Root hath been found already, by the former Approximation: agreeably to what Sir *Isaac* remarks Art. 22. So that in this Method of resolving affected Numeral Equations, every new Operation, as it were, doubles the Number of Decimal Places formerly found in the Root: even supposing all the Terms of the supplementary Equations to be rejected, wherein the Supplements p , q , r , &c. rise above the 1st Power; unless perhaps when b exceeds a : in which Case the supplementary Part $\frac{b}{a}r^2$ may affect the last Figures.

64. By the Consideration of these Things you may abridge the Work considerably towards the latter End, by neglecting such Terms or Parts of Terms, as you foresee will be of no Use within the Limits to which the Root is proposed to be carried: as you see done by the Author in the Prosecution of the Root y , in the Example proposed by him. However it may be observed that the Quotient $0.00004853 \text{ \&c.}$ errs in the last figure 3, by his omitting the Term q^3 altogether, in transforming the Equation of q : for it ought to be $0.00004851 \text{ \&c.}$ only: as you may find by Art. 63.

65. *Schol.* Dr. Halley's Rational Formula for extracting the Roots of Equations is easily deduced from what hath been shewn above. Let $fx + gx^2 + bx^3 + ix^4 \text{ \&c.} = K$ be any Equation; x the unknown Quantity, the other Letters standing for known Quantities: and let l be a Number as near a Value of x as may be; found by Trial, or any other Way: substitute $l + r$ instead of x in the given Equation $fx + gx^2 + bx^3 \text{ \&c.} = K$: and let the transformed Equation be $ar + br^2 + cr^3 + dr^4 \text{ \&c.} = A$: where r is the Supplement to l ; or the Difference betwixt l the approximate Value of x , and it's true Value: which therefore is supposed to be small. Wherefore $ar = A$ or $r = \frac{A}{a}$ nearly; or $ar + br^2 = A$ more nearly, that is $r = \frac{A}{a + br}$ fere, Substitute for r in the second Member of this last Equation, it's approximate Value deduced from the preceding $r = \frac{A}{a}$, and you'll have

$$r = \left(\frac{A}{a + \frac{bA}{a}} \right) = \frac{aA}{a^2 + bA} \text{ quam proxime.} \text{ Wherefore you have } x = l +$$

$\frac{aA}{a^2 + bA}$ nearly. Which is Dr. Halley's Rational Formula: published in Phil. Transactions for May 1694, N^o 210.

66. And if any one will be at Pains to consider the Business of the Resolution of Equations, there may be many Methods of approximating to their Roots discovered. Thus, retaining the same supplementary Equation $ar + br^2 + cr^3 + dr^4 \text{ \&c.} = A$, you have, as before $r = \frac{A}{a}$ nearly: or $r = \frac{A}{a} - \frac{br^2}{a}$ more nearly: insert in the latter Member of this last Equation, instead of r^2 it's approximate Value $\frac{A^2}{a^2}$, deduced from the preceding, and you'll obtain $r = \frac{A}{a} - \frac{b}{a} \times \frac{A^2}{a^2}$ for a nearer Approximation to the supplementary Root r . Which approximate Value of r being found, call it e : and in the Equation of r , insert

$$E \text{ e e} \qquad e + s$$

$e + s$ for r : from which you may find an approximate Value of s : and so on for the other subsequent Supplements.

67. Our Author tells us that when there is any Doubt whether the Value of p in the first supplementary Equation, approaches near enough the Truth, by supposing $10p - 1 = 0$, you may assume the three last Terms; thus, $6p^2 + 10p - 1 = 0$: and by the Solution of the Quadratic, find the first Figure of the Value of p . Now by the Solution of the Quadratic $6p^2 + 10p - 1 = 0$, you have $p = \frac{-5 \pm \sqrt{31}}{6}$: of

which two-fold Value you take $\frac{-5 + \sqrt{31}}{6} = + 0.09$ *quamproxime*; which is the least Root, and that which approaches nearest to the Value of p , by putting $10p - 1 = 0$. The Reason of which, shall be shewn Art. 71. And thus you find 0.09 a nearer Approach to the Value of p than 0.1 is: and it gives you one Decimal Place more in the Quotient. Moreover, as he tells us, it is convenient to ascertain the second, or third Figure, of the Quotient, the same way, *when in the last resulting Equation, the Square of the Coefficient of the Penult Term is not ten times greater than the Product of the last Term multiplied into the Coefficient of the Antepenult Term.*

68. To understand the Reason of this: suppose $ar + br^2 = A$, to represent an Equation resulting, when all the Terms except the three last of any supplementary Equation, are rejected: and $ar = A$, that which arises by rejecting all save the two last Terms: then we may easily find the Difference betwixt the two corresponding Values of r arising from these two Equations. From the last, we have $r = \frac{A}{a}$; from the first, $r = \frac{-a \pm \sqrt{a^2 + 4bA}}{2b}$. That we may compare these two Values of r , suppose $a^2 = z \times bA$ (where z represents the Number which multiplying bA , makes the Product equal to a^2) that is $b = \frac{a^2}{zA}$: insert this Value of b instead of it, in the Equation $r = \frac{-a \pm \sqrt{a^2 + 4bA}}{2b}$, and you'll have $r = \frac{-\frac{1}{2}z \pm \sqrt{\frac{1}{4}z^2 + z \times \frac{A}{a}}}{\frac{1}{2}z}$: of which two-fold Value, $-\frac{1}{2}z + \sqrt{\frac{1}{4}z^2 + z \times \frac{A}{a}}$ is to be taken, which is the least of the two Roots, and that which approaches nearest to the Root of the Equation $ar = A$, or $r = \frac{A}{a}$, betwixt which and it the Difference is sought. This Difference then is the same with the Difference betwixt $\frac{A}{a}$ and $\frac{A}{a} \times \sqrt{\frac{1}{4}z^2 + z} - \frac{1}{2}z$: which will vary according

according to the Value of z . If it be $z = 10$, then $\sqrt{\frac{1}{4}z^2 + z} - \frac{1}{2}z = 0.916 \text{ \&c.}$ and therefore the Difference betwixt the two Roots, in this Case is $1 - 0.916 \times \frac{A}{a} = 0.083 \text{ \&c.} \times \frac{A}{a}$, which is less than $\frac{1}{10}$ of $\frac{A}{a}$: but if $z = 9$, then $\sqrt{\frac{1}{4}z^2 + z} - \frac{1}{2}z = 0.908 \text{ \&c.}$ so that the Difference of the Roots would be nearly $0.091 \text{ \&c.} \times \frac{A}{a}$: which is nigh to $\frac{1}{10}$ of $\frac{A}{a}$. And if z be 8 or any thing less, the Difference will be more than $\frac{1}{10}$ of $\frac{A}{a}$ (the Difference still increasing as z diminishes)

Whence it appears, why, if z be less than 10, that is, if the Square of the Coefficient of the Penult Term be not ten times greater than the Product of the last Term and Coefficient of the Antepenult Term, it is proper to ascertain the first Figure of the Value of r , by assuming the three last Terms of the supplementary Equation; and taking the least Root of the Quadratic: because in this Case the approximate Root of r found by assuming only the two last Terms of the Equation, may differ from the true Root, by a Difference which may alter the first Figure of that approximate Value of r .

69. Nay, as Sir Isaac observes, if in the whole Course of the Operation, we assume the three last Terms of each supplementary Equation, instead of the two last only: and extract the least of the two Roots out of the Quadratic, we shall obtain about twice as many Figures in the Quotient the one way, as the other, at each Operation: as may be easily collected from what hath been said in Art. 63, 64: by which it appears that each new Operation; or supplementary Equation, will, as it were, triple the Number of Decimal Places already found in the Quotient. And it may be observed that by assuming the three last Terms of the supplementary Equation, as in the preceding Article, you have Dr. Halley's irrational Formula, published in the Philosophical Transactions in the Place mentioned already*. And as to * Art. 65. Mr. Raphson's Method of Approximation in the extracting the Roots of Equations, published in his *Analysis Equationum universalis*, it is, in Effect, the very same with our Author's Method here laid down; which proceeds by assuming only the two last Terms of the supplementary Equations, at each new Operation.

70. This Method of extracting the Roots of affected Numeral Equations, may be successfully applied to all kinds of them, how high soever: whether the Terms be all compleat; or any Number of them wanting: and therefore also serves for extracting the Roots of pure

Equations, that is the Roots of Numbers, especially the higher Ones. Moreover, after you have found the Quotient to any Degree of Exactness you please, the Quotient so found together with the Root of the last resulting Equation, must be equal to the Root of the Equation at first proposed: as appears from the Process itself. And therefore, having continued the Work to any desired Length, in approximating to the Root, you may prove or ascertain the Truth of the Operation here, as well as in any other arithmetical Operation, which proceeds by Resolution. For if you substitute for y , in the Original Equation, the Quotient already found + the Supplement q , r , or s , &c. the transformed Equation ought to be the same with the Supplementary Equation of q , r , or s , &c. Thus *e. g.* having found, in the Author's Example, $y = (2 + p) = 2.1 + q$: transform the original Equation $y^3 - 2y - 5 = 0$, by substituting $2.1 + q$ for y , and you have $0.061 + 11.23q + 6.3q^2 + q^3 = 0$, the same with the corresponding supplementary Equation: which ascertains the Truth of the Operation.

71. There seems to be no other Difficulty that can occur to the young Geometrician in this Matter: unless it be this. Every supplementary Equation is of the same Order or Rank with the original Equation proposed; and therefore the Roots or Values of the Supplement may be as many as the Dimensions of that Equation: how then doth it appear which Root of the supplementary Equation ought to be taken: and that the Value of the Supplement which arises by assuming the two last Terms of that Equation; or the least of the two Values of the Supplement, which arise by assuming the three last Terms, is an approximate Value of that which you want, to make up the true Root?

In order to clear up this Difficulty, let $ay + by^2 + cy^3$ &c. $= A$, represent any Numeral Equation, y being the unknown Quantity, and the other Letters standing for known Numbers; positive, or negative: and let the Roots of the Equation be called f , g , h , &c. positive, or negative. Then let n be a Number which approaches near to one of the Roots as f : which is found by Trial; or otherwise: and let it be $n + p = y$, where p is the Supplement. Substitute $n + p$ for y in the Equation $ay + by^2 + cy^3$ &c. $= A$: and thence arises a new Equation, *viz.*

$$\left. \begin{array}{l} + an + ap \\ + bn^2 + 2bnp + bp^2 \\ + cn^3 + 3cn^2p + 3cnp^2 + cp^3 \\ \text{\&c.} \quad \text{\&c.} \quad \text{\&c.} \quad \text{\&c.} \end{array} \right\} = 0, \text{ which you may call the Equation of } p;$$

and the other the Equation of y . In this Equation of p , the Roots or Values of p are just as many as the Roots or Values of y in the Equation

Values

of y , since $p = y - n$, that is $f - n$, $g - n$, $b - n$, &c. but of these Values of p , one, at least, is very small in Comparison of n , which is denoted by $f - n$, since n was supposed to differ but little from f ; and this therefore is the Value of p , which is the Supplement sought. Wherefore in the Equation of p , the Terms which include the Powers of p , that is all before the two last Terms, will be small in Comparison of these two last Terms, if p be small enough in Comparison of n : which is the Case when the Root $f - n$ is sought: therefore these Terms may be neglected, when you are seeking a near Value

of the Root $f - n$: so that you'll have $\left\{ \begin{array}{l} -A \\ +an \\ +bn^2 \\ +cn^3 \\ \text{\&c.} \end{array} \right\} + \left\{ \begin{array}{l} ap \\ +2bnp \\ +3cn^2p \\ \text{\&c.} \end{array} \right\} = 0 \text{ nearly,}$

or $p = \frac{A - an - bn^2 - cn^3 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}}$ a near Value of the Supplement p , that is of $f - n$. And you may easily discover how near this Approach will come to the true Value of the Supplement p or $f - n$: which (*ceteris paribus*) will be different according to the different Proportion that n bears to p or $f - n$: suppose $p (= f - n) = \frac{n}{10}$, then in

the preceding Equation of p , each Term, proceeding from the left to the right, abstracting from the numeral Coefficients, may be considered as ten times greater than the Term immediately succeeding it: in which Case the two last Terms of the Equation upon the left hand, are, abstracting from the Coefficients, more than $\frac{99}{100}$ of the whole; the three last Terms more than $\frac{999}{1000}$ of the whole, &c. And therefore the Value of the Supplement p deduced from the two last Terms of the Equation put equal to nothing, will not differ, generally speaking, from the true Value, but by a few hundred Parts. This will further appear if you compare the Value of p deduced from the abridg'd Equation, *viz.* $p = \frac{A - an - bn^2 - cn^3 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}}$, with it's Value derived from the

entire Equation, which will be $p = \frac{A - an - bn^2 - cn^3 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}} - \frac{bp^2 + 3cnp^2 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}} - \frac{cp^3 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}}$; that is (by inserting $\frac{n}{10}$ for p) $p = \frac{A - an - bn^2 - cn^3 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}} - \frac{b \times 0.01n^2 + 3c \times 0.01n^3 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}} - \frac{c \times 0.001n^3 \text{\&c.}}{a + 2bn + 3cn^2 \text{\&c.}}$: from whence it ap-

pears that the approximate Value differs from the full Value, generally speaking, but by a few hundred Parts; since the Value of the Places in the Numerator of the second Term of the Value of p , are but hundred Parts of the Value of the Places in the Numerator of the first Term, the Denominators being the same. And after the same Manner.

Manner it will appear, that, if $p = \frac{n}{100}$, the approximate Value will differ from the true Value of p , only by some thousand Parts, &c. but whatever be the Ratio of p to n , something depends upon the Coefficients: yet the exact Limits, in every particular Case, may easily be found, by applying what has been said in Art. 62, 63: by doing which, you may always discover to what Number of Places the Quotient may be continued, which arises from the Division of $A - an - bn^2 - cn^3$ &c. by $a + 2bn + 3cn^2$ &c. so as to give these Places all true; when you know the Ratio below which the Ratio of n to p doth not descend.

But if the Square of the Coefficient of the penult Term be not ten times greater than the Product of the last Term multiplied into the Coefficient of the antepenult Term, when n is not much more than 100 (of which Case I spoke Art. 68) then by assuming the three last Terms of the Equation of p , and putting them equal to nothing, the Solution of the Quadratic, gives a two-fold Value of p : now of these you take the least Value for the Supplement, because you still proceed upon the Supposition that the Value of p which you want is a small Value, viz. $f - n$, which may be supposed to be the least Value of p .

72. *Scholium.* There are some Things concerning our Author's Method of resolving numeral Equations, which it may be proper here to remark.

1°. He supposes that the approximate Value of y , which is assumed in the beginning, differs not a tenth part of itself from the true Value. Such an approximate Value may be found by the Methods for determining the Limits of the Roots of Equations, found in Algebraical Writers; or, perhaps by repeated Trials.

2°. This Method of resolving numeral Equations, equally extends to the discovering of all the possible Roots of any Equation, whether they be positive or negative, commensurable or incommensurable: for whether the approximate Value of y in the proposed Equation be taken positive or negative; if it be such an approximation to any real Root as has been mentioned, the Operation goes on after the same Manner. However, if any one chuse, he may transform any proposed Equation having one or more of it's Roots negative, into another, in which all the Roots shall be positive, by increasing the Roots by a Number greater than any of the negative Roots, by *Transformation*: and so he shall only have positive Roots to extract. Moreover this Transformation is of Use for determining the Limits of the Roots: for when you have found the two immediate Limits betwixt which

any

any Root lyes, whereof one is greater and the other less than the Root sought, in the Manner taught by our Author in his *Arith. Univ. Cap. de Limitibus Equationum*, you may easily discover whether any of these Limits doth or doth not differ by $\frac{1}{10}$ of itself from the true Root: for if you substitute the Limits in Place of the unknown Quantity in the Equation: and then substitute the least Limit increased by $\frac{1}{10}$ of itself; or the greatest Limit diminished by $\frac{1}{10}$ of itself, in Place of the unknown Quantity; and observe the Signs with which the Aggregates of all the Terms of the Equation in every Case are affected, you'll thereby determine what you want: for if the Aggregate arising from the Substitution of the least Limit, be affected with a different Sign from that with which the Aggregate is affected by substituting that Limit increased by $\frac{1}{10}$ of itself, the least Limit doth not differ $\frac{1}{10}$ of itself from the true Root; otherwise if they be affected with the same Sign, it differs more than $\frac{1}{10}$: the same Rule is to be observed with respect to the greatest Limit: but here I suppose, that the Difference of the two Limits is greater than $\frac{1}{10}$ of the least Limit in the first Case; and greater than $\frac{1}{10}$ of the greatest Limit in the other Case: for if the Limits don't differ that much from one another, they will not differ from the true Root by $\frac{1}{10}$ of themselves respectively.

3°. This Method of resolving Equations is most properly applied for finding the incommensurable Roots: because the commensurable Roots are found more conveniently and directly by the Methods taught in Algebra.

4°. If the last Term of any supplementary Equation of p, q, r , &c. vanish, then the preceding Quotient already found is a true Root; or full Value of the unknown Quantity y : for in this Case the least Root of the Supplementary Equation is equal to nothing, that is the Supplement is equal to nothing.

S E C T. IV.

Explication of the Resolution of affected specious or literal Equations, contain'd in Art. 30—34. 57.

73. **T**HE specious Equations, whose Resolution our Author treats of in this Place, contain two unknown Quantities, as x and y : which are to be conceived as the Absciss, and rectangular Ordinate, of

of a Curve; whose Relation to one another is exprest by any such Equation: the other letters standing for definite known Quantities. And it is proposed to exprest the Root y in a Series of Terms including the Quantity x together with the other known Quantities. Thus in the Equation $y^3 + a^2y - 2a^3 + axy - x^3 = 0$, which he proposes, x is considered as the Absciss, and y as the Ordinate, in a Curve, whose Nature is defined by that Equation. Now that the Value of y may be found in such a converging Series, it is necessary to consider x either, 1°. as very small; or, 2°. as very great; or then, 3°. as differing very little from some given Quantity. That so by supposing x to be very small, the Series may converge wherein the Root y is exprest by a Progression of Terms, in which the Dimensions of x increase in the Numerators: 2°. By supposing x to be very great, that Series may converge, in which the Dimensions of x continually increase in the Denominators of the Terms: Or that, 3°. By supposing x to differ but very little from some given Quantity, some Species or Letter, as z , being substituted for that Difference, it may come in Place of the Species x considered as very small. And all these three different Suppositions, may have each it's Conveniency, according to the Circumstances of the Case.

74. The first is the Case Sir *Isaac* chiefly insists upon: because most useful in the Affair of Quadratures: and to which the other Cases may be reduced.

In which he proceeds after this Manner. He supposes x entirely to vanish: and from the resulting Equation seeks a Value of y . This must be an approximate Value of y , because when x is very small, the Value of y cannot differ much from it's Value when x quite vanishes. But as this is but an approximate Value of y (suppose you call it A) he puts $y = A + p$: where p is the Supplement to the Root, as in the Resolution of Numeral Equations. Wherefore he substitutes $A + p$ instead of y , in the given Equation: whence arises a new Equation (which we shall call the Equation of p) from whence the Value of p is to be derived. Now although p may have several different Values, in this new Equation, yet it is the least Value that makes the Supplement (for which See Art. 71) Wherefore, since both x and p are very small, he rejects all the other Terms unless those where x and p are, separately, of the fewest Dimensions: and supposes them, as it were, equal to nothing from whence he finds an approximate Value of p : Which call Bx^n : and put $p = Bx^n + q$. Then he substitutes $Bx^n + q$ instead of p in the Equation of p ; q being the Supplement to the approximate Value of p , viz. Bx^n ; as p was formerly
the

the Supplement to A , the approximate Value of y . And from hence you obtain a third Equation for finding an approximate Value of q , viz. by the like Means, that is by rejecting all the Terms of the Equation of q , save those wherein q and x are separately of the fewest Dimensions, the other Terms being comparatively of very little Significancy. Which near Value call Cx^m , or $q = Cx^m + r$: r being the Supplement. And so he proceeds. Whence he finds $y = A + p = A + Bx^n + q = A + Bx^n + Cx^m + r$, &c. By which you find a Series of Terms continually approaching to the Root y , that is to one of it's Values. The Process may be seen in the Treatise itself.

75. In which you may observe the following Things ¹.

1°. That in the Equations of p , q , r , &c. upon the right Hand, those Parts of the last Term, in each Equation, will mutually destroy each other; in which x is of no higher Dimension than in the last, or immediately preceding Term of the Quotient. Thus in the Equation of p , the Parts $+a^3 + a^3 - 2a^3$ of the ultimate Term; in the Equation of q , the Parts $-a^2x + a^2x$; in the Equation of r , the Parts $+\frac{1}{16}ax^2 - \frac{1}{16}ax^2$ (and so for the rest) do destroy one another. The Reason of which is evident, by considering that those Parts which destroy one another in these last Terms, stand opposite to so many other Terms in the collateral Equation upon the left Hand, which were put equal to nothing. For since a , $\frac{1}{4}x$, $+\frac{x^2}{64a}$, &c. are the Values of y , p , q , &c. in the Equations $y^3 + a^2y - 2a^3 = 0$, $4aap + a^2x = 0$, $4a^2q - \frac{1}{16}ax^2 = 0$, &c. Hence it follows that the Substitution of a , $-\frac{1}{4}x$, $+\frac{x^2}{64a}$, &c. in place of y , p , q , &c. must make the Whole equal to nothing: So that those Parts of the last Terms of the Equations of p , q , r , &c. upon the right Hand of the Diagram, which have been mentioned, must destroy each other. Whence it appears

2°. That the Powers of x in the last Term of the transformed Equations of p , q , r , &c. upon the right Hand, increase more and more continually; the lowest of which, after rejecting those Parts

¹ To prevent being misunderstood, let it be observed that the Series expressing the Value of y , is called the Quotient: and with respect to the Equations of p , q , r , &c. which stand upon the right-hand Column of the Diagram, I call that the *first Term*, in which p , q , r , &c. is of the highest Dimension: although it be placed last in the Diagram; and that the *last Term* of the Equation, which stands first in that Part of the Diagram: because the Order of the Terms ought to be computed by the Dimensions of p , q , r , &c. Moreover when any Term of an Equation is complex, I call the several simple Quantities, which make it up, *Parts* of that Term: thus $+3a^2p$, $+a^2p$, $+axp$, are the Parts of the penult Term of the Equation of p in our Author's Diagram.

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which

which destroy one another (according to the last Remark) will be higher than the highest Dimension of x in the preceding part of the Quotient.

3°. Wherefore, when x is very small in Comparison of any other given Quantity, as $\bar{a}$, in the given Equation of y , the last Term of the supplementary Equation of $p, q, r, \&c.$ is continually diminishing, so as at length to vanish. And therefore one Value of the Quantities $p, q, r, \&c.$ continually diminishes, *viz.* the least Value, for that is the Supplement, as was observed already *: Consequently since that Value of $p, q, r, \&c.$ together with the Quotient as far as formerly found, makes up the Value of y , it follows, that, by this Operation, you approach continually to the Root y , so as to differ from it at length by less than any given Difference: Because the last Term of the supplementary Equation, becoming at length less than any given Quantity, and vanishing, some Root of that Equation must become less than any given Quantity, and vanish; since the last Term of any Equation is the Product of all the Roots with contrary Signs: which diminishing and evanescent Root is the Supplement, as hath been just now said.

* Art. 71.
73.

4°. Whereas in every new Step, by which a new Term of the Value of y is found, our Author desires you to assume the Terms in which x and p ; x and q ; are separately of the fewest Dimensions, it may be observed that the Species $p, q, r, \&c.$ will always be found separately of one Dimension, without being affected with x : because in the first Supposition, you assume an Equation in which the Letter x is not found, and therefore the approximate Value of y thence deduced doth not include x ; whence it is evident that the Parts of the Penult Term of the first transformed Equation, upon the right Hand Side of the Diagram, which stand collaterally to those Terms of the proposed Equation, wherein y is not affected with x , must contain p separately of one Dimension only, in the Equation of p : which in his Example are $3a^2p$ and a^2p : therefore in the next subsequent Transformation, *viz.* in the Equation of q , upon the right Hand Side of the Diagram, that part of the Penult Term which stands collateral to $4a^2p$ upon the left Hand, where p is separately but of one Dimension, must contain q separately of one Dimension only: and so the Reasoning is the same through all the subsequent Transformations *in inf.* So that in all Equations of this Kind, where you can deduce the first approximate Value of y , by supposing x to be nothing, the Parts of the transformed Equations to be selected, for deriving the Terms of the Quotient, in the Progress of the Operation, are manifest: and the Derivation of these

these Terms of the Quotient, requires no more but the dividing the Part of the last Term wherein x is of the least Dimension, with the Sign changed, by the Coefficient of such Part of the Penult Term, as includes but one Dimension of $p, q, r, \&c.$ separately.

76. And thus I have explained and demonstrated the general Process for deriving the several Terms of the converging Series for expressing the Value of y : but whereas in Art. 31, Sir *Isaac* mentions an Abridgment, which may serve to contract the Work, I must shew upon what Foundation it is built. In order to which, let it be observed that, when the Quotient is proposed to be continued only to a certain number of Terms; or (which is equivalent) to a given Dimension of x , all those Terms of any Equation upon the right Hand side of the Diagram, may be entirely omitted in the subsequent Transformation, in which, if, for the Supplement $p, q, r, \&c.$ that Power of x were substituted, which is found to be it's approximate Value, x would be of a higher Dimension than the proposed Limit¹. The Reason of which is this: since the Species x is supposed indefinitely small, any Term, that, by the Substitution just now mentioned, would arise to a higher Dimension of x , than the Dimension to which the Quotient is proposed to be continued, is vastly less than any of the Terms in which the Dimension of x rises not above that Limit: and as such Term in the subsequent Transformation, must make that Part of the last Term of the Equation upon the right Hand, which stands collateral to it, of a higher Dimension of x than the proposed Dimension; and every subsequent Quantity upon that same Line of the Transformation, of still a higher Dimension of x , after a proper Substitution, as is evident, hence it appears that no such Term can any way affect the Quotient within the proposed Limits: since every new Quotient Term must have that Dimension of x , which is the lowest in the last Term of the supplemental Equation. Thus in the Operation in the Author, after you have arrived at the Equation of q ; and found from it $q = \frac{x^2}{64a}$ nearly, the first Term q^3 is entirely neglected in the subsequent Transformation: because if $\frac{x^2}{64a} + r$ were substituted for

¹ The Reader must observe that this Rule which is given by Sir *Isaac*; and the Reason given for it here, can only be extended to those Cases in which the Supplements $p, q, r, \&c.$ are found in some Part of the penult Term of their respective Equations, without any Mixture of x : in other Cases the Rule will not hold. And therefore an universal Rule for all Cases shall be delivered afterwards.

q , it would give $q^3 = \frac{x^6}{262144a^3} + \frac{x^4r}{2048a^2} + \frac{x^2r^2}{32a} + r^3$, for it's collateral Value; in which the last Term $\frac{x^6}{262144a^3}$ rises above the fourth Power, which is the proposed Limit of the Dimension of x : and the other Terms would still rise to higher Dimensions of x , since the approximate Value of r must contain more Dimensions of x than q doth, which contains the Dimension x^2 : whence it appears, that none of the Terms of the Value of q^3 , in any subsequent Transformation, can give or produce any Quantity, but what shall contain either the 6th Dimension of x ; or one higher in the ultimate Term: which therefore can have no Influence upon the Quotient within the proposed Limit. And the reasoning is exactly the same with respect to the Part $-\frac{3}{4}xq^2$ of the second Term of the Equation of q upon the right hand: which therefore is likewise entirely neglected in the subsequent Transformation.

77. But sometimes all the Terms of the transformed Value of a Quantity are not to be omitted, but only some of them, in order to carry the Quotient to a proposed Limit, or Dimension of x : for which you are to observe this Rule, *That after the first resulting Term of the transformed Value of any Quantity, standing collateral to it upon the Right-hand, so many more Terms are to be added, as the Index of the highest Power of x , in the Quotient, exceeds the Index of x in such first resulting Term of the transformed Value: provided the Dimensions of x ascend only by the Difference of one, as in the Example of the Author: where the Dimensions of x in the Terms of the Quote are x^0, x^1, x^2, x^3 , &c. Thus if the Quotient is to be continued only to the 4th Dimension of x , you omit all the Terms after x^4 , and put one after x^3 ; and so in other Cases. Accordingly in the present Example, after you have arrived at the Equation of q , the transformed Value of the Quantity $3aq^2$ is $\frac{3x^4}{4096a} + \frac{3x^2r}{32} + 3ar^2$: of which the two Terms $\frac{3x^2}{r} + 3ar^2$ might be omitted, when the Quote is designed to be carried only to the 4th Power of x : for the same Reason, opposite to $\frac{3}{16}x^2q$, the Term $\frac{3}{16}x^2r$ might be omitted. The Reason of which may be easily deduced from what hath been said in the preceding Article. Which Rule includes the particular Rule given by Sir Isaac, Art. 31: as will be evident to one that considers it. But if the Dimensions of x in the Quote proceed by greater Differences, as x^0, x^2, x^4, x^6 , &c. or x^0, x^3, x^6, x^9 , &c. then, when*

when the Differences are 2, after the first resulting Term of the transformed Value of any Quantity, you need only subjoin half the Number of Terms, which denotes the Excess of the Index of the highest Power of x , to which the Quotient is to be continued, above the Index of x in the first resulting Term of the Value of the Quantity, to which that Term stands collateral: And when the Differences of the Dimensions of x in the Terms of the Quote are 3, you need only adjoin a third of that Number of Terms, &c. because now the Differences of the Dimensions of x in the Terms of the transformed Value of any Quantity taken from the last resulting Equation, will be 2, 3, &c. after substituting for p , q , r , &c. their approximate Values. And after the Operation, in this abridged way, is carried so far, as that the Supplement q , r , &c. rises no higher than the Root, in the transformed Equation upon the Right-hand Side of the Diagram, you obtain the rest of the Terms by Division: which is to be continued untill the Quote be brought to the proposed Limit ¹.

78. The Author's Example, with the Operation abridged, according to the preceding Rules, may stand thus: where the Mark * denotes the Parts of the Terms which are omitted: the Quotient or Value of y being proposed to be carried the Length of the 4th Power of x .

¹ These Things help to explain Art. 44. of the Author.

N. B. The Quantities which are dashed thus --- , are such as destroy one another.

$y^3 + aay - 2a^3 + axy - x^3 = 0.$		$y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \text{ \&c.}$	
$y = +a + p.$	$+y^3 =$	$+a^3 + 3a^2p + 3ap^2 + p^3$	$+4a^2p + a^2x = 0 \text{ fere, or } p = -\frac{x}{4} \text{ fere.}$
$+a^2y =$	$+a^2 + a^2p$		
$-2a^3 =$	$-2a^3$		
$+axy =$	$+a^2x + axp$		
$-x^3 =$	$-x^3$		
$p = -\frac{x}{4} + q.$	$+p^3 =$	$-\frac{1}{64}x^3 + \frac{3}{16}x^2q + \frac{3}{8}axq + 3aq^2$	$+4a^2q - \frac{1}{16}ax^2 = 0 \text{ fere, or } q = +\frac{x^2}{64a} \text{ fere.}$
	$+3ap^2 =$	$-\frac{3}{16}ax^2 - \frac{3}{8}axq + 4a^2q$	
	$+4a^2p =$	$-\frac{1}{4}ax^2 + axq$	
	$+axp =$	$-\frac{1}{4}ax^2 + axq$	
	$+a^2x =$	$-\frac{1}{4}ax^2 + axq$	
	$-x^3 =$	$-x^3$	
$q = +\frac{x^2}{64a} + r.$	$+3aq^2 =$	$+\frac{3x^4}{4096a} + \frac{3}{16}ax^2 + 4a^2r$	$+4a^2r - \frac{1}{2}axr - \frac{131x^3}{128a^2} - \frac{15x^4}{4096a} = 0 \text{ fere.}$
	$+4a^2q =$	$+\frac{1}{2}ax^2 + 4a^2r$	
	$-\frac{1}{2}axq =$	$-\frac{1}{2}axr - \frac{1}{2}axr$	$+ \frac{131x^3}{128a^2} + \frac{15x^4}{4096a} \text{ fere.}$
	$+ \frac{3}{16}x^2q =$	$+\frac{3x^4}{1024a}$	
	$-\frac{1}{16}ax^2 =$	$-\frac{1}{16}ax^2$	
	$-\frac{6}{64}x^3 =$	$-\frac{6}{64}x^3$	$r = +\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \text{ \&c.}$

Now

Now because in the last Transformation, which gives the Equation of r , the Supplement r rises no higher than the Root, the rest of the Terms of the Quotient, as far as the 4th Power of x , which is the Limit proposed, are found by Division, *viz.* by dividing $\frac{131}{128}x^3 + \frac{15}{4096}x^4$ by $4a^2 - \frac{1}{2}ax$, which gives the Quote $\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} - \mathcal{E}c.$ which must not be carried beyond the 4th Power of x ; in Regard the Terms affecting the 5th and higher Powers of x , were rejected.

79. Take another Example $x^2y^4 + c^2y^4 - x^4y^2 - c^4x^2 - c^6 = 0$, where the Dimensions of x in the Quotient, proceed by the Difference 2, and the highest Dimension of x to which the Quotient is proposed to be continued is the 6th: You assume the Terms $c^2y^4 - c^6 = 0$ *fere*, or $y = c + p$, and then proceed as you see:

$x^2y^4 + c^2y^4 - x^4y^2 + c^4x^2 - c^6 = 0$		$y = c - \frac{x^2}{2c} + \frac{3x^4}{8c^3} - \frac{5x^6}{16c^5} - \mathcal{E}c.$	
$+ x^2y^4 =$	$+ c^4x^2 + 4c^3x^2p + 6c^2x^2p^2 + 4cx^2p^3 + x^2p^4$	$4c^5p + 2c^4x^2 = 0$ <i>fere</i>	$p = -\frac{x^2}{2c}$ <i>fere</i>
$+ c^2y^4 =$	$+ c^6 - 4c^5p + 6c^4p^2 - 4c^3p^3 + c^2p^4$		
$- x^4y^2 =$	$- c^2x^4 - 2cx^4p - x^4p^2$		
$+ c^4x^2 =$	$+ c^4x^2$		
$- c^6 =$	$- c^6$		
$+ 4c^3p^3 =$	$- \frac{1}{2}x^6$		
$+ 6c^4p^2 =$	$+ \frac{3c^2x^4}{2} - 6c^3x^2q$	$4c^5q - 2c^3x^2q - \frac{1}{2}c^2x^4 + 2x^6 = 0$ <i>fere</i>	$q = \frac{+\frac{1}{2}c^2x^4 - 2x^6}{4c^5 - 2c^3x^2}$ <i>fere</i>
$+ 6c^2x^2p^2 =$	$+ \frac{3x^6}{2}$		
$- 2cx^4p =$	$+ x^6$		
$+ 4c^5p =$	$- 2c^4x^2 + 4c^5q$		
$+ 4c^3x^2p =$	$- 2c^2x^4 + 4c^3x^2q$		
$+ 2c^4x^2 =$	$+ 2c^4x^2$		
$- c^2x^4 =$	$- c^2x^4$		
		$4c^5 - 2c^3x^2 \left(\frac{\frac{1}{2}c^2x^4 - 2x^6}{4c^5 - 2c^3x^2} \right) + \frac{3x^4}{8c^3} - \frac{5x^6}{16c^5} - \mathcal{E}c. = 0$ $\frac{\frac{1}{2}c^2x^4 - 2x^6}{4c^5 - 2c^3x^2} = \frac{3x^4}{8c^3} - \frac{5x^6}{16c^5} - \mathcal{E}c.$	

Here

Here in this Example, because the Differences of the Dimensions of x in the Value of y are 2, and that Value or Quotient is to be carried forward only to the 6th Dimension of x , therefore in transforming the several Quantities of the Equation of p , you will observe that those are entirely omitted in which if x^2 , which is the Dimension of x in the approximate Value of p , were substituted for p , the Dimension would be higher than the 6th: no Term placed after the first resulting Term, when the Index of x in it is 6: and only one Term after it, when the Exponent of x is 4. By which the Equation of q rises not above the first Power of q : therefore you find other two Terms of the Quotient by Division, as appears by the Operation itself. And so you proceed in other like Cases, by which you shall obtain a Value of y in an infinite Series of simple Terms: and therefore if x and y denote the Absciss and perpendicular ordinate of a Curve, the correspondent Curvilinear Area is found by our Author's second Rule: which in this Case, where the Powers of x ascend in the Numerators, approaches the nearer to the Truth the less that x is.

80. But if x be supposed to be very great in Comparison of any other known Quantity in the given Equation (which is the second Case • Art. 72. we mentioned above *; and of which the Author treats in Art. 33, 34,) in this Case, the Quotient, or Value of y , must be such, that the Dimensions of x diminish in the Numerators; or increase in the Denominators of the Terms. Now in order to find such a Value of y , our Author directs to proceed thus: let $y^3 + axy + x^2y - a^3 - 2x^3 = 0$ be proposed, in which x is supposed very great in Comparison of the given Quantity a , then he desires you to select those Terms in which x and y either separately or multiplied together are of the most and at the same Time equal Dimensions, and put them as it were equal to nothing, and the Value of the Root y thence deduced, shall give the first Term of the Quotient: therefore in this Example you put $y^3 + x^2y - 2x^3 = 0$, whence you find $y = x$, an approximate Value of y , that is $y = x + p$, p being the Supplement. It will be the same Thing as Sir Isaac observes if you turn the assumed Equation $y^3 + x^2y - 2x^3 = 0$, into this other $y^3 + y - 2 = 0$, and extract the Root y out of this last Equation; which will be found $y = 1$, and then multiply this Value of y by x : the Reason of which is this, that if the given Equation $y^3 + x^2y - 2x^3 = 0$, be transformed by putting $z = \frac{y}{x}$ or $zx = y$, it will give $x^3z^3 + x^3z - 2x^3 = 0$; or, by dividing by x^3 , $z^3 + z - 2 = 0$, which is the same with $y^3 + y - 2 = 0$, whose Root 1 being multiplied by x , is the Root of the given Equation $y^3 + x^2y - 2x^3 = 0$, since $y = zx$. 81. Now

81. Now the first Term of the Quotient or Value of y being discovered in this Manner (the Reason of which shall be shewn afterwards) the subsequent Terms are found by substituting $x + p$ for y in the given Equation $y^3 + axy + x^2y - a^3 - 2x^3 = 0$, and transforming it thereby into a new Equation; from which an approximate Value of p is to be found, by assuming those Terms of this new Equation, which are vastly greater than all the rest, and putting them, as it were, equal to nothing: and so carrying on the Operation for the subsequent Terms, after the like manner as has been shewn already, in the other Case, in which x was supposed to be very small. The Process of which is here represented to the Reader,

$$y = x + p. \begin{array}{l} + y^3 = + x^3 + 3x^2p + 3xp^2 + p^3 \\ + axy = + ax^2 + axp \\ + x^2y = + x^3 + x^2p \\ - a^3 = - a^3 \\ - 2x^3 = - 2x^3 \end{array}$$

$$4x^2p + ax^2 = 0 \text{ fere}$$

$$p = \left(\frac{-ax^2}{4x^2} \right) = -\frac{1}{4}a \text{ fere}$$

$$p = -\frac{1}{4}a + q. \begin{array}{l} + p^3 = -\frac{1}{64}a^3 + \frac{3}{16}a^2q - \frac{3}{4}aq^2 + q^3 \\ + 3xp^2 = + \frac{3}{16}a^2x - \frac{3}{8}axq + 3xq^2 \\ + axp = -\frac{1}{4}a^2x + axq \\ + 4x^2p = -ax^2 + 4x^2q \\ - a^3 = - a^3 \\ + ax^2 = + ax^2 \end{array}$$

$$4x^2q - \frac{1}{8}a^2x = 0 \text{ fere}$$

$$q = + \frac{a^2}{64x} \text{ fere}$$

$$r = + \frac{a^2}{64x} + r. \begin{array}{l} + q^3 = + \frac{a^6}{262144x^3} + \frac{3a^4r}{4096x^2} + \frac{3a^2r^2}{64x} + r^3 \\ - \frac{3}{4}aq^2 = - \frac{3a^5}{16384x^2} - \frac{3a^3r}{128x} - \frac{3}{4}ar^2 \\ + 3xq^2 = + \frac{3a^4}{4096x} + \frac{3a^2r}{32} + 3xr^2 \\ + \frac{3}{16}a^2q = + \frac{3a^4}{1024x} + \frac{3}{16}a^2r \\ - \frac{1}{2}axq = - \frac{a^3}{128} - \frac{1}{2}axr \\ + 4x^2q = + \frac{a^2x}{16} + 4x^2r \\ - \frac{65a^3}{64} = - \frac{65a^3}{64} \\ - \frac{1}{8}a^2x = - \frac{1}{8}a^2x \end{array}$$

$$4x^2r - \frac{131a^3}{128} = 0 \text{ fere}$$

$$r = + \frac{131a^3}{512x^2} \text{ fere}$$

$$4x^2 - \frac{1}{2}ax \&c.) + \frac{131a^3}{128} - \frac{15a^4}{4096x} \&c. \left(+ \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3} \&c. = r \right.$$

$$+ \frac{131a^3}{128} - \frac{131a^4}{1024x} \&c.$$

$$+ \frac{509a^4}{4096x} \&c.$$

$$y = x - \frac{1}{4}a + \frac{a^2}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3} \&c. \text{ or } x - \frac{1}{4}a + \frac{1}{64}a^2x^{-1} + \frac{131}{512}a^3x^{-2} + \frac{509}{16384}a^4x^{-3} \&c.$$

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In

82. In which Operation, having obtained the Equation of p , I throw by the first and second Terms, as containing the Cube and Square of p : which being of small Value in Comparison of the other Terms, are to be neglected: likewise of the Penult Term, you only retain the Part $4x^2p$, in which x rises to the highest positive Dimension; or if there were no positive Dimension of x in it, you must take that Part only where x is of the lowest Negative Dimension¹, the rest you neglect, because by the Hypothesis x is supposed to be vastly great in Comparison of a ; likewise of the ultimate Term I retain (after throwing by repugnant Parts) only the Part ax^2 , viz. that in which x is of the highest positive Dimension; otherwise, if there be no positive Dimension, where it is of the lowest Negative Dimension: and so I put $4x^2p + ax^2 = 0$ almost, whence I deduce $p = -\frac{1}{4}a$ for an approximate Value of p ; and consequently, for the second Term of the converging Series for the Value of y : and then putting $p = -\frac{1}{4}a + q$, you transform the Equation of p into a new Equation; as you see; which you manage after the same Manner as the preceding, by putting $4x^2q - \frac{1}{16}a^2x = 0$ almost, whence I deduce $q = +\frac{a^2}{64x}$ nearly, which is the third Term of the converging Series: and so you proceed to find the other Terms by the like Means. But here it may be observed, that when the Series for the Value of y is proposed to be continued only to a certain Limit, or Dimension of x , many unnecessary Terms, or Parts of Terms, in the Transformations of the Equations, may be omitted, viz. those that cannot affect the Quotient within the proposed Limit, agreeably to what was shewn formerly*, allowing for the Difference of the Cases: according to which those Quantities marked with an Asterisk above them, might have been omitted, if the Value of y was proposed to be carried no further than the Dimension $\frac{1}{x^3}$, or x^{-3} . Moreover it is to be observed, that, after you have obtained a certain Number of Terms of the Series, by a continued Transformation, you may obtain so many more by Division, as you see in the Operation. The Division is always performed by dividing the last Term with the Signs changed, by the Complex

* Art. 78.
79.

¹ I call a negative Dimension of x , a Power having a negative Index in the Numerator; or a positive Index in the Denominator as $x^{-3} = \frac{1}{x^3}$: accordingly I call $x^{-3} = \frac{1}{x^3}$ a lower negative Dimension of x than $x^{-4} = \frac{1}{x^4}$: likewise $x^0 = 1$ is to be accounted the lowest Dimension of x , whether positive or negative.

Coefficient

Coefficient of the Supplement $q, r, \&c.$ in the Penult Term of the last Equation. And in order to know to how many Terms the Quotient of the Division may be carried, observe this Rule. In the last transformed Equation, suppose it be that of r , divide, or suppose to be divided, all the Terms of the Equation preceding the Penult, by r , some Power of which above the first, enters into them all: Then find the Power of x which enters into the first Term of the Value of r : which in the preceding Example I find to be $\frac{1}{x^2}$ or x^{-2} , by putting $4x^2r -$

$\frac{65a^3}{64} - \frac{a^3}{128} = 0$: substitute that Dimension of x , viz. $\frac{1}{x^2}$ in Place of r , or suppose it to be substituted, in the Terms preceding the Penult, after they have been supposed to be divided by r as above; and let the lowest positive, or yet negative Dimension of x , thence arising, be x^n or x^{-n} respectively, according as x is supposed to be very small, or very great: then you may continue the Division untill you obtain that Term of the Quotient whose positive, or negative Dimension, is the next below x^n or x^{-n} respectively: only it must be observed, that, if there be any positive Dimension of x in any of the Terms preceding the Penult Term of the Equation, after the Division and Substitution mentioned above, and the positive Dimensions of x decrease in the Value of y , then the Division is to be continued only until you obtain that Term of the Quotient, whose positive Dimension is the very next above the highest in these Terms of the Equation. And you must take Care, in setting down the Parts of the Divisor and Dividend, that you make the lower Dimensions to precede the higher, when they are positive, and x very small; but the higher positive Dimensions to precede the lower, and the lower negative Dimensions to precede the higher, when x is very great.

In this Rule it is supposed that all the Terms of the transformed Equations are compleat, although this be not always necessary¹: and the Rule will serve in all Cases, whether the Dimensions of x be all positive and increase from the beginning of the Series; or 2°. the Dimensions of x be first negative and decrease, and afterwards become positive, and then increase: in which Cases x must be supposed to be very small; or, 3°. the Dimensions of x be negative and increase from

¹ There is not any Necessity for annexing any of the Terms, of the last transformed Equation, which precede the Antepenult: because the Dimensions of x contained in the Quantities, which lye upon the same horizontal Line, proceed always in an arithmetical Progression, after the Substitution mentioned in the Rule: nay and of the Antepenult Term, there is no Necessity for annexing any other Part or Quantity, but that one where x is of the least positive or least negative Dimension, according as x is very small or very great; since the Continuation of the Terms of the Quotient, arising by the Division, is regulated by it only.

the beginning of the Series; or 4°. the Dimensions of x be first positive and decrease, and then become negative and afterwards increase: in which two last Cases, x must be supposed to be very great. And I need not take up Time in shewing the Foundation of this Rule: since the Reader will easily understand it from what was demonstrated formerly with respect to numeral Equations *, applied to this Case of literal Equations, which is very easy to do.

* Art. 63.
71.

83. From these Examples, I suppose, it will be manifest how you are to proceed in extracting the Roots of literal Equations including two unknown Quantities x and y ; so as to express the Value of y by a Series of Terms including different Powers of x , and that whether x be supposed to be very small, or very great. But, notwithstanding that it has been shewn in the last Example, how to find the Terms of a converging Series expressing the Value of y , when x is very great in comparison of any other Quantity, as a , in the Equation, by a Method differing in some respect from that whereby the Value of y was found, when x was supposed to be very small; yet the same thing might be obtained, by the first Method, if you only suppose that the given Quantity a , in Comparison of which x is supposed to be very great, be considered as the indefinitely small Quantity coming in Place of x : and that you proposed to yourself to find a converging Series expressing the Value of y in Terms proceeding according to the positive Dimensions of a , instead of those of x : which is taken Notice of by our Author at Art. 33: The Reason of which is, that the Smallness or Greatness of x is comparative, *viz.* with respect to the Magnitude of the known Quantity a ; so that the supposing of x to be very great is equivalent to the supposing of a to be very little.

84. And thus by either of these Methods the Value of y being extracted, and expressed by an infinite Series of simple Terms, including the Quantity x involved with known Quantities, the Area of the Curve, whose Absciss and Ordinate are x and y is found by the Author's second Rule. And the Area will approach the nearer to the Truth (*cæteris paribus*) the greater that x is, because this will evidently make both the Series for the Value of y ; and that for the Value of the Area, thence deduced, to converge the faster: thus having found in the preceding Example $y = x - \frac{a}{4} + \frac{a^2}{64x} + \frac{131a^3}{512x^2} +$

$\frac{509a^4}{16384x^3}$ &c. the Area belonging thereto is $\frac{x^2}{2} - \frac{ax}{4} + \left[\frac{a^2}{64x} - \frac{131a^3}{512x^2} - \frac{509a^4}{32768x^3} \right]$ &c. both of which converge the more quickly the greater

greater the Absciss x is. And as to the Curve, it approaches continually towards a rectilinear Affymptote, at which the Part of the Ordinate y denoted by the two first Terms of it's Value, viz. $x - \frac{1}{4}a$, always terminates; so that you may easily find the Position of that Affymptote, and any Portion of it, by assuming any two Points in the Absciss, and drawing two Perpendiculars at these two Points, equal to the Length of the Absciss less $\frac{1}{4}a$; the Line drawn betwixt the Extremities of the Perpendiculars, is the Portion of the rectilinear Affymptote, corresponding to the Part of the Absciss lying betwixt the two Points assumed therein. But that this, and likewise the way of computing the Area may appear the more clearly, I shall represent the thing to the Reader by an actual Description, since the Author himself has just only mentioned it, Art. 34; and it may be of some Use to the young Geometer.

Draw the Right Line AB produced indefinitely towards B, upon which take $AG = \frac{1}{4}a$, and calling the Absciss $AB = x$, let the Curve $\delta D\alpha$ be described by the

perpendicular Ordinate $BD = y$, so that it be

$$y = x - \frac{a}{4} + \frac{a^2}{64x} +$$

$$\frac{131a^3}{512x^2} \text{ \&c. upon BD take}$$

$$BE = BG = x - \frac{a}{4},$$

draw GE, and it is an Affymptote to the Curve $\delta D\alpha$, to which the Curve infinitely produced towards α , continually approaches. The Curvilinear Area was found to

$$\text{be } \frac{x^2}{2} - \frac{ax}{4} + \left[\frac{a^2}{64x} \right] -$$

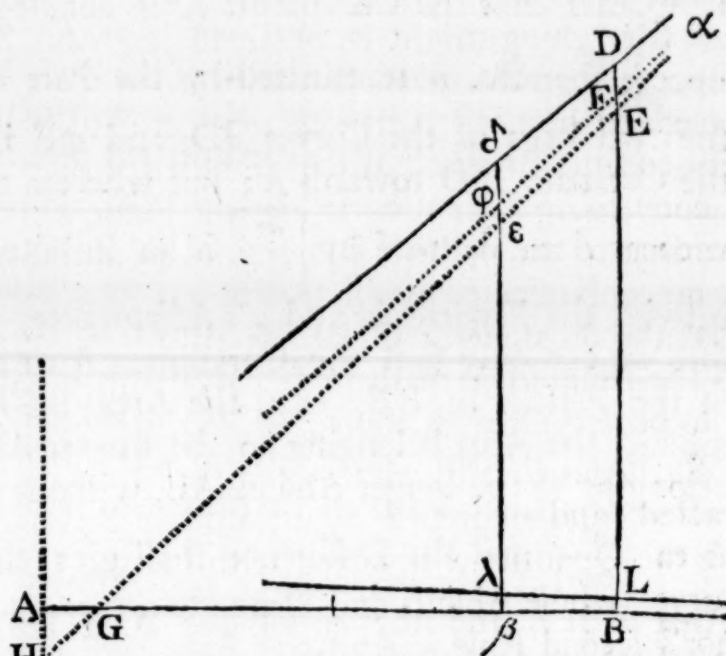
$$\frac{131a^3}{512x} - \frac{509a^4}{32768x^2}, \text{ \&c. of which that Part which is exprest by the two}$$

first Terms $\frac{x^2}{2} - \frac{ax}{4}$ denotes a rectilinear Area adjacent to the Absciss

AB: for if through A you draw AH perpendicular to AB, and meeting the Affymptote GE below the Absciss, in the point H, then the

Triangle GBE $= \frac{1}{2}x^2 - \frac{1}{4}ax + \frac{1}{32}a^2$, and the Triangle AGH $= \frac{1}{32}a^2$,

therefore GBE $-$ AGH $= \frac{x^2}{2} - \frac{ax}{4}$. Again the Part $-\frac{131a^3}{512x}$



$-\frac{509a^4}{32768x^2}$ &c. denotes a curvilinear Area lying upon the other side of the Ordinate, terminated at the Curve infinitely produced upon that Side: because the positive Powers of x are found in the Denominators of all the Terms: the other Terminations of this Curvilinear Area may be determined by comparing the Ordinate $+\frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$ &c. with which it is described, with the preceding Parts of the Value of y , viz. $x - \frac{a}{4}$, and $+\frac{a^2}{64x}$, of which the first Part, viz. $x - \frac{a}{4}$ is the right Line BE, which always touches the Affymptote GE; the other Part, viz. $\frac{a^2}{64x}$ will belong to an Hyperbola: for if in the Line ED you take $EF = \frac{a^2}{64x}$ the third Term of the Value of y , the Point F will touch an Hyperbola having HE and HA for it's Affymptotes; which let be described, and let it be F ϕ : from whence it appears that the Curvilinear Area denoted by the Terms $-\frac{131a^3}{512x^2} - \frac{509a^4}{32768x^3}$ &c. is terminated by the Part of the Ordinate FD and the two Legs of the Curves δD and ϕF infinitely extended beyond the Ordinate BD towards α : but whereas the third part of the Curvilinear Area denoted by $\left[\frac{a^2}{64x}\right]$ is an infinite hyperbolical Space, lying betwixt the Hyperbola and it's Affymptote, infinitely produced, therefore we can only find the Curvilinear Area adjacent to any given Part of the Absciss as B β , viz. the Area B $\beta\delta D$: which is done thus: subtract the Area belonging to the shorter Absciss A β from the Area belonging to the longer Absciss AB, without taking in the Hyperbolical Area $\left[\frac{a^2}{64x}\right]$ and the Difference shall give the Sum of the rectilinear Area B $\beta\epsilon E$ and the curvilinear Area F $\phi\delta D$: (for which see our Author's second Rule, Examp. 3. and Sect. 5. of the preceding Treatise) wherefore if to these you add the hyperbolical Area E $\epsilon\phi F$, you shall have the entire Area B $\beta\delta D$. To find that hyperbolical Area E $\epsilon\phi F$, you may proceed thus: Upon BD take BL = EF, and with the Center A and Affymptotes AB, AH describe the rectangular Hyperbola L λ through the Point L, cutting $\beta\delta$ in the Point λ , and the Area B $\beta\lambda L$ is equal to the Area E $\epsilon\phi F$, as is evident; wherefore let the Area B $\beta\lambda L$ belonging to the equilateral Hyperbola; and adjacent to the given part of the Affymptote B β be found by what hath been shewn

shewn formerly *: and this added to the Areas $B\beta E$ and $F\phi\delta D$ * Art. 25. found already, will make up the whole Area $B\beta\delta D$ required.

85. And thus I have endeavoured to illustrate the two different Cases mentioned by Sir *Isaac* in this Treatise, for discovering the Roots of affected literal Equations, and expressing them by infinite Series's, either when x is supposed to be very small; or when it is supposed to be very great in Comparison of any given Quantity in the proposed Equation: but I mentioned a third Case, which is when x is supposed to differ very little from any given Quantity: in this Case you substitute some letter for that small Difference in the proposed Equation, as z ; and so having the two unknown and variable Quantities y and z , of which z is considered as indefinitely small in Comparison of any given Quantity in the Equation, you may find a converging Series for the Value of y , in Terms made up of positive Powers of z , and known Quantities, as formerly: from which you may deduce a Series including x by replacing it in the Series for it's formerly substituted Value. Thus in the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, it being known or supposed that x is nearly of the same Quantity with a , you substitute z for their Difference, *i. e.* put $z = a - x$ or $x = a - z$, that is substitute $a - z$ or $a + z$ for x in the proposed Equation: by substituting the first of which, there arises this new Equation $y^3 + 2a^2y - azy + 3a^2z - 3az^2 + z^3 - 3a^3 = 0$, from whence the Value of y may be found by the first Method, z being supposed very small in Comparison of a : which Value being found, you may replace x for $a - z$, that is $a - x$ for z ; and so the Area may be found in Terms made up of a and x . And this third Method is proper to be taken when the Series for the Value of the Ordinate y , and consequently that for the Value of the Area, will converge the more quickly the nearer that the Length of the Absciss x is to the Length of the given Line a .

86. Moreover, hence it appears that all the three different Methods for extracting the Roots of such affected literal Equations, may be reduced to the first Method only: for this third Case mentioned in the preceding Art. falls in with it; and it was shewn in what Manner the second Case, which supposes x to be very great, falls in with it also *. * Art. 83.

Which Case likewise you may reduce to the first, by substituting $\frac{1}{x}$ for x in the proposed Equation, which must make z very small when x is very great. Thus in the preceding Example $y^3 + axy + x^2y - a^3 - 2x^3 = 0$, in which x is supposed very great in Comparison of a ,
substitute

substitute $\frac{1}{z}$ for x , and you will have $y^3 + \frac{ay}{z} + \frac{y}{z^2} - a^3 - \frac{2}{z^3} = 0$, that is, by multiplying by z^3 , $z^3y^3 + az^2y + zy - a^3z^3 - 2 = 0$, in which z is very small: and from which the Root may be extracted upon that Supposition, by what will be said afterwards upon the second Remark under Art. 35 of the Author. Which Article I should now enter upon the Explication of directly; were it not that it will fall in to be explained afterwards to more Advantage in the Progress of the following Section.

S E C T. V.

Of the various Methods of extracting and expressing the Roots of affected literal Equations, including two unknown Quantities, by Means of infinite Series's: in which the Author's 35th 42—46 Art. are explained.

87. **O**NE and the same Equation may afford different Series's for expressing the Root or Value of y , by *selecting* two or more Terms *differently*, out of the proposed Equation, and putting them equal to nothing, in order to discover from thence the first Term of a Series expressing the Value of y . In what goes before, it hath been shewn that one and the same Equation may afford two different Series's; one upon Supposition that x is very small, and another upon Supposition that x is very great, in Comparison of any known Quantity in the Equation: but it frequently happens that from the same Equation, there may be more Series's than one expressing the Value of y , when x is very small; and likewise more than one agreeing to the Supposition of x being very great: but then there is some Difficulty in finding out the first Terms of all those Series's: for understanding of which I shall demonstrate the following Lemmas.

L E M M A I.

88. In any Equation including two unknown Quantities x and y , from whence a converging Series is to be derived for expressing a Value of y in simple Terms involving different Powers of x ; let Ax^n denote the first Term of such a Series, where n denotes any indefinite Exponent,

ment, positive or negative, integral or fractional; and A any indeterminate Coefficient: then Ax^n being substituted for y throughout the several Terms of the proposed Equation, let the Dimensions of x in two or more of the Terms be the same, and either less than the Dimensions of x in all the other Terms; or greater than the Dimensions of x in all the other Terms¹, I say these two or more Terms are vastly greater than all the rest of the Terms of the Equation, according as x is supposed to be indefinitely less, or indefinitely greater, than any known Quantity in the Equation, *i. e.* if x be supposed to be indefinitely small, these two or more Terms in which the Dimensions of x are equal, and less than in all the rest, are vastly greater than all the rest: but if x be indefinitely great, the two or more Terms in which the Dimensions of x are equal and greater than in all the rest, are vastly greater than all the rest of the Terms of the Equation.

Dem. For when x is supposed indefinitely little, the Dimensions of x in the Terms of a converging Series, must be supposed to increase as the Series proceeds, so as to be least of all in the first Term, *viz.* Ax^n : wherefore Ax^n will be vastly greater than all the remaining Terms, which will vanish in Comparison of it, if you make x small enough, so that Ax^n is, in that Case, nearly the Value of y : consequently if Ax^n being substituted for y in the Terms of the Equation, raise two or more of them to equal and lower Dimensions of x than in all the rest, these two or more Terms must be vastly greater than all the rest taken together. This is when x is supposed to be indefinitely small.

And by the same Way of Reasoning, if x be vastly great, it may be shewn, that, if there be two or more Terms in which the Dimensions of x are equal and higher than the Dimensions of x in all the other Terms of the Equation, after Ax^n is substituted for y , such two or more Terms are vastly greater than all the rest of the Terms taken together: by considering that in a converging Series answering to this Case, the Dimensions of x must continually diminish; and that any higher Dimension of x is vastly greater in Value than all the lower Dimensions of x taken together. Q. E. D.

¹ Here it must be observed that any negative Dimension of x is considered as lower than any positive Dimension; and of negative Dimensions of x , the greater the negative Exponent is, the lower is the Dimension esteemed to be: thus x^{-4} or $\frac{1}{x^4}$ is reckoned a lower Dimension

of x than x^{-3} or $\frac{1}{x^3}$: so that $x^{-4} \cdot x^{-3} \cdot x^{-2} \cdot x^{-1} \cdot x^0 \cdot x^1 \cdot x^2 \cdot x^3$ &c. represents a Scale of Dimensions of x ascending.

LEMMA II.

89. If $x^p y^q \cdot x^r y^s$ exhibit the Dimensions of x and y in any two Terms of such an Equation as has been mentioned, in which Terms Ax^n (signifying the same Thing as before) being substituted for y , makes the Dimensions of x equal, and less in them; or greater in them, than the Dimensions of x in all the rest of the Terms of the Equation, after Substitution of Ax^n for y in them likewise, these two Terms being put equal to nothing, will give an approximate Value of y , in which the Dimension of x shall be x^n , the same as in Ax^n the first Term of the converging Series ².

Dem. For the two Terms including $x^p y^q$ and $x^r y^s$ being put equal to nothing, will give a Value of y , which contains this Dimension of

x , viz. $x^{\frac{p-r}{s-q}}$, as is evident: now I say $\frac{p-r}{s-q} = n$. For, by inserting x^n for y in the two Terms of the Equation in which the Dimensions of x and y are $x^p y^q$ and $x^r y^s$, their Dimensions of x will be x^{p+qn} and x^{r+sn} , which by the Hypothesis are equal, that is $p + qn = r + sn$, whence you'll have $\frac{p-r}{s-q} = n$. Q. E. D.

90. Cor. 1. From the last Lemma it follows, that if there be any two Terms of such an Equation as has been mentioned, which being put equal to nothing, give a possible Value of y , in which Value the Dimension of x is such, that, being substituted for y in all the Terms of the Equation, it makes them all of higher, or all of lower Dimensions of x than in the two Terms mentioned, these two Terms, so assumed and put equal to nothing, will give the first Term of a converging Series for the Value of y , according as x is supposed to be indefinitely small; or indefinitely great.

Example I. Let the Equation $y^3 - axy - x^3 = 0$ be proposed, (which is that proposed by the Author in Art. 35. Rem. 2.) put the Terms $-axy - x^3 = 0$, thence arises $y = -\frac{x^2}{a}$, in which the Dimension of x is x^2 : which being substituted for y in the first Term of the Equation, viz. y^3 would give x^6 for the Dimension of x ; and that being higher than x^3 ; which is the Dimension of x in the two Terms assumed, hence $-\frac{x^2}{a}$ is the first Term of a converging Series for the Value of y , whence x is very little.

The proposed Equation including x and y , is supposed here and all along to have all it's Terms brought to one Side, so that the whole Aggregate of the Terms may be equal to nothing.

After

After the same Manner, if you put $y^3 - x^3 = 0$, you have $y = x$, which will give the first Term of another converging Series for the Value of y , when x is very great: because if x be substituted for y in the middle Term of the Equation, viz. $-axy$, it gives the Dimension x^2 , lower than x^3 , which is it's Dimension in the two Terms assumed.

Example II. Let the Equation $y^3 - a^2y + axy - x^3 = 0$ be proposed: if you put $y^3 - a^2y = 0$, you have thence $y = +a$ or $-a$, where the Dimension of x is $x^0 = 1$, which being substituted for y through all the Terms, the Dimensions of x will be x^0, x^0, x^1, x^3 : which shews that $+a$ or $-a$ may, either of them, be the first Term of a converging Series, when x is very little.

91. From which Example it is evident, that, in every such Equation, if there be two or more Terms including no Dimension of x ; from which put equal to nothing a Value of y may be deduced, such Value will always give the first Term of a converging Series which shall be a Root of the Equation, when x is small. And this coincides with our Author's Rule, Art. 30.

92. In the same Equation, assuming the two Terms $-a^2y - x^3 = 0$, you have $y = -\frac{x^3}{a^2}$: by inserting of which Dimension of x for y through all the Terms, the Dimensions of x stand thus x^0, x^3, x^4, x^3 : whence it appears that $-\frac{x^3}{a^2}$ will be the first Term of another converging Series, when x is very little.

Again if you assume $y^3 - x^3 = 0$, you have $y = x$: by inserting of which for y , the Dimensions of x are x^3, x, x^2, x^3 : whence it appears that x will be the first Term of a converging Series for the Value of y , when x is very great: because the Dimensions of x in the two assumed Terms are higher than in any of the rest of the Terms of the Equation.

93. Cor. 2. But if any two Terms of such an Equation as has been mentioned, be assumed, and put equal to nothing, and from thence no possible Value of y can be deduced; or if a real Value can be deduced, but such a one that the Dimension of x it contains, being substituted in place of y through all the Terms, make the Dimensions of x in some higher, and in some lower than in the two assumed Terms, in either Case the two assumed Terms are improper from whence to derive the first Term of a converging Series, either when x is small or great. For in the first Case there is no real Value: and in the other Case where there is a real Value, it cannot be the first Term

of a converging Series, since the first Term of a converging Series must be deduced from those Terms only, which are much greater than all the rest : which the two Terms assumed would not be in this Case, whether you suppose x to be small or great : as easily appears from what hath been said above, especially in Lem. I. The same Thing is likewise evident of any two or more Terms in which the Dimension of y is the same : because the Value of y thence deduced is 0.

Thus in the last Equation $y^3 - a^2y + axy - x^3 = 0$, if you put the two Terms $y^3 + axy = 0$, you have $y = \sqrt{-ax}$, an *imaginary* Expression : therefore no first Term of a Series consisting of *real* Terms, can be thence deduced.

Suppose you put $axy - x^3 = 0$, thence you have $y = \frac{x^2}{a}$ by substituting of which Dimension of x through the Terms of the Equation, the Dimensions of x are these, x^5 , x^2 , x^3 , x^3 : where it appears that there is one Dimension of x higher, and another lower, than in the two assumed Terms : and therefore these two Terms are improper for deriving the first Term of a converging Series, either when x is very small ; or very great.

Finally, the two Terms $-a^2y + axy$ are improper : because the Value of y deduced from putting them equal to nothing is $y = 0$.

LEMMA III.

94. In any Equation including one *variable* or *flowing* Quantity only, as x , the rest being *constant* or *invariable* Quantities, all the Terms must include the same Dimension of x .^{*}

Dem. For, if possible, let $ax^4 + bx^4 + cx^2 + cx^2 = 0$ be an Equation whose Terms include different Dimensions of x , viz. x^4 and x^2 , and x at the same Time a variable or flowing Quantity : then divide all the Terms by the lowest Dimension of x , viz. x^2 , and it is $ax^2 + bx^2 + c = 0$: whence $x^2 = \frac{-c}{a+b}$ or $x = \sqrt{\frac{-c}{a+b}}$: therefore x is a constant Quantity, contrary to Hyp.

Universally let $ax^p + bx^q + cx^r = 0$ be an Equation including one variable Quantity x , which hath different Dimensions, if possible, and let x^r be the lowest Dimension, whether the others be equal Dimensions of x or not : divide by x^r , and you have $ax^{p-r} + bx^{q-r} + c = 0$: which being an Equation of a *definite* or certain degree, x can

^{*} In this Lemma it is supposed that the Coefficients a, b, c , &c. are not nothing.

only have a certain determinate Number of Values, which is contrary to the Nature of a *variable* Quantity; which admits of infinite different Values.

L E M M A IV.

95. Let ax^py^q , bx^ry^s , cx^ty^v , be three Terms of an Equation, which includes two unknown and *variable* Quantities x and y : and let these Terms be such, as that, when any Dimension of x , as x^n , is inserted for y in them, the Dimensions of x are equal: Moreover, when the three Terms $ax^py^q + bx^ry^s + cx^ty^v$ are put $= 0$, let a Value of y thence deduced be ax^n , I say $n = v$.

Dem. For if you insert ax^n for y in the Equation $ax^py^q + bx^ry^s + cx^ty^v = 0$, you shall have $aax^p+q + bax^r+s + cax^t+v = 0$: where x being a variable Quantity, it's Dimensions through all the Terms must be equal *; therefore $p + qn = r + sn$; whence $n = \frac{p-r}{s-q}$: but * Art. 94. by inserting x^n for y in the same three Terms, the Dimensions of x in them are x^{p+qn} , x^{r+sn} , x^{t+vn} : which by the Hyp. are equal, that is $p + qn = r + sn$ or $n = \frac{p-r}{s-q}$: therefore $n = v$. Q. E. D.

96. Cor. 1. The reasoning is the same if there be more than three Terms. And hence it follows, that, if there be three or more Terms of any proposed Equation, including two unknown Quantities x and y , which being put equal to nothing, give a possible Value of y , containing a certain Dimension of x ² such, as that the Dimension of x , being substituted for y through all the Terms of the given Equation, makes the Dimensions of x greater in all the other Terms, or less in them all, than in the three or more assumed Terms, the Value of y thence deduced will give the first Term of a converging Series, for a Value of y in the proposed Equation: according as x is supposed to be very little, or very great. But if the Dimensions of x rise higher in some Terms, and lower in other Terms, of the proposed Equation, than they do in the three or more assumed Terms, these assumed Terms are improper for deriving the first Term of a converging Series, either when x is very small, or very great.

97. Cor. 2. Let ax^py^q , bx^ry^s , cx^ty^v be three Terms selected out of the proposed Equation, which being put equal to nothing, give $y = Ax^n$, whether Ax^n be the first Term of a converging Series or not; then it shall be $p-r : s-q :: r-t : v-s$: or by Alternation, and

² Here $x^0 = 1$ is considered as a Dimension of x .

Composition $p-r : p-t :: s-q : v-q$: that is the Differences of the Indexes of x and y are proportional. For by inserting Ax^n for y , the Dimensions of x in the three Terms selected, are x^{p+qn} , x^{r+sn} , x^{t+vn} : which being equal *, you have 1°. $p+qn = r+sn$, 2°. $r+sn = t+vn$: from the first $n = \frac{p-r}{s-q}$; from the second $n = \frac{r-t}{v-s}$; whence $p-r : s-q :: r-t : v-s$. And the reasoning would be the same if the Terms were more than three. The Converse of which is likewise true, viz. If there be three or more Terms having the Indexes of x and y proportional, and if x^n be inserted for y in all of them, make the Dimension of x the same in any two of them, it will make it to be the same in all of them. For let $x^{p+qn} = x^{r+sn}$, then I say x^{t+vn} is equal to any of these: because since it is $p+qn = r+sn$, hence $n = \frac{p-r}{s-q}$: but $\frac{p-r}{s-q} = \frac{r-t}{v-s}$, whence $n = \frac{r-t}{v-s}$ or $vn - sn = r - t$, and by Transposition, $t + vn = r + sn$.

98. Cor. 3. If there be any Number of Terms more than two, of any proposed Equation, as the Terms ax^py^q , bx^ry^s , cx^ty^v , from which put equal to nothing, the first Term of a converging Series is derived, viz. Ax^n : then any two of these Terms being put equal to nothing, will give a Value of y in which the Dimension of x is the same as in Ax^n . For if you put any two of the Terms assumed, as $ax^py^q + bx^ry^s = 0$, you thence deduce $y = \frac{-a}{b} x^{\frac{p-r}{s-q}}$, where the Dimension of x is the same as in Ax^n by Lem. IV.

99. Cor. 4: From what has been said it appears, that, if any two Terms of a proposed Equation, being assumed, and put equal to nothing, a Value of y be thence deduced; and the Dimension of x contained in that Value, being substituted for y in all the Terms of the proposed Equation, make the Dimension of x the same in one or more Terms, as it is in the two assumed Terms; but higher in all the other Terms; or lower in all the other Terms; then such one or more Terms being joined to the two assumed Terms, and the Aggregate put equal to nothing, any real Value of y thence deduced, will make the first Term of a converging Series. Moreover any two Terms of the proposed Equation being taken, which belong to the Equation to be assumed for determining the first Term of a converging Series, you may always know if there are any more Terms to be joined with these two, from the Property of such Terms mentioned Art. 97. since the Differences of the Indexes of x and y must be proportional.

Examp.

Examp. Let the Equation $y^5 - 5xy^4 + \frac{x^2y^4}{a} - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$ be proposed: assume the two Terms $y^5 - 7a^2x^2y^2 = 0$, thence you have $y \pm \sqrt[5]{7 \times \sqrt{ax}}$, where the Dimension of x is $x^{\frac{1}{2}}$, which being substituted for y through the several Terms of the proposed Equation, makes the Dimensions of x these, viz. $x^3 \cdot x^{\frac{1}{2}} \cdot x^5$, $x^3 \cdot x^3 \cdot x^4$: wherefore you put the three Terms $y^5 - 7a^2x^2y^2 + 6a^3x^3 = 0$, and thence seek the first Term of a converging Series, which you will find to be fourfold, viz. $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, $-\sqrt{2ax}$, for all these are Roots of the Equation $y^5 - 7a^2x^2y^2 + 6a^3x^3$, as will appear by substituting them for y , one after another: therefore $+\sqrt{ax}$, or $-\sqrt{ax}$, or $+\sqrt{2ax}$, or $-\sqrt{2ax}$, may be the first Term of a converging Series, each of which Series's is a Root of the proposed Equation, when x is very little. Moreover you see that the Differences of the Dimensions of x and y are proportional in the three Terms $y^5 - 7a^2x^2y^2 + 6a^3x^3$ or $x^0y^5 - 7a^2x^2y^2 + 6a^3x^3y^0$, for $0 - 2 : 2 - 6 :: 2 - 3 : 0 - 2$ or $-2 : -4 :: -1 : -2$.

From the Symptoms or Properties of the Terms of any proposed specious Equation, including two unknown Quantities x and y , which are fit for deriving the first Term of a converging Series, which have been observed and demonstrated in the preceding Lemmas and Corollaries, there may be various *Methods* deduced for discovering such Terms.

METHOD I.

To be applied when there are two or more Terms of the proposed Equation, which involve not the Quantity x ; and one or more of which involve the Quantity y .

100. Put the Terms of the proposed Equation which include not the Quantity x , equal to nothing, giving each Term it's proper Sign: the Value of y deduced from this fictitious Equation, will be the first Term of a converging Series for the Value of y ; and if this fictitious Equation have more real Roots than one, each of these will give the first Term of a converging Series: when x is very small.

This is the Rule proposed by Sir *Isaac* himself Art. 30. See the Demonstration of it above, Art. 74. 91. and Examples in Art. 79. 80. and under Examp. 2. of Art 90.

But if y be of the same Dimension in all the Terms, which include not x , the first Term of a converging Series cannot be derived from them *.

* Art. 93. them *. And what may be done when all the Roots of the fictitious Equation are impossible shall be shewn afterwards.

M E T H O D II.

By which the first Terms of all the converging Series's may be discovered, which are the Roots of the proposed Equation, either when x is very small or very great.

101. This Method is by *Trial*, founded upon the Symptoms or Properties of the Terms to be selected for this Purpose, which have been explained and demonstrated above.

The Method of proceeding is this: Take the first Term of the proposed Equation, and join to it every one of the other Terms, each with it's proper Sign, one after another, forming so many fictitious Equations; from each of these seek the Value of y ; and observe the Dimensions of x which each of these Values contains: substitute the Dimension of x contained in the Value of y derived from the first of those fictitious Equations, instead of y , through all the Terms of the proposed Equation, and observe the Dimensions of x through all the Terms; and if you find them either higher in all the other Terms, or lower in all the other Terms, than the Dimension of x in the two assumed Terms, of which the first fictitious Equation was formed, then the Value of y deduced from that fictitious Equation, is the first Term of a converging Series*; or if there be more Values of y than one deducible from that fictitious Equation, each of them will be the first Term of a converging Series. But if, besides the two Terms forming the first fictitious Equation, there be any other, in which the Dimension of x is the same as in these two Terms, and it's Dimensions in all the rest of the Terms, either all higher or all lower, in that Case, the other Term or Terms, in which the Dimension of x is the same as in the two first assumed Terms, must be joined with them, all with their proper Signs, from which you are to form a new Equation, by putting the Aggregate equal to nothing; and the Root or Roots of it will give the first Term or Terms, of one or more converging Series's, which will be a Root, or Roots, of the proposed Equation*.

* Art. 96. 98. But if the Dimensions of x , after Substitution through all the Terms of the proposed Equation as above, be higher in some Terms, and lower in other Terms, than what the Dimension of x is in the two Terms of the first fictitious Equation, the assumed Terms are improper to derive the first Term of a converging Series, whether x be very little, or very great*.

* Art. 93. 96.

Terms

This done, you must next join the second Term of the proposed Equation with all the other Terms except the first; or perhaps such other as it was joined with formerly, in deriving the first Term of a converging Series; and thereby form so many new fictitious Equations, as there are distinct Terms to join it with: which are to be managed after the same manner as the former fictitious Equations. Then proceed to the third Term of the proposed Equation, and join it with the other Terms that follow it, one after another, with the exception just now mentioned; and use these fictitious Equations as the former ones. And thus proceed to do with all the remaining Terms; and so you shall have discovered the first Terms of all the Converging Series's, which express the Roots of the proposed Equation, *i. e.* the Values of y , either when x is very little; or very great.

The Demonstration of this Method is contained in the preceding Lemmas and Corollaries, particularly in the Articles referred to: And the following Example will illustrate it.

Ex. Let the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$ be proposed: I form an Equation by joining the first two Terms together, which gives $y^5 - by^2 = 0$, from whence you have $y = \sqrt[5]{b}$ or $\sqrt[5]{b} \times x^0$: where the Dimension of x being $x^0 = 1$, I substitute it for y in the Terms of the given Equation; and they will contain these Dimensions of x , *viz.* $x^0 \cdot x^0 \cdot x^2 \cdot x^3$: wherefore I conclude that $\sqrt[5]{b}$ is the first Term of a converging Series for y , when x is very small: since the Dimensions of x in the other Terms are greater than in the two first Terms, which were assumed.

But this I might have concluded immediately by the first Method.

Next I take the first and third Terms of the proposed Equation, and form this Equation $y^5 + 9bx^2 = 0$, from whence I have $y = \sqrt[5]{-9bx^2}$, in which the Dimension of x is $x^{\frac{2}{5}}$, which I substitute for y through all the Terms of the proposed Equation, from whence they have these Dimensions of x , *viz.* $x^2 \cdot x^{\frac{4}{5}} \cdot x^2 \cdot x^3$. where I observe that the Dimension of x in the second Term is less, and in the fourth Term greater than the Dimension of x in the first and third Terms, which were those assumed: wherefore I know those two Terms are improper for deducing the first Term of a converging Series, either when x is very small, or very great.

3°. I form an Equation of the first and last Terms, *viz.* $y^5 - x^3 = 0$: whence I have $y = x^{\frac{3}{5}}$; by substituting which Dimension of x for y , you have $x^3, x^{\frac{6}{5}}, x^2, x^3$: from which I conclude

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clude that $x^{\frac{1}{2}}$ is the first Term of a converging Series, when x is very great: because the Dimension of x in the two assumed Terms, is greater than in the other two.

And thus having joined the first Term of the proposed Equation with all the rest, I proceed to join the second Term with those that follow it; it having been joined with the first already. Wherefore by joining the second and third you form this Equation $-by^2 + 9bx^2 = 0$, whence I find $y = +3x$ or $-3x$; so that substituting this Dimension of x for y in the Terms of the proposed Equation, the Dimensions stand thus x^5, x^2, x^2, x^3 : whence, as before, I conclude that $+3x$ and $-3x$ are the first Terms of two converging Series's, each of which will be a Value of y when x is very small, in Comparison of b .

And now I join the second Term of the proposed Equation with the last, and thereby form this Equation, viz. $-by^2 - x^3 = 0$: in which I easily perceive that $y = \sqrt{-\frac{x^3}{b}} = \sqrt{-\frac{1}{b}} \times x^{\frac{3}{2}}$, which being the Square Root of a negative Quantity, supposing b to be positive, is impossible: from whence I conclude that these two Terms $-by^2 - x^3$ are unfit for finding the first Term of a converging Series, by themselves: and I know they cannot be joined with any other Term, so as to form with it an Equation fit for that Purpose, because that would have appeared in the foregoing Part of the Operation, either when the first and second Terms were assumed; or when the second and third Terms were assumed, for forming these fictitious Equations*.

* Art. 95.
98.

102. But if this had not appeared from the foregoing Part of the Operation, then I would have proceeded to insert $x^{\frac{1}{2}}$ for y through the Terms of the proposed Equation, in order to find whether there might not be some other Term, which joined with these two, might have been proper to form an Equation, admitting of some real Root, for the first Term of a converging Series; for although two Terms don't yield a real Root when they are alone, yet I cannot from that conclude that they may not be joined with some other Term, which with them may form an Equation with one or more real Roots, which will be the first Term or Terms of one or more converging Series's. Thus if the proposed Equation had been $-by^2 + 2b^{\frac{1}{2}}x^{\frac{1}{2}}y + y + 9bx^2 - x^3 = 0$, then by joining the two extreme Terms, which are the same with those just now mentioned in the foregoing Equation, you have $-by^2 - x^3 = 0$, whence $y = \sqrt{-\frac{1}{b}} \times x^{\frac{3}{2}}$, an impossible Value, when

when b is positive; yet I substitute $x^{\frac{1}{2}}$ for y through the Terms of the Equation, and I find the Dimensions these $x^3 \cdot x^3 \cdot x^{\frac{3}{2}} \cdot x^2 \cdot x^3$: whence I am directed to frame this Equation $-by^2 + 2b^{\frac{1}{2}}x^{\frac{3}{2}}y - x^3 = 0$, or by changing all the Signs $by^2 - 2b^{\frac{1}{2}}x^{\frac{3}{2}}y + x^3 = 0$, where y has a real Value, viz. $+\sqrt{\frac{x^3}{b}}$; which will therefore make the first Term of a converging Series expressing one of the Roots of the Equation $-by^2 + 2b^{\frac{1}{2}}x^{\frac{3}{2}}y + y + 9bx^2 - x^3 = 0$, viz. when x is supposed much greater than b .

103. And now I have shewn what will be the Result of forming Equations by joining together every two Terms of the proposed Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$, except the last two, from which it is plain no Value of y can be derived, since y doth not enter into any of them. And it appears that there may be three converging Series's when x is very little in Comparison of b , whose first Terms are $-b^{\frac{1}{5}} \cdot + 3x \cdot - 3x \cdot$; and one converging Series when x is very great in Comparison of b , whose first Term is $+x^{\frac{1}{5}}$.

104. Schol. But here I suppose the Reader will easily understand that the whole of this Trial, to know what Terms of any proposed Equation are fit to derive converging Series's from, in the most Part of Cases, may be made in his own Mind, without the Necessity of always writing down all the Steps.

Thus if the Equation $by^3 - 2b^{\frac{1}{2}}x^{\frac{3}{2}}y - y - 9bx^2 + x^3 = 0$ be proposed, I easily perceive, by running over the Terms in my own Mind, so as to join them two by two, that the only fictitious Equations that can serve for finding the first Terms of converging Series's, are 1°. $by^3 - y = 0$ *. 2°. $by^3 + x^3 = 0$. 3°. $-y - 9bx^2 = 0$; the first *Art. 100. and third when x is supposed to be very little; and the second, when it is supposed to be very great in Comparison of b ; which first Terms therefore are $+\frac{1}{\sqrt[5]{b}}$, $-\frac{1}{\sqrt[5]{b}}$, $-\frac{x}{\sqrt[5]{b}}$, and $-9bx^2$.

METHOD III.

By which, for most part, the first Terms of two converging Series's may be discovered, one of them for the Case of x being very small; the other for the Case of x being very great.

105. Let x^p represent the Dimension of x in that Term of the proposed Equation in which x is separately (*i. e.* without any Mixture of y) of the lowest Dimension ¹, and let $x^r y^s$ represent the Dimensions of x and y in every other Term of the Equation indifferently, one after another, (where r and s are supposed to put on as many different Values as the different Terms require) run over the several Terms, and observe where the Value of $\frac{p-r}{s}$ is greatest; then, that Term, or those Terms, if there be more of them, where this happens, being joined to the Term where you have the Dimension x^p , with their proper Signs, and put equal to nothing, the Root or Roots of this fictitious Equation will give the first Term or Terms of one or more converging Series's for the Value of y when x is very small.

Again if x^m represent the Dimension of x in that Term of the proposed Equation in which x is separately of the highest Dimension; and $x^r y^s$ represent the same thing as before, observe where the Expression $\frac{m-r}{s}$ is least of all, and such Term or Terms, being managed as before, will serve, in the same way, to discover the first Term or Terms of one or more converging Series's, when x is very great ².

Dem. Let x^p denote the same thing as in this Art. formerly; and let $x^r y^s$, $x^{r'} y^{s'}$, $x^{r''} y^{s''}$, &c. represent the Dimensions of x and y in the other Terms of the proposed Equation, where I have given the Letters r and s Marks of Distinction, according as they are supposed to denote the Indexes of x and y in the different Terms: Now I say if $\frac{p-r}{s}$ be the greatest of all the corresponding Expressions, that is $\frac{p-r'}{s'}$, $\frac{p-r''}{s''}$ &c.

¹ Here it is supposed that the Exponents of the Powers of x and y are positive: and that x^p therefore is the lowest Dimension of x in the proposed Equation.

² It must be observed that in computing the Value of the Expressions $\frac{p-r}{s}$ and $\frac{m-r}{s}$, a negative Value is less than any positive Value; and a greater negative is a less Value than a less Negative.

then

then the two Terms having x^p and $x^r y^s$ for their Dimensions, being put equal to nothing, are proper to derive the first Term of a converging Series from, when x is very small. For if these two Terms be put equal to nothing, the Value of y thence deduced, will contain

this Dimension of x , viz. $x^{\frac{p-r}{s}}$: put $\frac{p-r}{s} = n$; insert x^n for y in the other Terms of the Equation, and the Dimensions of x in them will be $x^{r'+ns'} + x^{r''+ns''} \&c.$ Now by Hyp. $n (= \frac{p-r}{s}) > \frac{p-r'}{s'}$ or $\frac{p-r''}{s''} \&c.$ wherefore, if you multiply by s' or $s'' \&c.$ you'll have $ns' > p - r'$; $ns'' > p - r'' \&c.$ (and this is true whether $n = \frac{p-r}{s}$ be positive or negative) and by Transposition $r' - ns' > p$, $r'' - ns'' > p$, $\&c.$ whence the thing is evident from Art. 90. And the Reader will easily understand, that, if there be any other Term, whose Dimensions of x and y are denoted by $x^t y^v$, such, that $\frac{p-t}{v} = \frac{p-r}{s}$, such Term must be joined with the other two, in forming the fictitious Equation for deriving the first Term of the converging Series *.

* Art. 97.

Again if x^m denote the highest Dimension of x separately, and the other Symbols be used in the same Sense as before; by putting the Terms having the Dimensions $x^m \cdot x^r y^s$, equal to nothing, you

have a Value of y whose Dimension of x is $x^{\frac{m-r}{s}}$; put $\frac{m-r}{s} = n$, then by the Hyp. $n < \frac{m-r'}{s'}$; multiply both by s' , and it is $ns' < m - r'$ (and that whether $n = \frac{m-r}{s}$ be positive or negative) wherefore, by Transposition, $r' + ns' < m$: and the rest of the Reasoning is as before, by Help of Art. 30. and 99.

Ex. 1. Let the Equation $y^3 - axy + x^3 = 0$ be proposed, to find the first Term of a converging Series, when x is very small in Comparison of a , then $p = 3$, and $m = 3$ likewise, $r = 1$. and $s = 1$, when applied to the middle Term $-axy$, and $r = 0$, $s = 3$, when applied to the first Term y^3 , so that the Values of $\frac{p-r}{s}$ or $\frac{m-r}{s}$ are $\frac{3-1}{1} = 2$, and $\frac{3-0}{3} = 1$; wherefore I put $-axy + x^3 = 0$ for finding the first Term of a converging Series, when x is very small; and $y^3 + x^3 = 0$, when x is very great: which give $y = + \frac{x^2}{a}$; and $y = -x$ respectively.

Ex.

Ex. 2. Let $y^3 - a^2y + axy - x^3 = 0$ be proposed: here you find, by the same Method, the Terms $-a^2y - x^3$ proper for deriving the first Term of a converging Series, when x is very small; and $y^3 - x^3$ when x is very great: which therefore are $-\frac{x^3}{a^2}$, and x respectively.

Ex. 3. Let the Equation $by^3 - 2b^{\frac{1}{2}}x^{\frac{3}{2}}y - y - 9bx^2 + x^3 = 0$ be proposed: here you have $-9bx^2$ the Term wherein x is of the lowest Dimension separately, and therefore $p = 2$, and the Term $-y$ joined to it, is that which gives the Expression $\frac{p-r}{s}$ the greatest, which is $\frac{2-0}{1} = 2$, therefore the Equation $-y - 9bx^2 = 0$ gives the first Term of a converging Series, when x is very little, viz. $y = -9bx^2$. Again $+x^3$ is the Term where x is separately of the highest Dimension, therefore $m = 3$, and the Term by^3 is that which compared with $+x^3$, gives the Expression $\frac{m-r}{s}$ the least, wherefore I put $by^3 + x^3 = 0$, and thence deduce $y = -\sqrt[3]{\frac{x}{b}}$, for the first Term of a converging Series, when x is very great.

136. Schol. 1. This Method supposes that the Exponents of the Powers or Dimensions of x and y , through all the Terms of the proposed Equation, are positive; otherwise the Method will not answer, without other Limitations; as will appear, if you suppose that, in the first Part of the Demonstration, $n = \frac{p-r}{s}$ is positive; and $\frac{p-r}{s}$ and s' both negative.

107. Scol. 2. The Expression $\frac{p-r}{s}$, is always the greater the less that r is, if s be the same in different Terms of the proposed Equation: and if r be the same in different Terms of the Equation, $\frac{p-r}{s}$ is always the greater the less that s be, when $p-r$ is positive; but the greater that s be, if $p-r$ be negative: which is of Use in the Practice of selecting the Terms, to be used for discovering the first Term of a converging Series, when x is very small; and the like Observations may be made with Respect to the Expression $\frac{m-r}{s}$, to direct you in the selecting of the Terms proper for finding the first Term of such a Series, when x is very great. So that what has been said under the Head of this third Method, serves to illustrate our Author's second Remark under Art.

EXPLAINED.

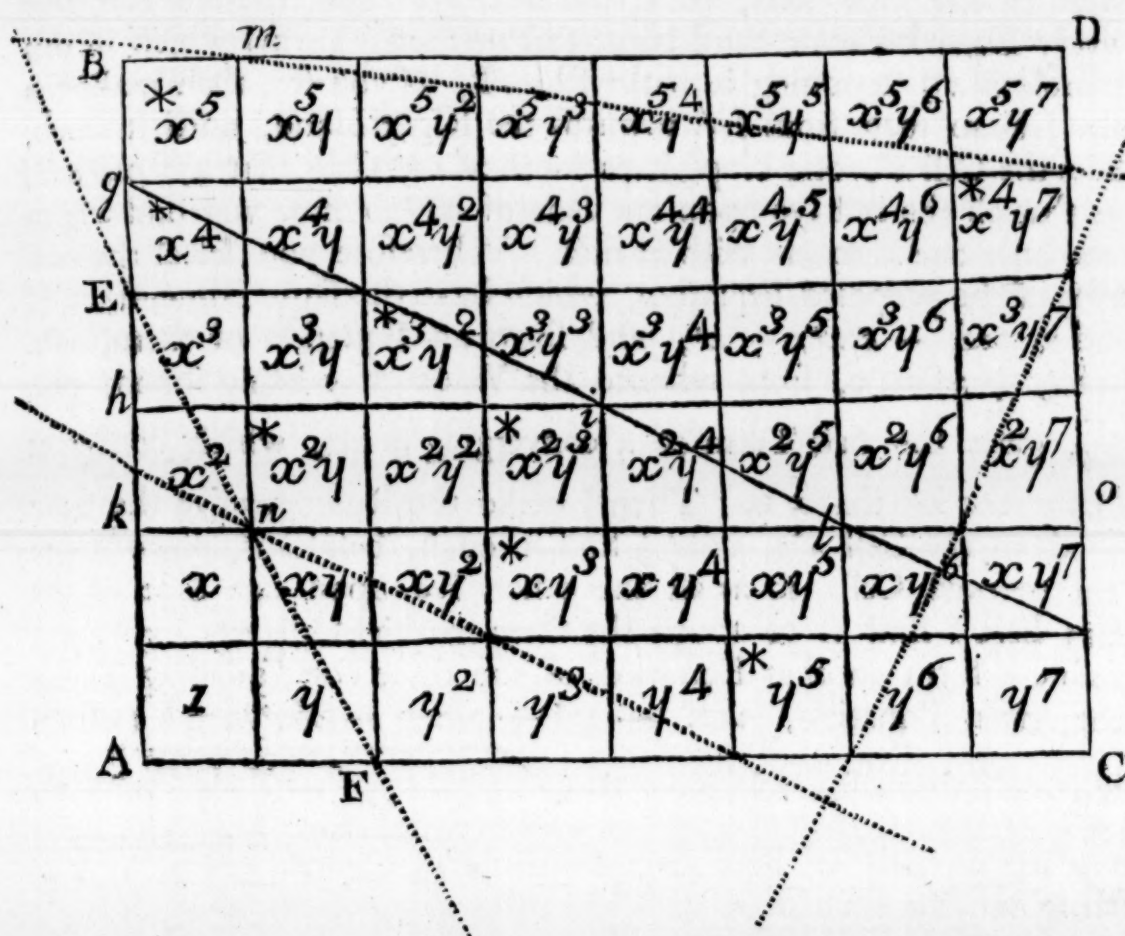
431

Art. 35. and, in general, his whole Method of proceeding in discovering the several Terms of the converging Series for the Value of y : which he calls the Quotient. And this leads us to

METHOD IV.

By Means of which the first Terms of all the converging Series's may be found which are Roots of the given Equation, whether x be very small, or very great.

108. This Method is by the help of a rectangular Parallelogram, ingeniously contrived by our celebrated Author.



Let AB, AC be two right Lines at right Angles to one another, divided each into some Number of equal Parts: and the Parallelogram AD being completed, through all the Points of Section of AB and AC draw Perpendiculars to these Lines, as you see, dividing the whole into equal Squares: which let be denominated from the Dimensions of

of x and y ; so that those lying along AB may contain the Dimensions of x separately, viz. $1 = x^0 \cdot x \cdot x^2 \cdot x^3 \cdot x^4$ &c. those along AC, the separate Dimensions of y , viz. $1 = y^0 \cdot y \cdot y^2 \cdot y^3 \cdot y^4$ &c. and every other Cell or Square, that Dimension of x that stands at the beginning of the same horizontal Row, and that Dimension of y that stands at the foot of the same perpendicular Row with itself, in the Manner the Figure represents. Then any Equation being proposed, as the Equation $x^4y^7 - 4y^5 + 2axy^3 - 2bx^2y^3 + 3x^3y^2 - abx^2y + x^4 + 2ax^5 = 0$, mark such of the small Squares or Cells, as correspond to the several Terms of the proposed Equation, according to the Dimensions of x and y they contain: which you see here done, by placing this Mark * in them: then let a Ruler as EF be applied to the lowest of the cells that, being marked, lye along the line AB, (for in every Equation there must be at least one such Term) which in this Example is x^4 , touching it in the Point E, as you see, then make the Ruler EF to turn from EA towards the Right Hand, until it touch, besides the Cell x^4 , one or perhaps more of the Cells that are marked, so as that all the rest lye upon one Side of it: so here you find x^2y to be the only one it meets with at first. Wherefore you select the two Terms of the proposed Equation, which have the same Dimensions of x and y , viz. $-abx^2y + x^4$, and form an Equation of them, viz. $-abx^2y + x^4 = 0$, from whence the Value of y being found, viz. $+\frac{x^2}{ab}$, gives the first Term of a converging Series, which converges the faster the less that x is. Then I make the Ruler revolve the same Way about the Cell x^2y , until I find it touch, besides it, one or more of the marked Cells, so as all the others lye upon one Side of the Ruler: here I find it to touch the three marked Cells $x^2y \cdot xy^3 \cdot y^5$: therefore I form another Equation of the three corresponding Terms of the given Equation, viz. $-4y^5 + 2axy^3 - abx^2y = 0$: from which you'll find a fourfold Root $+\sqrt{\frac{1}{4}a + \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, $+\sqrt{\frac{1}{4}a - \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, $-\sqrt{\frac{1}{4}a + \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, and $-\sqrt{\frac{1}{4}a - \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, any of which will be the first Term of a converging Series, when x is very small in Comparison of a and b .

Next I make the Ruler, according to the same Method, to touch the two marked Cells y^5 and x^4y^7 , so as to leave all the rest of the marked Cells

Cells upon the same Side of the Ruler, as you see by the Figure ; and so I form an Equation of the two corresponding Terms, viz.

$x^4y^7 - 4y^5 = 0$: whence you have $y = + \frac{2}{x^1}$ or $- \frac{2}{x^1}$, for the first Terms of other two Series's, which will also converge when x is very small.

Hitherto the Ruler has been made to touch the outward marked Cells below, so as to have all the rest still lying as it were above it : and as no more of the extreme marked Cells can be touched by it in this Manner, you have got all the fictitious Equations from which the first Terms of Series's can be formed, which converge when x is very small. And therefore you now begin to lay the Ruler to that marked Cell, which is the highest of such that lye along the Line A B, which here is x^5 , and making the Ruler to touch it above, let it revolve the contrary Way, and observe what extreme marked Cells can be touched by it at once, so as to leave all the others below it ; and these will afford fictitious Equations for finding the first Terms of Series's, which will converge the faster the greater that x is : and of such I find only one, formed of the two Terms, which correspond to the two Cells x^5 and x^4y^7 : therefore I put $x^4y^7 + 2ax^5 = 0$, from whence you have $y = \sqrt[7]{2a \times x^1}$, which will be the first Term of a converging Series, wherein the Dimensions of x shall continually diminish, corresponding to the Case of x being very great.

Here you may conceive the Ruler beginning to revolve from the Position E A towards E F, until it has made a half of an entire revolution, making it to touch the outward marked Cells below, as it revolves ; and so you shall find the Terms of the proposed Equation, which are proper for deriving the first Terms of all those Series's, which converge the faster the less that x is. (And again, if you conceive the Ruler, beginning at the point B, and lying in the direction of the Side A B of the Parallelogram, to turn about the contrary Way to what it did before, until it shall have turned about 180 Degrees, so as to make it touch the extreme marked Cells above, always leaving all the rest entirely upon one Side of it, and as it were below it, you shall thereby obtain the Terms of the proposed Equation, which are proper for forming fictitious Equations from which the first Terms of all such Series's are to be derived, which converge when x is very great ; that is in which the Dimensions of x continually decrease. The same Thing may be conceived to be done by drawing a right lined Figure, every one of whose Sides shall touch two or more of the outward marked Cells, so as to leave the rest entirely within the Figure, in the Manner represented in the Diagram itself.

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109. The

109. The Demonstration of these Things may be easily made out from a due Consideration of the Parallelogram, and what has been formerly said. In order to which, all that is necessary to be done, is to shew that, if the Cells which are touched by the lower Sides of the right-lined Figure, that is the Sides lying betwixt BA and CD upon the lower Sides of the Figure, be put equal to nothing; or (which, in this Case, is the same) the Terms of the proposed Equation, which correspond to these Cells, and the Dimension of x in the Value of y thence deduced, be inserted for y in all the rest of the marked Cells, they shall contain higher Dimensions of x than those Cells which the Line touches, after the Dimension of x is inserted in place of y in them also: and that these last mentioned shall contain the same Dimension of x .

And after the like Manner, that, if the Cells which are touched by the upper Sides of the right-lined Figure above, *i. e.* the Sides which lye betwixt BA and DC upon the upper Side of the Figure, be put equal to nothing, the Dimension of x contained in the Value of y thence deduced, being inserted instead of y in all the marked Cells, will produce lower Dimensions of x in them all, than in those which the Side of the Polygon touches: in which last the Dimensions of x

* Art. 90, shall be the same *.
96.

Now in order to shew this, let it be considered that if any Number of Cells are touched by any Line, as the Cells x^4 , xy^2 , and y^2 are by the Line EF ; and as the Cells x^2y , xy^3 and y^5 are touched by the next Side of the Polygon, and so for any others, whatever way the Line touch them, whether above or below, and you take any Number of such Cells more than two, and that in any Order or at any Distances from each other, the Differences of the Indexes of x and y will be proportional, in the Sense of Art. 97. For let the right Line gil touch the Cells x^5 , x^4y^2 , x^3y^4 , x^2y^6 upon the same Side: compare the Indexes of x and y in the three Cells x^5 , x^3y^4 , x^2y^6 together, and you find $5-3:4-0::5-2:6-0$. and so it is universally: for let the Cells x^5 , x^3y^4 , x^2y^6 be touched by the Line gil in the Points g , i , l , and let ib , lk , be the horizontal Lines drawn through i and l , meeting AB in the Points b and k , then the Triangles gbi , gkl being similar, you have $gb:bi::gk:kl$: but gb and gk measure the Differences of the Indexes of x , and bi and kl the Differences of the Indexes of y , as is evident by considering the Diagram: therefore the Differences of the Indexes of x and y are proportional, in the Terms x^5 , x^3y^4 , x^2y^6 . And the Reasoning is the same, in every other Case.

110. From

110. From this it follows, that, if any Dimension of x , being inserted in place of y in all the Cells which are similarly touched by any right Line, make the Dimensions of x in two of these Cells equal, then the Dimensions of x in them all shall be equal *.

• Art. 97.

111. Again, if a right Line touch any Number of Cells similarly below them (as the Line EF touches the Cells x^4 , x^2y , y^2) then the Dimension of x , mentioned in the last Art. viz. that which being inserted for y in the touched Cells, makes them, or the Dimensions of x in them, equal; when it is inserted for y in the Cells that lye *entirely* above the Line EF , shall give higher Dimensions of x . For take any of the Cells which the Line EF touches below, as x^2y , which it touches in the Point n : let nm and no be the perpendicular and horizontal Lines upwards and towards the right Hand, then it will be evident, that all the Cells contained within the right Angle mno have either a greater Dimension of x , or a greater Dimension of y , than x^2y has; or both a greater Dimension of x and y : and the like must hold with respect to every other Cell that is similarly touched by EF , or by any other right Line which slopes downward from left to right: and these Cells which are contained within such angular Spaces, are all that lye *entirely* above the Line: but in this Case the Value of y that arises by putting any two of the Quantities contained in two of the touched Cells, equal to nothing, is a positive Dimension of x , as will be evident to any one that considers the Construction of the Parallelogram: therefore, when this positive Dimension of x is inserted for y in the other Cells which lye *entirely* above EF , the Dimension of x must be higher in every one of them.

But if the Line or Ruler, still touching below, slope downward from right to left, the Value of y which arises by putting any two of the Quantities, contained in two of the Cells which it touches similarly, equal to nothing, must contain a negative Dimension of x , as will be evident to one that considers it; and all the Cells contained within the right Angle which proceeds from any of the touched Cells towards the left Hand and upwards, contain either a higher Dimension of x with the same Dimension of y , or a lower Dimension of y with the same Dimension of x ; or then, both a higher Dimension of x and a lower Dimension of y ; in all which Cases, the negative Dimension of x inserted for y in such Cells, must make the Dimension of x higher in them than in the touched Cell: and this being true with respect to every one of the Cells, which are touched by a Line sloping downwards from right to left, hence the same Thing appears in this Case likewise, viz. that the negative Dimension of x , which inserted for y in the Cells

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which

which the Ruler touches similarly below, makes the Dimensions of x the same in them, will make the Dimensions of x higher in all the Cells, which lye entirely above the Line, when it is inserted for y in them.

112. And therefore, this being universally true, it plainly appears that the Cells which correspond to the Terms of any given Equation, which are touched by the lower Sides of the Polygon, must point out the Terms of the Equation, which are proper for deriving the first Terms of those Series's, which converge when x is very small. And by a like Way of Reasoning it may be demonstrated that the upper Sides of the Polygon placed as has been explained, must point out those Terms of the proposed Equation, which serve to discover the first Terms of those Series's which converge, when x is very great: which were the Things we proposed to shew.

* Art.
100.

113. Schol. 1. In the foregoing Demonstration I have not particularly mentioned the Cases in which the Ruler, touching any of the extreme marked Cells, either lies parallel to AB ; or to AC : because in the first Case, it's plain no Value of y can be deduced from the fictitious Equation *; and the Reasoning in the other Case is most apparent.

* Art. 80.

114. Schol. 2. It may be observed likewise that Sir *I. Newton* makes use of the rectangular Parallelogram, only for discovering the Terms of any Equation, which are proper for finding the first Term of a converging Series, that will converge when x is very small; because he reduces all other Cases to this one *. And in particular, he uses it only for finding the first Term of such a converging Series, as may be deduced from the Supposition of the Ruler touching that Cell which contains the lowest Dimension of x separately, represented in the Diagram by the Line EF touching the Cells x^4 and x^2y^1 : which, I suppose, he doth, partly that he might not perplex his Readers with too great a Variety of Cases; and likewise because he uses the same uniform Method in finding the first, and all the other subsequent Terms of the Quotient or Value of y , as we shall see just now.

Of the different Methods of finding the subsequent Terms of the Quotient or converging Series, for the Value of y .

115. The first Term of the Quotient being found by any of the preceding Methods, the subsequent Terms may be discovered in diffe-

* See his Method of Fluxions and infinite Series, Introd. Art. 29, 30. and Com. Epist. No. 63.

rent Ways : one of which is by Means of the Parallelogram already described. Thus, in any proposed Equation, having found the first Term of any Root or Value of y , when x is very small, to be Ax^n , you put $y = Ax^n + p$; and then transform the proposed Equation of y , by substituting $Ax^n + p$ for y , whence arises a new Equation, *viz.* that of p : which you now suppose to come in place of y : then mark those Cells of the Parallelogram, which correspond to the Terms of this secondary Equation: and laying a Ruler to the lowest of these Cells which lye adjacent to the Side AB , make it to revolve towards the right Hand and upwards, until it touch one or more of the marked Cells besides the first, in the Manner already explained; the Terms of the Equation of p , which correspond to the Cells which the Ruler touches, are to be feigned equal to nothing, and an approximate Value of p thence deduced, which will give the second Term of the Quotient: which let be denoted by Bx^{n+m} : then you suppose $p = Bx^{n+m} + q$, and transforming the former Equation of p into a new one, by substituting $Bx^{n+m} + q$, in place of p , you must find an approximate Value of q , by Means of the Parallelogram, after the same Manner as you find an approximate Value of p already, *i. e.* by applying the Ruler to the lowest marked Cell adjacent to AB . And so you proceed to find the subsequent Terms of the Quotient as far as you please to produce it. By proceeding thus, the Dimensions of x which are in the marked Cells upon the left Hand, adjacent to AB , will continually increase, as will be evident to one that considers it, and therefore you will continually approach to the true Root, by diminishing constantly the Supplement, p , q , &c. *

* Art. 75.

116. Schol. In this Method of continuing the Terms of the Quotient, although you might find out the first Term variously, by different Applications of the Ruler; yet you find all the subsequent Terms in the same uniform Manner, *viz.* by applying the Ruler always to the lowest marked Cell adjacent to the Line AB : the Supplement p , q , r , &c. being supposed to be very small; as well as x . I have only considered the Case of a Root which converges when x is very small; because all other Cases may be reduced to this: and this Method of finding the subsequent Terms of the Quote cannot be so properly applied to the Case of x being very great: upon the account of the increasing negative Indexes of x .

117. But whereas the Parallelogram contains no negative nor fracted Powers of x and y , therefore it is proper to observe that, when any such occur, either in the Equation proposed at first, or in any of the secondary or supplementary Equations arising in the Progress of the Operation,

Operation, they ought to be removed, before you proceed, by such Methods as are practised in Algebra: thus, if we had the Equation $x^3 + 3x^2y^{-1} - 2x^{-1} - 16y^{-3} = 0$, you free it from the negative Dimensions of x and y by multiplying by the positive Dimensions, which correspond to the highest negative Dimension, viz. x and y^3 , by which Means it becomes $x^4y^3 + 3x^3y^2 - 2y^3 - 16x = 0$. After the same Manner the Equation $x = \frac{aa}{y} - \frac{2a^3}{y^2} + \frac{3a^4}{y^3}$ must be converted into $xy^3 = a^2y^2 - 2a^3y + 3a^4$ or $xy^3 - a^2y^2 + 2a^3y - 3a^4 = 0$. Which change being made, either upon the primary, or any of the secondary Equations, when necessary, the Operation will proceed.

118. Likewise in the Case of fractional Indexes, remove them by making a proper Substitution: thus if the Equation $y^{\frac{1}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}} = 0$, were proposed, put $y^{\frac{1}{2}} = v$ and $x^{\frac{1}{2}} = z$, and by Substitution, you shall have $v^6 - z^3v + z^4 = 0$, out of which having extracted the Root v , you replace $y^{\frac{1}{2}}$ for v and $x^{\frac{1}{2}}$ for z : By extracting the Root v you would find $v = z + z^3 + 6z^5 + 51z^7$ &c. that is $y^{\frac{1}{2}} = x^{\frac{1}{2}} + x + 6x^{\frac{3}{2}} + 51x^{\frac{5}{2}}$ &c. and by squaring both Sides, $y = x^2 + 2x^{\frac{3}{2}} + 13x^2 + 114x^{\frac{3}{2}}$ &c.

119. Moreover when any complicate Fractions, or furd Quantities, happen to occur, either in the proposed Equation; or to arise afterwards, they ought to be removed by the Methods used in common Algebra.

Thus if the Equation $y^3 + \frac{a^2}{a-x}y^2 - x^3 = 0$, was proposed, you must free it of the complicate Fraction, by multiplying by $a-x$, whence arises $ay^3 - xy^3 + a^2y^2 - ax^3 + x^4 = 0$, out of which you extract the Root y : or you might substitute $\frac{v}{a-x}$ for y , and thence would arise $\frac{v^3}{a-x} + \frac{a^2v^2}{a-x} - x^3 = 0$, or $v^3 + a^2v^2 - a^3x^3 + 3a^2x^4 - 3ax^5 + x^6 = 0$: from whence having found the Value of v , divide it by $a-x$, and the Quotient gives the Root y .

Likewise the Equation $y^3 - \sqrt{a-x} \times y^2 + x^3 = 0$ being given, to extract the Root y , you may substitute z for $\sqrt{a-x}$, whence you will have $y^3 - zy^2 + a^3 - 3a^2z^2 + 3az^4 + z^5 = 0$, out of which having extracted the Root y , replace $\sqrt{a-x}$ for z . Thus also the Equation $y^3 - \sqrt{a-x} \times y^2 - \sqrt{a-x} \times x^3 = 0$ being given, put $y = \frac{v}{\sqrt{a-x}}$, and thence arises $v^3 - av^2 + xv^2 - a^2x^3 + 2ax^4 -$

x^5

$x^5 = 0$, the Root of which last Equation being extracted, divide it by $\sqrt{a - x} = a^{\frac{1}{2}} - \frac{1}{2}a^{-\frac{1}{2}}x - \frac{1}{8}a^{-\frac{3}{2}}x^2 \&c.$ and so you shall have a Value of y . Understand also that in the other Example which precedes, if the Area of the Curve be required, you should find the Expression of that Area in Terms of x before you replace $\sqrt{a - x}$ for x .

The Reduction and Preparation of Equations, mentioned in this Article, is always necessary in all the Methods of extracting the Roots of affected Equations, by Series; but those mentioned in the two preceding Articles are not always necessary in the other Methods; but are so in that which proceeds by Means of the Parallelogram.

120. Another Method for discovering the subsequent Terms of the Root, is that of our Author's, which he delivers in this Treatise, and which was formerly explained *. But because there are some Things * Sect. 4. in it about which the Reader may still require some further Satisfaction; and which may be of use in Practice, I shall illustrate it a little further. Wherefore let $ay^m + bx^ny^r + cx^sy^t \&c. + bx^v = 0$ be a general Equation with indefinite Exponents, the Root y being proposed to be extracted, when x is supposed to be very small; to which all the other Cases may be reduced *: in which there will be one Term, * Art. 86. at least, which do not include x , represented here by ay^m , and another one at least, that includes not y , which is here represented by $+ bx^v$: let the Terms $cx^sy^t + bx^v$ be those from which, put equal to nothing, the first Term of the Root may be deduced; found by any of the

preceding Methods; which therefore is $\sqrt[m]{\frac{-b}{c}} x^{\frac{v-m}{r}}$: put $\frac{v-m}{r} = n$,

then $y = \sqrt[m]{\frac{-b}{c}} x^n + p$: let the Equation be transformed, as follows,

by the Help of the Binomial Theorem; omitting the numeral Coefficients,

$$\begin{aligned} + ay^m &= + a \times \sqrt[m]{\frac{-b}{c}} x^n + a \times \sqrt[m]{\frac{-b}{c}}^{m-1} x^{n-m} p + a \times \sqrt[m]{\frac{-b}{c}}^{m-2} x^{n-2m} p^2 \&c. \\ + bx^ny^r &= + b \times \sqrt[r]{\frac{-b}{c}} x^{n+r} + b \times \sqrt[r]{\frac{-b}{c}}^{r-1} x^{n+r-n} p + b \times \sqrt[r]{\frac{-b}{c}}^{r-2} x^{n+r-2n} p^2 \&c. \\ + cx^sy^t &= + c \times \sqrt[t]{\frac{-b}{c}} x^{n+s} + c \times \sqrt[t]{\frac{-b}{c}}^{t-1} x^{n+s-n} p + c \times \sqrt[t]{\frac{-b}{c}}^{t-2} x^{n+s-2n} p^2 \&c. \\ &\&c. \quad \&c. \\ + bx^v &= + bx^v \end{aligned}$$

By

By considering of which the following Things will appear :

1°. That in the transformed Equation, viz. the Equation of p , those Parts of the ultimate Term ¹ which are collateral to the Terms
 * Art. 75. $bx^v + cx'y'$ in the given Equation, must destroy one another. *
 Rem. 1.

2°. These same Parts of the ultimate Term contain a lower Dimension of x , than any of the other Parts of that ultimate Term do : because they are the Quantities which arise by inserting the Dimension of x , which the Value of y contains, deduced from the fictitious Equation $bx^v + cx'y' = 0$, instead of y , in the several Terms of the proposed Equation *.

* Art. 88. 3°. The Dimensions of x in the several Quantities lying in the same horizontal Line of the Equation, constantly diminish by the same Difference ; and those of p increase by the same Difference ; so that, abstracting from the Coefficients, these Quantities go on in a continued Proportion, by the common Multiplier $x^{-n}p$: which is also the same in all the Lines. This appears from the Nature of Involution of Powers.

4°. From the last two Observations it follows, that the Part of the

Penult Term, viz. $c \times \frac{\overline{b}}{c} \times x^{v-n}p = cx^{v-n}p$, contains a lower Dimension of x , than any of the other Parts of that Penult

Term : that the Part $c \times \frac{\overline{b}}{c} \times x^{v-2n}p^2$ of the antepenult Term upon that same Line contains a lower Dimension of x than any other Part or Quantity belonging to the Antepenult Term, &c. And moreover that, since the approximate Value of p ; or the first Term of that Value, must contain a higher Dimension of x than x^n , which is it's Dimension in the first Term of the Quotient or Root, expressing the Value of y *, therefore, if the Dimension of x contained in that approximate Value of p , were substituted for p through all the Quantities, belonging to the Equation of p , the Dimensions of x will increase, and that by the same common Difference in all the horizontal Lines, reckoning the Terms from the left Hand to the right : consequently

* Art. 73.

5°. If $+ b \times \frac{\overline{b}}{c} x^{q+m}$ be that Part of the Ultimate Term, which, after throwing by those Parts which destroy each other, contains

¹ See Note 1. Art. 75.

the lowest Dimension of x , the approximate Value of the Supplement p must be found from the fictitious Equation $cx^{v-n}p + b \times \frac{-b}{c} x^{q+r} : \text{and the same is to be understood of all the other secondary or supplementary Equations. Therefore you have this general Rule for finding all the subsequent Terms of the Value of } y \text{ after the first, viz. Let that Part of the ultimate Term of every supplementary Equation, which contains the lowest Dimension of } x, \text{ having it's Sign changed, be divided by the Coefficient of } p \text{ in that Part of the Penult Term, where } x \text{ is of the lowest Dimension, and that shall be the next Term of the Quote to be annexed to what hath been formerly found. The Observation of which, together with a Rule formerly given *, will mightily facilitate the } \bullet \text{ Art. 32. Extraction of the Roots of such specious Equations.}$

6°. The Coefficient of $p, q, r, \&c.$ in the Part of the Penult Term of any supplementary Equation, by which you divide, as in the last Remark, contains the same invariable Dimension of x throughout all these Equations: as will be evident if you consider, that, in the next supplementary Equation, viz. that of q , it will be found in the Transformation of the Equation of p , opposite to, or upon the same Line with, the Quantity $cx^{v-n}p$: so that if that Coefficient contain the Dimension x^{v-n} in the first supplementary Equation, it will contain the same in all the subsequent Ones.

7°. The Difference betwixt the Exponents of x in the two first Terms of the Series expressing the Root, is the same with the Difference of the Exponents of x , in the Parts of the ultimate Term of the Equation of p , in which x is of the lowest Dimension, and of the

next higher, which are here bx^v and $b \times \frac{-b}{c} x^{q+r}$: For the difference of the Dimensions of x in these Parts is $q + rn - v$: and the Value of the first Term of p , deduced from the fictitious Equ-

ation $+ c \times x^{v-n}p + b \times \frac{-b}{c} x^{q+r} = 0$, contains this Power of x , viz. $x^{q+r-n-v+n}$, whose Exponent exceeds the Exponent of x^n , which is the Power of x contained in the first Term of the Series, by $q + rn - v$, the same as the former: and the Reasoning is the same, although the Terms of the proposed Equation from which, the first Term of Series is derived, be any other than $cx^v + bx^v$.

121. Although we have hitherto spoke of the converging Series, or Quotient, expressing the Value of y , as if the Exponents of the Powers of x in the Terms of the Series proceeded in an Arithmetical Progression; yet the Reader must know that, albeit this most frequently happens, yet it is not always so: for which Reason I shall resolve the following

P R O B L E M,

To find whether the Exponents of the Powers of x in the Terms of the converging Series go on in an Arithmetical Progression, or not: and if so, what the common Difference of that Progression is.

122. Having found the Dimension or Power of x contained in the first Term of the Series, (*viz.* when x is very small) by any of the preceding Methods, which let be x^n , substitute x^n instead of y through all the Terms of the proposed Equation; and let the Exponents of the Powers of x after Substitution be $A, A, B, C, E, \&c.$ of which A is the least (and therefore found at least in two Terms) and the others $B, C, E, \&c.$ of which B is the least; C the next; E the next, $\&c.$ in an ascending Scale: then if $B - A$ measure $C - B, E - C, \&c.$ throughout all the rest, the Exponents of x in the Terms of the Series proceed in an Arithmetical Progression, whose common Difference is $B - A$: but if $B - A$ do not exactly measure all the Differences $C - B, E - C, \&c.$ the Dimensions of x in the Terms of the converging Series, will not constitute an Arithmetical Progression. Which will be plain by Examples.

Ex. 1. Take the Equation $y^3 + a^2y - 2a^3 + axy - x^3 = 0$, (which is the first of our Author's) by supposing $y^3 + a^2y - 2a^3 = 0$ *, we have a or ax^0 for the first Term of the Root; whose Dimension of x , *viz.* $x^0 = 1$, being put for y , in the Terms of the proposed Equation, the Exponents of x in the several Terms shall be 0, 0, 0, 1, 3, where the Difference betwixt 0 and 1, *viz.* 1. measures the Difference betwixt 1 and 3, *viz.* 2, therefore I conclude that the Exponents of the Powers of x in the Terms of the Quotient or Series, proceed in an Arithmetical Progression, whose Difference is 1.

Ex.

Ex. 2. Let the Equation $x^2y^4 + c^2y^4 - x^4y^2 + c^4x^2 - c^6 = 0$ be proposed: by feigning $c^2y^4 - c^6 = 0$ *, you find $+c$ for the first Term of a converging Series; wherefore substitute x^0 for y , you have these Exponents of the Powers of x , viz. 2, 0, 4, 2, 0; which reduced to Order, and the coincident ones thrown into one, gives this Progression of distinct Terms 0, 2, 4, where $2 - 0 (= 2)$ measures $4 - 2 (= 2)$ therefore the Exponents of the Powers of x in the Terms of the Root, go on in an Arithmetical Progression, having 2 for the common Difference.

* Ex. 3. Let the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$ be proposed: then you may form the first Term of the Root either from the fictitious Equation $y^5 - by^2 = 0$; or $-by^2 + 9bx^2 = 0$ *: the first gives $y = \sqrt[3]{b}$ or $\sqrt[3]{b} \times x^0$; wherefore x^0 being inserted every where for y , the different Exponents of x in the several Terms, will be 0. 2. 3: here 2 being the Difference betwixt the least Exponents, measures not the Difference $3 - 2 = 1$, therefore the Exponents of x in the Terms of the Series will not proceed in an Arithmetical Progression.

But if the first Term of the Root be found from the other fictitious Equation $-by^2 + 9bx^2 = 0$; it is $+3x$ or $-3x$: where the Dimension of x , viz. x^1 , being inserted for y gives these Exponents of x in the Terms of the proposed Equation, viz. 5. 2. 3; which shews that $3 - 2 = 1$, measuring $5 - 3 = 2$, the Exponents of x in the Terms of this Root, will constitute an Arithmetical Progression, whose common Difference is 1.

Ex. 4. If $y^4 + x^3y - ax^3 - a^3x = 0$ be proposed, the first Term of the Root may be derived from putting $y^4 - a^3x = 0$ *, which will be $a^{\frac{3}{4}}x^{\frac{1}{4}}$; insert $x^{\frac{1}{4}}$ for y , and the Dimensions of x are 1. $\frac{1}{4}$. 3. where the Difference betwixt the two least 1 and $\frac{1}{4}$, doth not measure the Difference betwixt 3 and $\frac{1}{4} = 3\frac{3}{4}$, therefore the Exponents of x in the Terms of the Quotient will not proceed in an Arithmetical

Progression: for the Root is $a^{\frac{3}{4}}x^{\frac{1}{4}} + \frac{x^{\frac{9}{4}}}{4a^{\frac{1}{4}}} - \frac{x^{\frac{13}{4}}}{4a^{\frac{5}{4}}} - \frac{3x^{\frac{17}{4}}}{32a^{\frac{9}{4}}} \&c.$

The Process of which, according to this Method, I shall here set down for the Reader's Satisfaction. Putting then $y = a^{\frac{3}{4}}x^{\frac{1}{4}} + p$, we have as follows:

$$\begin{aligned}
 +y^4 &= +a^{\frac{1}{2}}x + 4a^{\frac{3}{2}}x^{\frac{1}{2}}p + 6a^{\frac{5}{2}}x^{\frac{3}{2}}p^2 + 4a^{\frac{7}{2}}x^{\frac{5}{2}}p^3 + p^4 & 4a^{\frac{3}{2}}x^{\frac{1}{2}}p - ax^3 &= 0 \text{ * fere} \\
 +x^3y &= +a^{\frac{1}{2}}x^{\frac{3}{2}} + x^3p & p &= +\frac{x^{\frac{3}{2}}}{4a^{\frac{1}{2}}} + q \text{ * Art. 120. Rem. 5.} \\
 -ax^3 &= -ax^3 \\
 -a^{\frac{1}{2}}x &= -a^{\frac{1}{2}}x
 \end{aligned}$$

$$\begin{aligned}
 +6a^{\frac{3}{2}}x^{\frac{1}{2}}p^2 &= +\frac{3x^5}{8a} + 3a^{\frac{1}{2}}x^{\frac{11}{2}}q \text{ *} \\
 +4a^{\frac{5}{2}}x^{\frac{3}{2}}p &= +\cancel{ax^3} + 4a^{\frac{3}{2}}x^{\frac{3}{2}}q \\
 -ax^3 &= -\cancel{ax^3} \\
 +a^{\frac{1}{2}}x^{\frac{1}{2}} &= +a^{\frac{1}{2}}x^{\frac{1}{2}}
 \end{aligned}$$

$$(4a^{\frac{3}{2}}x^{\frac{1}{2}} + 3a^{\frac{1}{2}}x^{\frac{11}{2}}) - a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{3x^5}{8a} \left(-\frac{x^{\frac{1}{2}}}{4a^{\frac{1}{2}}} - \frac{3x^{\frac{1}{2}}}{32a^{\frac{1}{2}}} \right) \&c. = r.$$

$$y = a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{4a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{4a^{\frac{1}{2}}} - \frac{3x^{\frac{7}{2}}}{32a^{\frac{1}{2}}} \&c.$$

In this Process the Reader will observe, that I have omitted all those Quantities which cannot affect the Quotient within the Limit of that Power of x which is denoted by $x^{\frac{1}{2}}$ inclusive, which Quantities may be easily known and distinguished by what was formerly shewn *, especially when joined with the Remarks in Art. 120.

* Art. 82.

123. Having illustrated this Problem by Examples, I come now to give the Demonstration of it. Wherefore let $ay^m + bx^ny + cx^py$ &c. $+ bx^v = 0$ represent any specious Equation as formerly: from which the first Term of the Series being found by any of the preceding Methods, let the Equation be transformed into the first supplemental Equation, called that of p ; in which Equation of p , let the Exponents of the Powers of x in the ultimate Term be A, A, B, C, E,

Ult.	Penult.	Antepen.	&c.
A	* A + D	* A + 2D	* A + 3D &c.
A	A + D	A + 2D	A + 3D &c.
B	B + D	B + 2D	B + 3D &c.
C	C + D	C + 2D	C + 3D &c.
E	E + D	E + 2D	E + 3D &c.

* Art. 75.
Rem. 1.

&c.

&c. as in the Margin: of which two at least must be equal, which are here represented by A and A: the Parts of the ultimate Term corresponding to which, do always destroy one another *. Then suppose

pose the Exponent of the Power of x , which is contained in the first Term of the Value of p , found as above, to be D : so that after Substitution of such Power of x instead of p in the several Terms of the transformed Equation, viz. that of p , the Exponents of the Powers of x in the Penult Term may be $A + D$, $B + D$, $E + D$ &c. in the Antepenult $A + 2D$, $B + 2D$, &c. as in the Margin: for where there are Parts of these Terms of the Equation of p upon any horizontal Line, the Exponents of x in them, must be such as here represented, as appears by Remark 3 and 4, Art. 120. Wherefore it is evident, that, when the Equation of p is transformed into the second supplementary Equation, viz. that of q , the ultimate Term of this new Equation can contain no other Exponents of x but $A + D$, $A + 2D$, $A + 3D$, &c. B , $B + D$, $B + 2D$ &c. C , $C + D$, &c. E , $E + D$ &c. But in the Equation of p , after throwing by A , the Exponent B is supposed to be the least of all the remaining ones B , C , E , &c. in the ultimate Term, therefore $A + D = B^*$; consequently $A + 2D = B + D$, $A + 3D = B + 2D$ &c. so that in the Equation of q , there shall be no other Exponents of x in the ultimate Term but what may be designed thus, B , $A + D$, $A + 2D$, $A + 3D$, &c. C , $C + D$, $C + 2D$, &c. E , $E + D$, $E + 2D$, &c. as in the Margin; of which B and $A + D$ being equal, the Parts of the ultimate Term corresponding

*Art. 120.
Rem. 4, 5.

to these must destroy one another, as before. Now after throwing by these, let $A + 2D$ be the lowest Quantity in the ultimate Term, then it is evident that opposite to $A + D$, in this Equation of q , there must be a Part or Quantity belonging to the penult Term; but no Part belonging to any Term preceding the penult, because it is the Part of the Equation of q , which arises by the Transformation of the Part of the penult Term of the

Ult.	Penult.	Antepen.
B		
$A + D$	$A + 2D$	
$A + 2D$	$A + 3D$	$A + 4D$
$A + 3D$	$A + 4D$	$A + 5D$ &c.
&c.	&c.	&c.
C		
$C + D$	$C + 2D$	
$C + 2D$	$C + 3D$	$C + 4D$
&c.	&c.	&c.
E		
$E + D$	$E + 2D$	
$E + 2D$	$E + 3D$	$E + 4D$
&c.	&c.	&c.

Equation of p , which corresponds to $A + D$, in which the Supplement p is in the first Power. And if you suppose that Power of x to be inserted for q , which is contained in the approximate Value of q found as above, the Exponent of x in this Part of the penult Term, must

* Ibid. must be $A + 2D$, viz. the same as it is in that Part of the ultimate Term where it is lowest *: consequently the Exponents of x in the other Parts of the Penult Term, where there are such Parts, must exceed the Exponents of x in the corresponding Parts of the ultimate Term, by the same Difference D , after Substitution for q : and the Exponents of x in the Antepenult Term, must exceed those in the Penult by the same Difference still, viz. D ; where there are such Parts belonging to the Antepenult Term: and so on for the rest of the Terms *, as you see represented in the Margin. And thus, by supposing the Equation of q to be transformed into the new supplemental Equation, viz. that of r ; and so the Transformation to go on continually, it appears that the ultimate Term of every such Equation *in inf.* can contain no other Exponents of x but what fall in with some one or other of the Arithmetical Progressions $A, A + D, A + 2D, A + 3D, \&c. C, C + D, C + 2D, \&c. E, E + D, \&c.$ and that therefore if the first Progression $A, A + D = B, \&c.$ fall upon $C, E, \&c.$ the lowest Exponent of x in the ultimate Term of any supplemental Equation, will exceed the lowest Exponent of x in the ultimate Term of the immediately preceding supplemental Equation, by the constant Difference $D = B - A$: therefore the Exponents of the Powers of x in the converging Series for the Root y of the proposed Equation, ought likewise to proceed in an Arithmetical Progression (by Rem. 6. above) and that having the same Difference D or $B - A$ (by Rem. 7.)

But if the Arithmetical Progression $A, A + D = B, A + 2D, \&c.$ do not fall upon the Terms $C, E, \&c.$ contained in the ultimate Term of the Equation of p , it appears from what has been said, that the Exponents of x , in the Terms of the Series which expresses the Root y , will not proceed in an Arithmetical Progression; but the Difference of the Exponents, after certain Intervals, become less.

And therefore, since the ultimate Term of the Equation of p , is the very same with what arises by substituting the first Term of the converging Series in place of y , throughout the Terms of the proposed Equation, the Thing we proposed is demonstrated.

123. Schol. It makes no Difference in the Demonstration, from whatever Terms selected out of the proposed Equation, the first Term of the Series be deduced, provided they be proper Terms for this Purpose: although the preceding Demonstration represents the first Term

* Art. 105. as deduced by the third Method preceding *.

124. Moreover it may be proper to advertise the Reader, that, although the Difference $D = B - A$ do measure the other Differences $C - B,$

C — B, E — C, &c. yet it may happen, that the known Coefficient of one or more Terms of the Series for the Root of the proposed Equation, may become equal to nothing : which may happen from certain Parts of the ultimate Term of any secondary or supplemental Equation destroying one another, sooner than they ought to do : which may make a Chasm in the Arithmetical Progression of the Exponents of x in the Series for the Root of the proposed Equation, when that happens; but the Exponents of the Powers of x through all the Terms of the Series, even in such a Case, must belong to the Arithmetical Progression mentioned in this Problem. And the Problem must therefore be understood to be taken with this Limitation.

125. And it may likewise happen that all the Parts, or Quantities, constituting the ultimate Term of any supplemental Equation, may totally destroy one another : in which Case, the Series for the Root of the proposed Equation terminates ; and that Root is fully expressed by the Terms of the Quotient already found : since the remaining Supplement vanishes *.

* Art. 75.
Rem. 3.

126. Besides the two Methods I have* already mentioned, and explained, for extracting the Roots of affected specious Equations, containing two variable and unknown Quantities, by Means of infinite Series ; there is a third, which I shall explain a little, lest the Reader should find Fault with the omitting of it entirely. It consists in the Assumption of a Series of Terms with indeterminate Coefficients, in which the Exponents of the Powers of x proceed in an Arithmetical Progression, either ascending or descending, according as x is supposed to be very small, or very great, in Comparison of the other known Quantities in the given Equation. Such a Series may be represented thus $Ax^n + Bx^{n+p} + Cx^{n+2p} + Dx^{n+3p}$, &c. which Series is supposed to express the Root of the proposed Equation ; and therefore to be equal to y . And in order to determine the Values of the Coefficients A, B, C, D, &c. the proposed Equation is transformed into a new Equation, by substituting $Ax^n + Bx^{n+p} + Cx^{n+2p}$, &c. instead of y ; and the Powers of it instead of the like Powers of y , in the Terms of the proposed Equation ; whence there arises a new Equation, including only one variable Quantity, viz. x : of which Equation all the Terms together being equal to nothing ; each Term by itself must be equal to nothing ; that is the complex Coefficient of each Term must be equal to nothing : so that by putting the complex Coefficient of each Term equal to nothing, there will arise so many Equations, by the Resolution of which, the Values of the indeterminate Coefficients A, B, C, &c. may be found : and consequently the Root of the proposed

proposed Equation; provided the assumed Series be taken of a right Form.

127. To illustrate the Thing by an Example, let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$ be proposed; to find y : then assume $y = A + Bx + Cx^2 + Dx^3$, &c. and let the given Equation be transformed as you see

$$\begin{array}{rcl}
 + y^3 & = & + A^3 + 3A^2Bx + 3A^2Cx^2 + 3A^2Dx^3 + 3A^2Ex^4 \text{ \&c.} \\
 & & + 3AB^2 + 6ABC + 6ABD \\
 & & + BBB + 3ACC \\
 & & + 3BBC \\
 + a^2y & = & + a^2A + a^2Bx + a^2Cx^2 + a^2Dx^3 + a^2Ex^4 \text{ \&c.} \\
 + axy & = & * + aAx + aBx^2 + aCx^3 + aDx^4 \text{ \&c.} \\
 - 2a^3 & = & - 2a^3 \\
 - x^3 & = & * \quad * \quad * \quad - x^3
 \end{array} \quad \left. \vphantom{\begin{array}{l} + y^3 \\ + a^2y \\ + axy \\ - 2a^3 \\ - x^3 \end{array}} \right\} = 0.$$

In which transformed Equation, the complex Coefficient of each Term must be equal to nothing, otherwise the Whole could not be equal to nothing, as we shall see by and by: therefore, by putting these Coefficients equal to nothing, there arises so many Equations, viz. 1°. $A^3 + a^2A - 2a^3 = 0$. 2°. $3A^2B + a^2B + aA = 0$. 3°. $3A^2C + a^2C + 3AB^2 + aB = 0$. 4°. $3A^2D + a^2D + 6ABC + B^3 + aC - 1 = 0$, &c. From the first of which find the Value of A : which Value of A being inserted for it in all the rest of the Equations; you may, from the second, find the Value of B : which Value of B being inserted for it in all the subsequent Equations, you may, from the third, find the Value of C ; and so on *in inf.* From the first Equation $A^3 + a^2A - 2a^3 = 0$, you find $A = +a$: from the second you will find $B = -\frac{1}{4}$: from the third, $C = +\frac{1}{64a}$: from the fourth, $D = +\frac{131}{512a^2}$, &c. therefore, by substituting these Values of A, B, C, D , &c. instead of them in the assumed Series $y = A + Bx + Cx^2 + Dx^3$, &c. We have $y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c.

* Art. 78. the same as formerly *.

128. The chief Difficulty in this Method is to find the right Form of a Series: for although the Series $A + Bx + Cx^2 + Dx^3$, &c. was that which we assumed in the preceding Example; and which indeed is the right Form for finding the Root of the Equation there proposed; yet in other Examples, the Series to be assumed, may be of a different

a different Form. And it's evident, that the Difference of the Forms depends upon these two Things, viz. 1°. The Dimension or Power of x in the first Term of the Series. 2°. The Difference betwixt the Exponents of the Powers of x in any two Terms of the Series that are next to another. The first of these Things may be found by any of the Methods that were formerly explained for discovering the first Term of a converging Series, which shall be the Root of the proposed Equation.

129. The second Thing, viz. the common Difference of the Exponents, may be found thus: insert the Power or Dimension of x contained in the first Term of the Series, in place of y , in the several Terms of the proposed Equation; then set down all the different Exponents of x (among which 0 is included) in order, which let be a, b, c, e , &c. find the greatest Number, which will exactly measure all the Differences $b - a, c - b, e - c$, &c. call it p ; then is p the Difference of the Indexes of x in the Terms of the converging Series: so that if n be the Exponent of x in the first Term of the Series, the right Form of the Series to be assumed shall be $Ax^n + Bx^{n+p} + Cx^{n+2p} + Dx^{n+3p}$, &c. The Thing will appear plainer by Examples.

Ex. 1. Let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, mentioned just now, be proposed: then by feigning $y^3 + a^2y - 2a^3 = 0$, we have $y = +a$ or $+ax^0$; which is the first Term of the Root, when x is very small *: next insert x^0 instead of y in the Terms of the proposed Equation, and the Indexes of the Powers of x are 0, 0, 1, 0, 3; so that the different ones, reduced into order, are 0, 1, 3, whose Differences 1, 2, have 1 for their greatest common Measure: therefore $A + Bx + Cx^2 + Dx^3$, &c. is the proper Form of the Series to be assumed. * Art. 100.

Ex. 2. Let the Equation $x^4y^7 - 4y^5 + 2axy^3 - 26x^2y^3 + 3x^3y^2 - abx^2y + x^4 + 2ax^5 = 0$, be given: by putting the Terms $-abx^2y + x^4 = 0$, you will have $+ \frac{x^2}{ab}$ for the first Term of a converging Series, when x is very small *: wherefore, insert x^2 for y , through the several Terms of the proposed Equation; and the Exponents of x will be these, 18, 10, 7, 8, 7, 4, 4, 5, that is the different ones, reduced to order, are 4, 5, 7, 8, 10, 18, whereof the Differences are, 1, 2, 1, 2, 8, whose greatest common Measure is 1: therefore the Form of the Series is $Ax^4 + Bx^5 + Cx^7$, &c. * Art. 105. 108.

But if the proposed Equation remaining the same, you deduce the first Term of the Series from the fictitious Equation $-4y^5 + 2axy^3 - abx^2y$
M m m — abx^2y

— $abx^2y = 0$, you'll find it to be $+\sqrt{\frac{1}{4}a} \pm \sqrt{-\frac{ab}{4} + \frac{aa}{16}} \times x^{\frac{1}{2}}$ or

*Art. 108. — $\sqrt{\frac{1}{4}a} \pm \sqrt{-\frac{ab}{4} + \frac{aa}{16}} \times x^{\frac{1}{2}}$ *, which gives the first Term of the Series, fourfold; but $n = \frac{1}{2}$ in all of them: and by inserting $x^{\frac{1}{2}}$ for y , the Exponents of the Powers of x will be $7\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4, 2\frac{1}{2}, 4, 5$; that is the different ones when reduced to Order, are $2\frac{1}{2}, 3\frac{1}{2}, 4, 5, 7\frac{1}{2}$; whose Differences are $1, \frac{1}{2}, 1, 2\frac{1}{2}$ or $\frac{1}{2}, 1, 2\frac{1}{2}$, whose greatest common Measure is $\frac{1}{2}$: Therefore the proper Form of a Series for any of these Roots will be $Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2$, &c.

Ex. 3. Let the Equation $y^3 + 5ay^2 - 34a^2y + x^2y - 80a^3 = 0$ be proposed: by assuming the fictitious Equation $y^3 + 5ay^2 - 34a^2y - 80a^3 = 0$, and resolving it, you'll find it to have these three Roots $+5a, -2a, -8a$; any of which will be the first Term of a converging Series, when x is very small *: therefore insert x^0 for y , and the Indexes of the Powers of x in the proposed Equation, which are different, are none other but 0 and 2, whose Difference 2 has 2 for it's greatest measure: whence it follows that $A + Bx^2 + Cx^4 + Dx^6$ &c. is the true Form of the Series. Which Example, as to the Method of Operation, is here represented to the Reader.

$$\begin{aligned}
 +y^3 &= +A^3 + 3A^2Bx^2 + 3A^2Cx^4 + 3A^2Dx^6 + 3A^2Ex^8 \text{ \&c.} \\
 &\quad + 3AB^2 + 6ABC + 6ABD \\
 &\quad \quad + BBB + 3ACC + 3B^2C \\
 +5ay^2 &= +5aA^2 + 10aABx^2 + 10aACx^4 + 10aADx^6 + 10aAEx^8 \text{ \&c.} \\
 &\quad + 5aBB + 10aBC + 10aBD \\
 &\quad \quad + 5aCC \\
 -34a^2y &= -34a^2A - 34a^2Bx^2 - 34a^2Cx^4 - 34a^2Dx^6 - 34a^2Ex^8 \text{ \&c.} \\
 +x^2y &= * + Ax^2 + Bx^4 + Cx^6 + Dx^8 \text{ \&c.} \\
 -80a^3 &= -80a^3
 \end{aligned}
 \quad = 0$$

Where, by putting the complex Coefficients of each Term of the transformed Equation, equal to nothing, you'll have 1^o $A^3 + 5aA^2 - 34a^2A - 80a^3 = 0$. 2^o $3A^2B + 10aAB - 34a^2B + A = 0$. 3^o $3A^2C + 3AB^2 + 10aAC + 5aB^2 - 34a^2C + B = 0$; and so on, for the Coefficients of the subsequent Terms. The first of these Equations, viz. $A^3 + 5aA^2 - 34a^2A - 80a^3 = 0$, is the same with

with the fictitious Equation at first assumed for finding the first Term of the Series, viz. $y^3 + 5ay^2 - 34a^2y - 80a^3$ (which will always be the Case) therefore A will have the same Values here, as y had there; that is $A = + 5a$ or $- 2a$ or $- 8a$: any of which may be taken, according to the Root that is proposed to be extracted. Let the positive Value $+ 5a$ be taken: which therefore must be substituted in Place of A, in the subsequent Equations for discovering the Values of B, C, D, &c. by substituting it for A in the second Equation, viz. $3A^2B + 10aAB - 34a^2B + A = 0$, it becomes $75a^2B + 50a^3B - 34a^2B + 5a = 0$, whence $B = -\frac{5}{91a}$. Again, by substituting these Values of A and B, in Place of them, in the third Equation for determining C, viz. $3A^2C + 3AB^2 + 10aAC + 5aB^2 - 34a^2C + B = 0$, it becomes $75a^2C + \frac{375}{8281a} + 50a^2C + \frac{125}{8281a} - 34a^2C - \frac{5}{91a} = 0$; whence you have $C = -\frac{45}{753571a^3}$. After the same Manner, by substituting these known Values of A, B and C, in the fourth Equation, you may discover the Value of D, viz. $D = +\frac{6470}{91a^5}$; and by Means of the fifth Equation, in like Manner, you would find $E = +\frac{971855}{91a^7}$: and so on *in inf.* Therefore, by inserting these Values of A, B, C, &c. in the Equation $y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8$ &c. $= 0$, we obtain $y = 5a - \frac{5x^2}{91a} - \frac{45x^4}{91a^3} + \frac{6470x^6}{91a^5} + \frac{971855x^8}{91a^7}$ &c. which is an Approximation to one of the Roots of the proposed Equation, when x is very small.

130. The Demonstration of this Practice may be easily derived from what was demonstrated with respect to the preceding Method*: For ^{*Art. 122.} when any specious Equation, including two unknown Quantities x and ^{123.} y , is proposed, to find the Value of y in a converging Series; of which Series Ax^m is the first Term; and Ax^m is inserted for y through the Terms of the given Equation; if the assumed Series $Ax^m + Bx^{m+p} + Cx^{m+2p}$ &c. be of such a Form, as that the Dimension of x in every one of the Terms of the proposed Equation (after substituting Ax^m for y in them) fall in with the Dimension of x in some Term of the assumed Series: I say if this be the Case, the young Geometrician will easily understand, from what hath been shewn in the foresaid Place, that there can be no Term of the Root of the proposed Equation, but such, whose Dimension of x must fall in with the Dimension of x

in some Term or other of the assumed Series; since p , which is the common Difference of the Exponents of x in the assumed Series, is a common Measure of the Differences of the Dimensions of x in the Terms of the proposed Equation, after Substitution of Ax^n for y in them: and since p is the greatest common Measure of these Differences, there cannot be any other Series assumed, having the Exponents of x in an arithmetical Progression, whose common Difference is greater, which shall have these Properties. Consequently, if the Values of the indeterminate Coefficients $A, B, C, D, \&c.$ in the assumed Series can be found, which we have shewn how to do, by the Examples we have adduced, then the several Terms of the Root of the proposed Equation, will thereby be found. Which is all that is required.

131. All that the young Geometrician can require to know further, for his Satisfaction with respect to the Grounds of this Method of extracting the Roots of Equations, by the *Assumption* of a Series with indeterminate Coefficients is, 1°. Why the Exponents of x in the Terms of the transformed Equation, must constitute an arithmetical Progression having the same common Difference as the Exponents of x in the Terms of the assumed Series: which is absolutely necessary for finding the Values of $A, B, C, D, \&c.$? This Question will be easily satisfied, by considering the Nature of the Involution of Powers: by which it appears, that, if there be any Quantity consisting of any Number of Terms, finite or infinite, in which the Exponents of the Powers of any Letter as x , are in arithmetical Progression, any *perfect* Power of that Quantity, as the Square, Cube, &c. will consist of Terms in which the Exponents of x will constitute an arithmetical Progression having that same Difference: so that any Power of $Ax^n + Bx^{n+p} + Cx^{n+2p} \&c.$ or even any Power of it multiplied by any Power of x , will constitute a Series of Terms, in which the Exponents of x will make an arithmetical Progression, having the same Difference as in $Ax^n + Bx^{n+p} \&c.$ viz. p . From which it follows, that the Exponents of the Powers of x , in the several horizontal Lines of the transformed Equation, as in the preceding Examples, will proceed by the same common Difference in all, where there are more Terms than one upon any such Line: and where there is but one, the Dimension of x in it, must fall in with the Dimension of x in some one, or more Quantities, of the other horizontal Lines; since the Progression of the Exponents of x , in the Series $Ax^n + Bx^{n+p} + Cx^{n+2p} \&c.$ falls upon the Exponent of x in such Terms by the Hyp. Which Considerations, *by themselves*, may furnish a Proof of this Method.

132. The

132. The other Thing he may want to know (which I likewise promised to shew) is this: how it appears that, in any Equation such as $\overline{a-b} \times x^n + \overline{c-d} \times x^p + \overline{e-f} \times x^q \&c. = 0$, in which x is a variable or flowing Quantity, and $a, b, c, d, e, f, \&c.$ constant Quantities, if the Terms contain different Powers of x , the Coefficients of every one of the Terms, that is $a-b, c-d, e-f, \&c.$ must be equal to nothing? The Thing appears thus: divide all the Terms of the Equation, by the lowest Power of x , which let be x^n , and you'll have $\overline{a-b} + \overline{c-d} \times x^{p-n} + \overline{e-f} \times x^{q-n} = 0$: which being true whatever be the Value of x , it is true when x vanishes: wherefore, let x vanish, and then it will become $a-b=0$: therefore $\overline{c-d} \times x^{p-n} + \overline{e-f} \times x^{q-n} \&c. = 0$: consequently, by the same way of reasoning, we prove that $c-d=0$: and so on for all the rest: as was to be shewn.

133. The Reader will easily perceive that sometimes the Values of some of the indeterminate Coefficients $B, C, D, \&c.$ may be equal to nothing: which will always happen when the Exponents of x in the Terms of the Series for the Root, do not constitute an arithmetical Progression: which we shew'd how to discover formerly. In which Cases, there may perhaps Methods be fallen upon to abridge the Operation, by the Assumption of Series's with different Properties, when the Root is proposed to be carried only to a certain Limit: but upon this, and some other useful Things relating to this Method, I will not now insist, lest I should be thought to be too particular. Only the Reader may observe, that this Method of resolving specious Equations, serves for extracting the Roots, as well when the Species x is supposed very great, as when it is supposed very small: and the same Rules will serve for determining the Form of the Series to be assumed; only the Exponents of x in the Numerators of it's Terms must constitute a descending arithmetical Progression: thus $Ax^n + Bx^{n-p} + Cx^{n-2p} \&c.$ where n may be positive or negative, and p always positive. The Operation, in one Example, I shall here set down. Let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, be proposed, to find the Value of y in a converging Series, when x is very great: the same Equation, which hath been proposed formerly, particularly under this third Method. I take the fictitious Equation $y^3 - x^3 = 0$, from whence I find $+x$ for the first Term of the Root, and likewise of the Series to be assumed: and by substituting x^1 for y in the proposed Equation, I find 1 for the greatest common Measure of the Exponents of x : whence I know it must be $y = Ax + Bx^{-1} + Cx^{-2} \&c.$ Wherefore

$+y^3$

$$\begin{array}{rcl}
 + y^3 & = & + x^3 + 3Bx^2 + 3Cx + 3Dx^0 + 3Ex^{-1} \quad \&c. \\
 & & + 3B^2 + 6BC + 6BD \\
 & & + BBB + 3CC \\
 & & + 3B^2C \\
 \\
 + a^2y & = & * \quad * \quad + aax + a^2Bx^0 + a^2Cx^{-1} \quad \&c. \\
 + axy & = & * \quad + ax^2 + aBx + aCx^0 + aDx^{-1} \quad \&c. \\
 - 2a^3 & = & * \quad * \quad * \quad - 2a^3x^0 \\
 - x^3 & = & - x^3
 \end{array}
 \quad \left. \vphantom{\begin{array}{rcl} + y^3 \\ + a^2y \\ + axy \\ - 2a^3 \\ - x^3 \end{array}} \right\} = 0.$$

In which transformed Equation, 1 is inserted instead of A, because it was found already that $Ax = 1x$: then for deriving the Values of B, C, D, E &c. we have these Equations, 1°. $3B + a = 0$. 2°. $3C + 3B^2 + aa + aB = 0$. 3°. $3D + 6BC + B^3 + aaB + aC - 2a^3 = 0$. 4°. $3E + 6BD + 3C^2 + 3B^2C + a^2C + aD = 0$. &c. by resolving of which, you'll find 1°. $B = -\frac{a}{3}$. 2°. $C = -\frac{a^2}{3}$. 3°. $D = +\frac{55a^3}{81}$. 4°. $E = +\frac{64a^4}{243}$, &c. whence you have $y = (Ax + Bx^{-1} + Cx^{-2} + Dx^{-3} + Ex^{-4} \&c.) = x - \frac{a}{3x} - \frac{a^2}{3x^2} + \frac{55a^3}{81x^3} + \frac{64a^4}{243x^4} \&c.$ the Root sought.

134. It may likewise be observed, that, in the foregoing Methods of resolving Equations into infinite Series, the Reduction of complex Fractions and Radicals into infinite Series, is *virtually* included. Thus if it were required to reduce the Fraction $\frac{aa}{b+x}$ into an infinite Series of simple Terms, you suppose $y = \frac{aa}{b+x}$, or (by multiplying by $b+x$) by $+xy = a^2$: from which you extract the Value of y . After the same Manner if it were proposed to reduce $\sqrt{aa - xx}$ into an infinite Series, you might put $y = \sqrt{a^2 - x^2}$, that is $y^2 = a^2 - x^2$: and if $\sqrt[3]{a - x}$ were proposed, you may put $y = \sqrt[3]{a - x}$, whence $y^3 = a - x$ or $y^3 = a^3 - 3a^2x + 3ax^2 - x^3$: and so the Values of y being found by the Resolution of Equations into Series, you'll obtain the Values of the Radicals in such a Series as was proposed.

Art.
43-46.

135. The preceding Doctrine of the Extraction of the Roots of specious Equations, may be conveniently applied to that Operation which goes under the Name of the *Reversion of Series*: which is nothing else but this: the Value of one of the indefinite Quantities, e.g. y , being exprest by a Series of simple Terms, including different Powers of the

the other indefinite Quantity x , thence, by a kind of *reverse* Operation, the value of x is found, exprest in a Series of simple Terms, including different Powers of the Quantity y . By Means of this Operation, the Areas or Lengths of Curves being given in a converging Series, the Bases or Abscisses, and consequently the Ordinates, of the Curves may be found: as our Author shews in Art. 43—46: likewise from the Logarithm given, we return to the Ratio of which it is the Logarithm.

Let CD (see the Fig. in the Author Art. 43) be an equilateral Hyperbola, AB = x , a Part of one of the Asymptotes, so that the Distance of the Point A from the Centre be called 1; AC and BD being two Ordinates: then the Area ABDC is the hyperbolical Logarithm of the Ratio of AC to BD, that is of $1 + x$ to 1: which Area, as has been shewn formerly, is $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \&c.$ so that, if you put l for the known Logarithm of the Ratio of $1 + x$ to x , that is, if you put $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \&c. - l = 0$, then by the preceding Methods, you may extract the Root x , after the Manner Sir Isaac teaches in this Art. which is nothing else but the Application of his Method of resolving specious Equations, applied to this particular Case: by which you find $x = l + \frac{1}{2}l^2 + \frac{1}{6}l^3 + \frac{1}{24}l^4 + \frac{1}{120}l^5 \&c.$ so that the Ratio, of which l was the given Logarithm is that of $1 + l + \frac{1}{2}l^2 + \frac{1}{6}l^3 + \frac{1}{24}l^4 \&c.$ to 1, that is according to the common way of speaking, the Logarithm of the Number $1 + l + \frac{1}{2}l^2 + \frac{1}{6}l^3 \&c.$ But if the Logarithm that is given be the common Tabular Logarithm, you must first reduce it to the hyperbolical Logarithm by multiplying by 2.30258 &c. which Doctrine we explained at fuller Length in our Notes upon the Quadrature of Curves.

Sir Isaac having set before the Reader the Reversion of this Series, by the Method he lays down for extracting the Roots of Equations in this Treatise; I shall shew how it is done by the Assumption of a Series with indeterminate Coefficients. Since x is supposed to be very small, you'll find by Art. 128, 129, that the Form of the Series must be $x = A/l + B/l^2 + C/l^3 + D/l^4 \&c.$ wherefore transform thus:

$$\begin{array}{l} -l = -l \\ +x = +A/l + B/l^2 + C/l^3 + D/l^4 + E/l^5 \&c. \\ -\frac{1}{2}x^2 = * -\frac{1}{2}A^2/l^2 - AB/l^3 - \frac{1}{2}AC/l^4 - AD/l^5 \&c. \\ +\frac{1}{3}x^3 = * * +\frac{1}{3}A^3/l^3 + A^2B/l^4 + A^2C/l^5 + AB^2 \&c. \\ -\frac{1}{4}x^4 = * * * -\frac{1}{4}A^4/l^4 - A^3B/l^5 \&c. \\ +\frac{1}{5}x^5 = * * * * +\frac{1}{5}A^5/l^5 \&c. \end{array} \quad \left. \vphantom{\begin{array}{l} -l = -l \\ +x = +A/l + B/l^2 + C/l^3 + D/l^4 + E/l^5 \&c. \\ -\frac{1}{2}x^2 = * -\frac{1}{2}A^2/l^2 - AB/l^3 - \frac{1}{2}AC/l^4 - AD/l^5 \&c. \\ +\frac{1}{3}x^3 = * * +\frac{1}{3}A^3/l^3 + A^2B/l^4 + A^2C/l^5 + AB^2 \&c. \\ -\frac{1}{4}x^4 = * * * -\frac{1}{4}A^4/l^4 - A^3B/l^5 \&c. \\ +\frac{1}{5}x^5 = * * * * +\frac{1}{5}A^5/l^5 \&c. \end{array}} \right\} = 0.$$

Whence

Whence you obtain so many Equations, for determining the Values of the Coefficients A, B, C, &c. viz. 1°. $A - 1 = 0$. 2°. $B - \frac{1}{2}A^2 = 0$. 3°. $C - AB + \frac{1}{3}A^3 = 0$, &c. by reducing of which you'll find $A = 1$, $B = \frac{1}{2}$, $C = \frac{1}{6}$, $D = \frac{1}{24}$, $E = \frac{1}{120}$ &c. and therefore $x = 1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5$ &c. in inf.

136. There is, moreover, another ingenious Method, which our Author makes use of for the Reversion of Series: which is this¹: let the Equation $z = t - \frac{1}{3}\frac{t^3}{r^2} + \frac{1}{5}\frac{t^5}{r^4} - \frac{1}{7}\frac{t^7}{r^6} + \frac{1}{9}\frac{t^9}{r^8}$ &c. be proposed, which was given by Mr. James Gregory²: by which the Length of the Arch of a Circle is found, from the Radius r and Tangent t being given, the Arch being called z : and it is required, conversely, to find the Tangent t from the Arch z given, in a converging Series extending to the 9th Power of z . In order to this, let the given Series $t - \frac{1}{3}\frac{t^3}{r^2} + \frac{1}{5}\frac{t^5}{r^4}$ &c. be involved to the 1st, 3d, 5th, 7th and 9th Powers, as follow:

$$\begin{aligned} z &= + t - \frac{1}{3}\frac{t^3}{r^2} + \frac{1}{5}\frac{t^5}{r^4} - \frac{1}{7}\frac{t^7}{r^6} + \frac{1}{9}\frac{t^9}{r^8} \text{ \&c.} \\ z^3 &= * + t^3 - \frac{t^5}{r^2} + \frac{4}{5}\frac{t^7}{r^4} - \frac{8}{9}\frac{t^9}{r^6} \text{ \&c.} \\ z^5 &= * * + t^5 - \frac{5}{3}\frac{t^7}{r^2} + \frac{1}{9}\frac{t^9}{r^4} \text{ \&c.} \\ z^7 &= * * * + t^7 - \frac{7}{3}\frac{t^9}{r^2} \text{ \&c.} \\ z^9 &= * * * * + t^9 \end{aligned}$$

Here I observe, that, if the second Line were multiplied by $\frac{1}{3r^2}$, and then added to the first, it would make the second Term of the Series, viz. $-\frac{1}{3}\frac{t^3}{r^2}$ to disappear: by doing which you will obtain this Equation $z + \frac{z^3}{3r^2} = t * - \frac{2}{15}\frac{t^5}{r^4} + \frac{1}{9}\frac{t^7}{r^6} - \frac{5}{81}\frac{t^9}{r^8}$ &c. then I observe, with respect to this last Equation, that the next Term $-\frac{2}{15}\frac{t^5}{r^4}$ would be destroyed, by multiplying both Sides of the Equation that comes

¹ See Com. Epist. N°. 64.

² See Com. Epist. N°. 20.

next in the Example, viz. $z^5 = t^5 - \frac{1}{3} \frac{t^7}{r^2} + \frac{1}{9} \frac{t^9}{r^4} \&c.$ by $\frac{2}{15r^4}$ and then adding the two Equations together; whence arises $z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} = t^{***} - \frac{1}{3} \frac{t^7}{r^2} + \frac{5}{9} \frac{t^9}{r^4} \&c.$ Next I perceive that the Term $-\frac{1}{3} \frac{t^7}{r^2}$ in this last Equation may be destroyed by multiplying the fourth Equation in the Example, viz. $z^7 = t^7 - \frac{7}{3} \frac{t^9}{r^2}$ by $\frac{17}{315r^6}$, and then adding as before: for hence will arise $z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} + \frac{17z^7}{315r^6} = t^{***} - \frac{6}{2835} \frac{t^9}{r^6}$: finally, by multiplying $z^9 = t^9 \&c.$ by $\frac{62}{2835r^8}$, and adding as before, you'll destroy the Term $-\frac{6}{2835} \frac{t^9}{r^6}$: for there will arise $z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} + \frac{17z^7}{315r^6} + \frac{62z^9}{2835r^8} \&c. = t$, the Equation sought; whereby the Tangent is found from the Arch; which is the same with Mr. Gregory's¹. And after the same Manner may any other Series be reverted.

137. By this Method of extracting the Roots of affected Equations, we are enabled to resolve Equations though infinitely affected: thus in the Equation:

$$\left. \begin{array}{l} - 8 + z^2 - 4z^4 + 9z^6 - 16z^8 \&c. \\ + y \times z^2 - 2z^4 + 3z^6 - 4z^8 \&c. \\ - y^2 \times z^2 - z^4 + z^6 - z^8 \&c. \\ + y^3 \times z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6 - \frac{1}{4}z^8 \&c. \end{array} \right\} = 0.$$

Let the Value of y be required as far as the 4th Dimension of z : then because z is supposed to be small, the first Term of the Root may be derived from this fictitious Equation $-8 + y^3z^2 = 0^*$; which ^{*Art. 105.} therefore must be $+2z^{-\frac{2}{3}}$: therefore, I consider to what Length I propose to have the Root y extracted, that is, to what Dimension of z ; that so the unnecessary Terms may be thrown by: and it being proposed to carry the Quote only to the 4th Dimension of z , all those Terms may be neglected from the beginning, in which the Dimension

¹ See Com. Epist. N^o. 20.

of z , after Substitution of $z^{-\frac{2}{3}}$ for y , exceeds $z^{4+\frac{2}{3}}$, because they cannot affect the Quote within the proposed Limit, as appears from what
 * Art. 82. was shewn formerly * 2 : so that the Equation to be resolved, comes
 120. to this $\frac{1}{3}z^6y^3 - \frac{1}{2}z^4y^3 + z^2y^3 - z^6y^2 + z^4y^2 - z^2y^2 - 2z^4y + z^2y - 4z^4 + z^2 - 8 = 0$: out of which the Root y may be extracted by the preceding Methods.

138. Hence also the Reader may see, how the Root y may be extracted out of this infinite Equation $ay + by^2 + cy^3 + dy^4$ &c. $= gx + bx^2 + ix^3 + kx^4$ &c. the Resolution of which will give the Theorem for this Purpose, published by that famous Analyst Mr. *Abraham de Moivre*, in the Phil. Transf. N^o. 249: and likewise, how still more general Theorems may be investigated, for the Solution of Equations in which the indefinite Species x and y are infinitely affected.

139. One of the chief Difficulties in the Practice of resolving affected Equations, by the Assumption of an infinite Series with indeterminate Coefficients, is, the raising of such an infinite Series to any Power required: and therefore, to serve this, as well as other Purposes, I shall here set down at length the distinct Terms, until the 7th, inclusive, of any indefinite Power of the infinite universal Series $Ax^m + Bx^{m+p} + Cx^{m+2p} + Dx^{m+3p} + Ex^{m+4p} + Fx^{m+5p} + Gx^{m+6p}$ &c. having m for it's Exponent, by Help of our Author's famous Binomial Theorem: by which duly applied we have

$$Ax^m + Bx^{m+p} + Cx^{m+2p} + Dx^{m+3p} + Ex^{m+4p} + Fx^{m+5p} + Gx^{m+6p} \&c. \uparrow^m =$$

$$A^m \times x^{mm} + \frac{m}{1} \times A^{m-1} \times B \times x^{mm+p}$$

$$+ \frac{m}{1} \times A^{m-1} \times C$$

$$+ \frac{m}{1} \times \frac{m-1}{2} \times A^{m-2} \times B^2$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \times x^{mm+2p}$$

* When the proposed Equation, viz. that of y , is transformed (by putting $y = +zx^{-\frac{2}{3}}$ $+p$) into the supplementary Equation of p , that Part of the penult Term of the Equation of p , which stands opposite to the Term $+y^3z^2$ in the Equation of y , is $+12z^{\frac{2}{3}}p$; in which the Dimension of z , viz. $z^{\frac{2}{3}}$, is lower than in any other Part of the penult Term of the Equation of p ; hence it appears, that, in the Prosecution of the Root of the proposed Equation, the several Parts of the ultimate Terms of the supplementary Equations, must always be divided by that Power of z , that is, the Dimension of z in them must be lessened by $\frac{2}{3}$: whence it follows, that, when the Root of the proposed Equation is to be carried no further than z^4 , all those Terms of that Equation, may be neglected from the beginning, where the Dimension of z rises above $z^{4+\frac{2}{3}}$, after substituting $+zx^{-\frac{2}{3}}$ for y , in those Terms. See Art. 120.

†

$$\left. \begin{aligned} & + \frac{m}{1} \times A^{m-1} \times D \\ & + \frac{m}{1} \times \frac{m-1}{2} \times A^{m-2} \times 2BC \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times A^{m-3} \times B^3 \end{aligned} \right\} \times x^{m+3p}$$

$$\left. \begin{aligned} & + \frac{m}{1} \times A^{m-1} \times E \\ & + \frac{m}{1} \times \frac{m-1}{2} \times A^{m-2} \times \begin{cases} 2BD \\ CC \end{cases} \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times A^{m-3} \times 3B^2C \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times A^{m-4} \times B^4 \end{aligned} \right\} \times x^{m+4p}$$

$$\left. \begin{aligned} & + \frac{m}{1} \times A^{m-1} \times F \\ & + \frac{m}{1} \times \frac{m-1}{2} \times A^{m-2} \times \begin{cases} 2BE \\ 2CD \end{cases} \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times A^{m-3} \times \begin{cases} 3B^2D \\ 3BC^2 \end{cases} \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times A^{m-4} \times 4B^3C \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times A^{m-5} \times B^5 \end{aligned} \right\} \times x^{m+5p}$$

$$\left. \begin{aligned} & + \frac{m}{1} \times A^{m-1} \times G \\ & + \frac{m}{1} \times \frac{m-1}{2} \times A^{m-2} \times \begin{cases} 2BF \\ 2CE \\ D^2 \end{cases} \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times A^{m-3} \times \begin{cases} 3B^2E \\ 6BCD \\ C^3 \end{cases} \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times A^{m-4} \times \begin{cases} 4B^3D \\ 6B^2C^2 \end{cases} \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times A^{m-5} \times 5B^4C \\ & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \frac{m-5}{6} \times A^{m-6} \times B^6 \end{aligned} \right\} \times x^{m+6p}$$

&c.

&c.

&c.

By Help of which any Power of the universal infinite Series $Ax^n + Bx^{n+p} + Cx^{n+2p}$ &c. may be raised the Length of seven Terms, whatever be the particular Values of the Numbers n, p, m , which belong to the Exponents: *e. g.* if it was required to raise the infinite Series $A + Bx^2 + Cx^4 + Dx^6$ &c. to the third Power, you'll have $n=0, p=2, m=3$: by inserting of which Values into the general Theorem, it becomes

$$\begin{aligned} A^3 + 3A^2Bx^2 + 3A^2Cx^4 + 3A^2Dx^6 & \text{ \&c.} \\ + 3AB^2 & + 6ABC \\ + BBB \end{aligned}$$

The same Theorem will serve for extracting any Root of an infinite Series; and universally for exhibiting any Power, Root, Dimension, or Dignity whatsoever, of such a Series, whatever it's Exponent be, positive or negative, integral or fractional: *viz.* by inserting such it's Exponent in Place of m : and although the Exponents of x in the Terms of the Series, decrease instead of increasing, it will, notwithstanding, give you any desired Power, by only supposing p to be negative; and n also negative if necessary. Which therefore, because of it's great Excellency and Usefulness, I have laid before the Reader, to apply according to Occasion. The Demonstration of the Binomial Theorem shall conclude this Work, after we have explained a little the first and third Observations under our Author's 35th Art. and shewn by one Example, how this Doctrine of the Resolution of affected Equations is to be applied for finding the Areas of Curves.

Art. 35.

140. Sir *Isaac* takes Notice in the first of these Observations, that, although the Root of the fictitious Equation, by means of which the first Term of the converging Series for the Value of y in the proposed Equation is to be found, were a surd Quantity; or wholly unknown, as being perhaps the inextricable Root of an high affected Equation, you may notwithstanding, even in this Case, resolve the proposed Equation, provided the Root of the fictitious Equation be not impossible (of which immediately) by substituting some Letter for that unknown Root, and then proceeding with it as before, after the Manner he shews.

Thus if the Equation $y^3 - 3a^2y + axy + a^3 - x^3 = 0$ were proposed, to find the Value of y in a converging Series of simple Terms, when x is very small: I find, by the Methods formerly taught, that there can be no other Equation assumed for finding the first Term of the Series, but $y^3 - 3a^2y + a^3 = 0$, whose Root is
a Surd;

a Surd; therefore, instead of the Letter a , which denotes a known invariable Quantity, I substitute any Number I think proper, e.g. 1, by which it becomes $y^3 - 3y + 1 = 0$, which is to be resolved by our Author's Method of resolving affected numeral Equations: by which one of it's Roots will be found $y = + 1.53208$ &c. so that, having substituted any Letter, such as b , for the Root of that fictitious Equation $y^3 - 3a^2y + a^3 = 0$, and extracted the Root of the proposed Equation $y^3 - 3a^2y + axy + a^3 - x^3 = 0$, after the Manner our Author shews, to any desired Length, you substitute $+ 1.53208$ &c. every where for b ; and 1 for a , which was the Value assumed for it; and so you shall have the required Number of Terms of a converging Series, for one of the Roots of the proposed Equation $y^3 - 3a^2y + axy + a^3 - x^3 = 0$. But, if you put $a = 1$, the proposed Equation may be first converted into this $y^3 - 3y + xy + 1 - x^3 = 0$: which being resolved by putting b for the first Term of the Series, as representing the Root of the fictitious Equation $y^3 - 3y + 1 = 0$, you'll obtain $y = b + \frac{b}{c}x + \frac{3b}{c^2}x^2 + \frac{18b^3 + cb^3 + 3cb - c^4}{c^3}x^3$ &c.

where c stands for $3 - 3b^2$: and therefore, having found one of the Roots of $y^3 - 3y + 1 = 0$ to be $+ 1.53208$ &c. insert it instead of the fictitious Root b : and thus you shall have so many initial Terms of the converging Series for the Value of y : in which Value, Unity is equal to the given Quantity, or Line, a .

141. Yet perhaps in such Cases, some of the known Species or Letters belonging to the proposed Equation, may be retained in the Root: thus in the preceding Equation, having found that the first Term of the converging Series ought to be derived from $y^3 - 3a^2y + a^3 = 0$, you may divide the Root of this fictitious Equation by a , according to the known Rules of Algebra, by which it will become $y^3 - 3y + 1 = 0$, and having found any of the Values of y in this numeral Equation, multiply that Value by a , and you shall have an approximate Root of the Equation $y^3 - 3a^2y + a^3 = 0$, for the first Term of the converging Series.

Therefore, all literal Equations are to be freed as much as possible from the Letters they contain, by the Rules of Algebra: but if, after all, there remain one or more known Letters in the fictitious Equation, from which the first Term of the Series is to be derived, and the Root of that Equation be surd or wholly unknown, you may substitute Numbers for such Letters: which Numbers must bear the same Proportion to one another, that the Quantities represented by such Letters do bear, either exactly, if possible, or, if not, as near as may be;

viz.

viz. when there are more than one: but if there be but one, such Letter remaining, you may put any Number for it, you think proper: and then you proceed with the Solution, as you have been directed.

142. I come now to speak a little of that Case, where the Equation to be assumed for deriving the first Term of the Series has no real or possible Root, mentioned by the Author in Obs. 3. Art. 35. In this Case he directs to increase, or diminish, the Quantity x , until you find such a fictitious Equation, as admits of one or more possible Roots.

Thus, if the Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, were proposed; the Equation from which the first Term of a converging Series ought to be derived, when x is very small, is $y^4 + 2y^2 - 2y + 1 = 0$: whose Roots are all *imaginary*, therefore I suppose x to be increased, or diminished, by any given Quantity, which you may call a , and then substitute the Quantity x so increased, or diminished, in place of the former x : thus if you put $z = x - a$, or $z + a = x$, in the preceding Equation, it becomes $y^4 - z^2y^2 - 2azy^2 + zy^2 - a^2y^2 + ay^2 + 2y^2 - 2y + 1 = 0$: where, if you suppose z to vanish, this Equation results $y^4 - a^2y^2 + ay^2 + 2y^2 - 2y + 1 = 0$: in which Equation, if there be any Number, which being put in for a , will give an Equation with one or more real and commensurable Roots, I insert such Number, and then take the Root of that Equation for the first Term of the converging Series: thus I find that if a was supposed to be 2, the last fictitious Equation would become $y^4 - 2y + 1 = 0$, in which one of the Values of y is $+1$: therefore I suppose $z + 2 = x$, and so the transformed Equation $y^4 - z^2y^2 - 2azy^2 - a^2y^2 + zy^2 + ay^2 + 2y^2 - 2y + 1 = 0$, is converted into $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$, where the first Term of a converging Series for the Value of y , is $+1$; and if you prosecute the Extraction, you'll find $y = 1 + \frac{3z}{2} - 4z^2 + \frac{99z^3}{4}$ &c. which therefore is one of the Values of y , z being equal to $x - 2$; and x being supposed to differ very little from 2.

143. The Reader may conceive the Matter thus: the proposed Equation defines the Relation betwixt the Absciss x of some Curve, and it's Ordinate y : now if the Ordinate drawn through the beginning of the Absciss, cannot meet the Curve; it is evident y can have no possible Value, when x the Absciss is supposed to vanish: therefore the Author's Rule for finding the first Term of the Value of y cannot be applied (although perhaps some of the other Methods demonstrated above may) immediately and directly: yet if you suppose the

the beginning of the Absciss to be changed, by removing it to a proper and sufficient Distance, the Ordinate passing through this new beginning of the Absciss, will meet the Curve in one or more Points; and then his Method for finding the first Term of the Root of the proposed Equation, may be applied. Which may be done by inserting $a + z$, or $a - z$, instead of x in the proposed Equation; by Means of which, a new Equation arises, wherein y shall have one or more possible Values when z vanishes, if the Distance a betwixt the two beginnings of the Absciss, be taken of a proper Length. And thus the Root of the proposed Equation may be found in a converging Series, which shall converge the faster the less that z is; or (which is the same Thing) the less that the Difference betwixt x and a is.

144. It is with a Design to explain this Affair that Sir *Isaac* introduces the Figure belonging to this Article, where it appears, that, when the Absciss AC is supposed to begin at the given Point A, there is no possible Value of the Ordinate; yet if that Absciss AC be lessened by the given Line AB; so that the Point B be conceived as the Commencement of the Absciss BC, the Ordinate CD shall have a four fold Value, when BC vanishes: each of which Values may be the first Term of the Root or Quotient.

Suppose the Equation to the Curve to be $50y^4 + \frac{5}{\sqrt{51}}xy^3 - 2\sqrt{51} \times bxy^2 + x^2y^2 - \frac{10}{\sqrt{51}}b^2xy + \frac{5}{\sqrt{51}}bx^2y + b^4 = 0$: here if you put $x = 0$, you have $50y^4 + b^4 = 0$; where y has no possible Value: but if you suppose x to be diminished by the given Quantity $\sqrt{51} \times b$; or suppose $x = z + \sqrt{51} \times b$: by substituting this for that, the given Equation will become $50y^4 + 5by^3 + \frac{5}{\sqrt{51}}zy^3 - 51b^2y^2 + z^2y^2 - 5b^3y + \frac{5}{\sqrt{51}}bz^2y + b^4 = 0$: where, when z vanishes, you have $50y^4 + 5by^3 - 51b^2y^2 - 5b^3y + b^4 = 0$, which has four real Roots, viz. 1°. $y = +b$. 2°. $y = +\frac{b}{10}$. 3°. $y = -\frac{b}{5}$. 4°. $y = -b$, corresponding to the four Values of the Ordinate, viz. BE, BF, BG, BH. Wherefore any of these may be made the first Term of a Series for expressing the Value of y , that is the Ordinate CD; and the other Terms must be found by the preceding Methods: And here it is evident, that, the nearer the Ordinate CD is to EBH, so much the faster must any of these Series's converge: for when the Points B and C coincide, the first Terms of the Series's already found, are the full Values of the Ordinate; and therefore, the less the Distance BC = z is,

z is, the nearer will the Lengths of the Ordinate CD approach to these: and accordingly the approximate Values of the Area's, thence deduced, viz. $BCDE$, $BCDF$, $BCDG$, $BCDH$, will approach the nearer to the Truth, in any given Number of Terms, the less that $BC = z$ is; or the less the Difference betwixt b and x is.

145. Schol. The Reader may observe from what hath been said, from first to last, upon this Subject, that the Roots of specious Equations, including two variable Quantities, may be extracted and expressed after a variety of different Ways. And if one would desire to find after how many Ways this may be done, you must try what are the Quantities, which being substituted for the indefinite Species x , in the proposed Equation, will make it divisible by $y +$ or $-$ any known Quantity; or perhaps by y alone: thus, in the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, if you substitute $+a$, $-a$, $-2a$, or $-\sqrt[3]{2 \times a}$ for x , there will result these Equations $y^3 + 2a^2y - 3a^3 = 0$, $y^3 - 3a^3 = 0$, $y^3 - a^2y + 6a^3 = 0$, and $y^3 - \sqrt[3]{2 \times a^2y} + a^2y = 0$, respectively; the first of which Equations is divisible by $y - a = 0$; the second by $y - \sqrt[3]{3 \times a} = 0$; the third by $y + 2a = 0$; the fourth by $y = 0$, and besides by $y - \sqrt{-1 + \sqrt[3]{2 \times a}} = 0$, and $y + \sqrt{-1 + \sqrt[3]{2 \times a}} = 0$: whence it follows that the known Quantities joined to y , in the several lateral Equations, having their Signs changed, may be the first Terms of so many converging Series's for the Value of y in the proposed Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$; according as you suppose x to differ very little from $+a$ or $-a$ or $-2a$, or $-\sqrt[3]{2 \times a}$, which were the Quantities substituted for x : the Reason of which is evident, since these will be Values of y when x is of the supposed Values: and therefore the several Roots may be prosecuted by inserting $z + a$, or $z - a$, or $z - 2a$, or $z - \sqrt[3]{2 \times a}$ for x respectively, in the proposed Equation; as has been shewn already: after which, x , if you please may be restored.

And these you obtain by supposing the Differences betwixt x and $+a$, x and $-a$, x and $-2a$, x and $-\sqrt[3]{2 \times a}$, that is z , to be very small; but you may obtain other Series's, which shall be Roots of the proposed Equation, by supposing these same Differences, that is z , to be indefinitely great, after the Manner shewn above. And all

ing the Extraction of the Root whose first Term is $+5a$, you would find $y = +5a - \frac{5x^2}{91a} - \frac{45x^4}{91a^3} + \frac{6470x^6}{91a^5} + \frac{971855x^8}{91a^7} + \frac{76184570x^{10}}{91a^9} \&c.$
 $= BC$. Wherefore by our Author's second Rule the Curvilinear Area $ABCD = 5ax - \frac{5x^3}{3 \times 91a} - \frac{9x^5}{91a^3} + \frac{6470x^7}{7 \times 91a^5} \&c.$ which converges the faster the less that x ($= AB$) is.

After the same Manner, by prosecuting the Extraction of the Roots whose first Terms are $-2a = Ad$, and $-8a = Ad$, you would find $y = -2a - \frac{x^2}{21a} - \frac{11x^4}{21a^3} \&c. = Bc$, and $y = -8a + \frac{4x^2}{39a} + \frac{74x^4}{39a^3} \&c. = Bx$: and therefore the Area $ABcd = -2ax - \frac{x^3}{3 \times 21a} - \frac{11x^5}{5 \times 21a^3} \&c.$ and the Area $ABxd = -8ax + \frac{4x^3}{3 \times 39a} + \frac{74x^5}{5 \times 39a^3} \&c.$ which Areas lying below the Base AB , are represented negative; and therefore their true Values are obtained by changing the Signs of all the Terms: which Expressions are so much the nearer the Truth, the less that x ($= AB$) is.

147. But besides the foregoing Value of y , express'd by Series's which converge when x is very small; there is another Value of y , the first Term of which is found by assuming the fictitious Equation $+x^2y - 80a^3 = 0$, viz. $+ \frac{80a^3}{x^2}$, which agrees to the Supposition of x being very great: and which Value must denote the Ordinate BC lying above the Base, since the first Term is positive: this Root may be further prosecuted, and it will give $y = + \frac{80a^3}{x^2} + \frac{2720a^5}{x^4} - \frac{60480a^7}{x^6} + \frac{631680a^9}{x^8} \&c. = BC$: which converges only when x is great in Comparison of a ; and, by the Dimensions of x increasing continually in the Denominators of the Terms, shews that the Line AB is an Asymptote to that Branch of the Curve, viz. DCE . And the Area derived from this Expression of the Ordinate (by the Author's second Rule) is $- \frac{80a^3}{x^2} - \frac{2720a^5}{3x^3} + \frac{60480a^7}{5x^5} \&c.$ which denotes the Area BCa lying upon the other Side of the Ordinate, and adjacent to the Absciss infinitely extended with the Curve upon that Side. But upon the account of that Position, the Expression gives it negative; and therefore the Signs of the Terms must be changed: but it will only be of Use, when x (AB) is very great.

148. But if the curvilinear Area $ilmk$, adjacent to any given Part of the Base il , be required, then you may find the Areas $Almd$ and

and $Aikd$ separately, according to Art. 146, provided the Series's by which these Areas are expressed, converge sufficiently; and subtract the latter from the former: or if these Series's do not converge sufficiently, then find the Areas ika and lma according to Art. 147, if the Series's for these Areas converge sufficiently, and subtract the one from the other. But if none of these two Methods will serve your Purpose, you must suppose the beginning of the Absciss to be changed, and brought to some other Point, such as you find to be most convenient for the Purpose. In chusing this new beginning of the Absciss, it is proper to enquire whether there be any known Length, which substituted for x in the given Equation $y^3 + 5ay^2 - 34a^2y + x^2y - 80a^3 = 0$, will produce an Equation having a commensurable Root, at least a Root which is not involved in a complex Radical Expression, according to Art. 145, by which given Length if the Absciss x be shortened, by substituting $z +$ that Length for x , and the Root of the new transformed Equation extracted, by supposing z to be very little, the Root may converge sufficiently for your Purpose: in which Case, you should proceed by supposing the Absciss x to be shortened by that known Length. But if this cannot be obtained, you must substitute $z +$ any known Length for x , that will produce an Equation such as is mentioned, and resolved Art. 140, 141: which known Length should be taken as convenient for the Purpose as possible.

149. That this may be explained by the Example before us; let it be $Ai = 6a$, and $Al = 8a$: the Area $ilmk$ might be found by finding the Areas $Aikd$ and $Almd$ separately, if the Series which arises by subtracting the two Series's denoting these two Areas, did sufficiently converge: but you would find it by Trial to converge so slowly as to make this Method improper. Wherefore, you may next try to find the Area $ilmk$ by Means of the Areas ika and lma : for finding the Area ika , substitute $6a = Ai$ for x in the general Series belonging to this Case, Article 147, viz. $-\frac{80a^3}{x} - \frac{2720a^5}{3x^3} + \frac{60480a^7}{5x^5} - \frac{631680a^9}{7x^7} \&c.$ by which it becomes $-\frac{80a^2}{6} - \frac{2720a^2}{3 \times 216} + \frac{60480a^2}{5 \times 7776} - \frac{631680a^2}{7 \times 279936} \&c.$ that is, when reduced, $-\frac{40}{3} - \frac{340}{81} + \frac{14}{9} - \frac{235}{729} \&c. \times a^2$. Then you find the Series for the Value of the Area lma , by the like Means, by substituting $8a = Al$ for x in the same general Expression; which, after due Reduction, becomes $-10 - \frac{85}{48} + \frac{189}{512} - \frac{705}{16384} \&c. \times a^2$: both which Expressions are

O o o 2

negative,

negative, and therefore if you subtract the first, which is greatest, from the last, you'll obtain $+\frac{10}{3} + \frac{3145}{1296} - \frac{5467}{4608} + \frac{3336259}{11943936} \&c. \times a^2 = 4.8 \&c. \times a^2$ for an approximate Value of the Area $ilmk$ required.

But if none of the preceding Methods would answer (or even allowing that any of them did answer in some Measure, but so as the Convergency of the Series were very slow) you must suppose the beginning of the Absciss to be changed, so as it may fall in with the Point i or l , or not be far distant from them, by the Method pointed out in Art. 145: but since I do not find any known Quantity such as $5a$, $6a$, $7a$, $8a$, or $\sqrt{35} \times a$ $\sqrt{48} \times a$ &c. which substituted for x in the given Equation $y^3 + 5ay^2 - 34a^2y + x^2y - 80a^3 = 0$, would make it divisible by y — any known Quantity, according to that Art. (which if there were and I could find that Quantity, and it were such as the Absciss being shortened by, would bring the beginning of it to be near to, or in the Line il , I would shorten the Absciss by that known Length) therefore you may alter the Beginning of the Absciss variously; but most conveniently by bringing it to the Point n , which bisects il : draw the Ordinate no ; then is $An = 7a$: therefore, calling the new Absciss z , to be reckoned from the given Point n , substitute $7a + z$ for x in the proposed Equation, and manage it as directed in Art. 142. By such Substitution, the proposed Equation becomes $y^3 + 5ay^2 + 15a^2y + 14azy + z^2y - 80a^3 = 0$; from which assuming the fictitious Equation $y^3 + 5ay^2 + 15a^2y - 80a^3 = 0$, I put any letter as b for it's Root; and so extract the Root of the Equation $y^3 + 5ay^2 + 15a^2y + 14azy + z^2y - 80a^3 = 0$, in the Manner our Author shews Art. 35. Rem. 1. by which you find $y = b - \frac{14ab}{c}z + \frac{2940a^4b + 980a^3b^2 - c^2}{c^3}z^2 + \frac{42ab^2d - 210a^3d + 14abc^3}{c^5}z^3 \&c.$ where c stands for $15a^2 + 10ab + 3b^2$, and d for $2940a^4b + 980a^3b^2 - c^2$. And therefore the Area $n l m o = bz + \frac{7ab}{c}z^2 + \frac{2940a^4b + 980a^3b^2 - c^2}{3c^3}z^3 + \frac{42ab^2d - 210a^3d + 14abc^3}{4c^5}z^4 \&c.$ Wherefore since $ni = nl$, but negative, as lying upon the other Side of the Point n , where the Absciss z begins, the Area $no ki$ will be denoted by the same Expression, only by making z negative, which is done by changing the Signs of the first, third, fifth, &c. Terms into their Opposites; that is the Area $no ki = -bz + \frac{7ab}{c}z^2 - \frac{2940a^4b + 980a^3b^2 - c^2}{3c^3}z^3 + \frac{42ab^2d - 210a^3d + 14abc^3}{4c^5}z^4 \&c.$ Therefore, if you subtract this negative Value of $no ki$; or add it's

it's positive Value, which is the opposite of the former, you'll have the Area $ilmk = 2bz + \frac{5880a^2b + 1960a^3b - 2c^2}{3c^3} z^3 \&c.$ And in order to find the Value of b , you must extract the Root of the Equation $y^3 + 5y^2 + 15y - 80 = 0$, by our Author's Method of extracting the Roots of numeral Equations, and multiply it by a ; by which you'll find $b = 2.42432070 \&c. \times a$. So that if you substitute it for b , and a for $z = nl$, in the preceding Value, you shall have $ilmk = 4.848 \&c. + \frac{19006.674 \&c. - 6469.576 \&c.}{561937.669 \&c.} \&c. \times a^2 = 4.86 \&c. \times a^2$ the Area required.

150. Schol. 1. By thus making the new Beginning of the Absciss to fall exactly in the middle of the given Base il , the Series will be made to converge faster than if you made it to fall in the Point i or l ; and it is likewise more convenient for Calculation, than if it were brought to any other Point; since the same Series serves for the two different Parts $nlmo$ and $niko$, with the Change only of the Signs of some Terms: by the Addition of which Series to itself with such Change of Signs, the resulting Series, which expresses the Area $ilmk$ will converge the more quickly.

152. Schol. 2. The Reader may observe that this Curve belongs to the 40th Species of our Author's Enumeration of Lines of the third Order: and that the Areas belonging to the different Branches of the Curve, might much more easily be found by Prop. 5. of the Quad. of Curves, by taking the Absciss upon AD perpendicular to AB. It may be observed likewise that Cor. 7 and 8. Prop. 9. *ibid.* may be of considerable Use; and when Occasion requires, ought to be applied, for reducing affected Equations, either to such as are not affected; or such as are less affected, before we attempt to find the Areas of those Curves, the Relations of whose Ordinates and Abscisses are defined by such Equations.

153. Schol. 3. With respect to the Doctrine of the Extraction of the Roots of affected specious Equations, contained in this Section, it is to be observed, that although, in order to find an infinite converging Series, which approaches continually to the true Value of the Root y , the Quantity x must be supposed to be indefinitely small; or indefinitely great, in Comparison of any other known Quantity in the Equation, as appears from the Methods of extracting these Roots; yet after such Series for the Root is thus investigated, it will still denote the Value of y , whatever be the Value of x : only the Series will not converge so fast; or perhaps not at all. This appears from the
Analogy

Analogy of the Thing, especially when x and y are conceived as denoting the Absciss and Ordinate of a Curve. Only, in these Equations, the Value of x may be so great, or so small, as to render the Root impossible: which will happen when the Ordinate y cannot meet that Branch of the Curve, to which the infinite Equation belongs: and in this Case the Aggregate of all the Terms of the Series becomes infinitely great.

S E C T. VI.

Containing the Investigation and Demonstration of the
Binomial Theorem.

THE Demonstration of this famous Theorem I promised to give Art. 45, having there illustrated it by several Examples.

154. I say then that
$$\overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} \times$$

$$1 + \frac{\frac{m}{n}}{n} Q + \frac{\frac{m}{n}}{n} \times \frac{m-n}{2n} Q^2 + \frac{\frac{m}{n}}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} Q^3 + \frac{\frac{m}{n}}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} Q^4 \text{ \&c.}$$

or
$$P^{\frac{m}{n}} + \frac{\frac{m}{n}}{n} A Q + \frac{m-n}{2n} B Q + \frac{m-2n}{3n} C Q + \frac{m-3n}{4n} D Q \text{ \&c.}$$
 where $P+PQ$ denotes the Quantity whose Power, Root, or Dimension whatsoever, is to be investigated; P the first Term, Q the Quotient of all the other Terms divided by the first; $\frac{m}{n}$ the Index or Exponent of the Power, Root or Dimension whatsoever, which denotes any Number whatsoever, integral or fractional, positive or negative. Moreover, in the second Form the Capitals $A, B, C, \text{ \&c.}$ denote the first, second, third, &c. Terms of the Series. It likewise appears evident that both the Forms amount to the same Thing: only when it is to be applied to particular Examples, the first Form may be more convenient for continuing the Terms of the Series, by shewing more evidently the Law of the Progression; the other is more convenient for summing up the Terms of the Series, when the Letters have the Numbers which denote their particular Values, put in Place of them.

155. Case 1. When $n=1$ and m positive: then it shall appear in this Shape
$$\overline{P+PQ}^m = P^m \times 1 + mQ + m \times \frac{m-1}{2} Q^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} Q^3 \text{ \&c.}$$

In

In order to demonstrate this first Case of the Prop. let $1+x$ denote a simple Binomial having Unity for it's first Term and x for the other; and let $p = m+1$: then if $(1+x)^m = 1 + mx + m \times \frac{m-1}{2} x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 \&c.$ I say $(1+x)^p = 1 + px + p \times \frac{p-1}{2} x^2 + p \times \frac{p-1}{2} \times \frac{p-2}{3} x^3 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} x^4 \&c.$

For it is evident from the Involution of Powers by Multiplication, that the several Terms of the Power of $1+x$ whose Exponent is $p = m+1$, will be had by setting down in one Line the several Terms of the Power of $1+x$, whose Exponent is m , in Order, and then these several Terms, each multiplied by x , in the same Order in another Line, advancing each Term one Step, and so adding the two Lines together as follows.

$$\begin{array}{r} 1 + mx + m \times \frac{m-1}{2} x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 \&c. \\ 1x + mx^2 + m \times \frac{m-1}{2} x^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^4 \&c. \\ \hline \end{array}$$

$$\text{Sum. } 1 + 1 + mx + m + m \times \frac{m-1}{2} x^2 + m \times \frac{m-1}{2} + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 \&c.$$

Here the Constitution of the Coefficient of any Term of the Sum is manifest; which may be generally expressed thus:

$$\frac{m \times m-1 \times m-2 \times m-3 \times m-4 \&c.}{1 \times 2 \times 3 \times 4 \times 5 \&c.} + \frac{m \times m-1 \times m-2 \times m-3 \&c.}{1 \times 2 \times 3 \times 4 \&c.}, \text{ which I say is } = \frac{p \times p-1 \times p-2 \times p-3 \times p-4 \&c.}{1 \times 2 \times 3 \times 4 \times 5 \&c.}$$

where the Number of Factors in the Numerators and Denominators of the Terms must be the same; the Number of Factors in the first Term of the first Side of the Equation, the same with those in the Term standing upon the other Side of the Equation; but the Factors in the Numerator and Denominator of the second Term of the first Side of the Equation, one fewer.

For by inserting $p-1$ for m in the first Side of the Equation, that Side stands thus

$$\frac{p-1 \times p-2 \times p-3 \times p-4 \times p-5 \&c.}{1 \times 2 \times 3 \times 4 \times 5 \&c.} + \frac{p-1 \times p-2 \times p-3 \times p-4 \&c.}{1 \times 2 \times 3 \times 4 \&c.}$$

$$= \frac{p-1 \times p-2 \times p-3 \times p-4}{1 \times 2 \times 3 \times 4} \times \frac{p-5}{5} + 1 = \frac{p-1 \times p-2 \times p-3 \times p-4}{1 \times 2 \times 3 \times 4} \times \frac{p}{5}$$

$$= \frac{p \times p-1 \times p-2 \times p-3 \times p-4}{1 \times 2 \times 3 \times 4 \times 5}. \text{ And it is evident that this will be true, whatever}$$

whatever be the Number of Factors by whose Multiplication the Terms of the Equation are made up, according to the forementioned Conditions. Now let us suppose $m=2$, then it is true that $\overline{1+x}^m = 1 + mx + m \times \frac{m-1}{2} x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^3$ &c. $= 1 + 2x + x^2$, all the Terms after the third vanishing, because they involve the Factor $\frac{m-2}{3} = 0$:

therefore $p (= m + 1)$ being 3, it follows that $\overline{1+x}^p = 1 + px + p \times \frac{p-1}{2} x^2 + p \times \frac{p-1}{2} \times \frac{p-2}{3} x^3$. And by the same way of reasoning, if $q = p + 1$ then $\overline{1+x}^q = 1 + qx + q \times \frac{q-1}{2} x^2 + q \times \frac{q-1}{2} \times \frac{q-2}{3} x^3 + q \times \frac{q-1}{2} \times \frac{q-2}{3} \times \frac{q-3}{4} x^4$: and so on continually. Whence it appears that, if m be made to stand for any positive Integer whatsoever, it shall be true that $\overline{1+x}^m = 1 + mx + m \times \frac{m-1}{2} x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4$ &c.

Now the Letter x may stand for any Quantity whatsoever, wherefore $\overline{1+Q}^m = 1 + mQ + m \times \frac{m-1}{2} Q^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} Q^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} Q^4$ &c. consequently $\overline{P+PQ}^m = (P^m \times \overline{1+Q}^m) = P^m \times 1 + mQ + m \times \frac{m-1}{2} Q^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} Q^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} Q^4$ &c. Which is the first Case of the Prop.

156. Schol. 1. Q may denote a Quantity consisting of any Number of Terms as $a + b + c + d$ &c. yet if we know how to raise the simple Binomial $1 + a$ to any positive integral Power, in which Case Q is only one Term; the same Rule will serve to raise the Quantity $1 + a + b + c + d$ &c. to any such Power, where Q consists of any Number of Terms whatsoever $a + b + c + d$ &c. For 1°. suppose $Q = a + b$ a Binomial, then $Q^2 = a^2 + 2ab + b^2$; $Q^3 = a^3 + 3a^2b + 3ab^2 + b^3$; $Q^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$: and so on, by what hath been demonstrated. Again 2°. suppose $Q = a + b + c$, a Trinomial, you may conceive it as a Binomial, whose two Members are a and $\overline{b+c}$: in which Case $Q^2 = a^2 + 2a \times \overline{b+c} + \overline{b+c}^2$; $Q^3 = a^3 + 3a^2 \times \overline{b+c} + 3a \times \overline{b+c}^2 + \overline{b+c}^3$: and so on for other higher Powers in this Case: and thus, by supposing the Number of Terms of which Q consists, to be increased continually, you may perceive by what Means the Quantity $P + PQ$

$P+PQ$ or $P \times 1+Q$, may be raised to any positive integral Power, whatever Number of Terms Q or PQ consist of; and that by the Help of what hath been demonstrated: by which you'll obtain such a Canon as we have exhibited in Art. 139.

157. Schol. 2. It is evident that, in this first Case, the Series for the Power must terminate, and the Number of Terms be one more than the Exponent m .

158. Case 2. When the Exponent $\frac{m}{n}$ is a positive Fraction, proper or improper, the Theorem will appear in this Form:

$$P+PQ^{\frac{m}{n}} = P^{\frac{m}{n}} \times 1 + \frac{m}{n} Q + \frac{m}{n} \times \frac{m-n}{2n} Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} Q^3 \&c.$$

For take the simple Binomial $1+x$ as before: then I say $(1+x)^{\frac{m}{n}} = 1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 \&c. \text{ in inf.}$

To demonstrate which suppose x to be a variable or flowing Quantity, and put $y = 1+x^{\frac{1}{n}}$: raise both Sides of this last Equation to

the Power whose Exponent is n , and it is $y^n = (1+x^{\frac{1}{n}})^n =$ (by Case 1.)

$$1 + mx + m \times \frac{m-1}{2}x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4$$

&c. wherefore if you extract the Root y of this Equation $y^n - 1 - mx$

$$- m \times \frac{m-1}{2}x^2 - m \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 - m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 \&c. =$$

0, by supposing x to be very small: you shall have 1 for the first

Term of a Series for that Root, by assuming the fictitious Equation

$$y^n - 1 = 0^* \text{; and the Form of the Series for that Root shall be } y = 1 + Bx + Cx^2 + Dx^3 + Ex^4 \&c. \text{ * Art. 128.}$$

Wherefore, by Help of the first Case of this Prop. let the Equation $y^n - 1 - mx - m \times$

$$\frac{m-1}{2}x^2 \&c. = 0, \text{ be transformed thus:}$$

$$+y^n = 1 + nBx + nCx^2 + nDx^3 \&c.$$

$$+ n \times \frac{n-1}{2}B^2x^2 + n \times \frac{n-1}{2} \times 2BCx^3 \&c.$$

$$+ n \times \frac{n-1}{2} \times \frac{n-2}{3}B^3x^3 \&c.$$

$$- mx = \frac{mx}{y^n} = \frac{mx}{1 + nBx + nCx^2 + nDx^3 \&c.}$$

$$= mx \times \frac{1}{1 + nBx + nCx^2 + nDx^3 \&c.} = mx \times \frac{1}{1 + nBx + nCx^2 + nDx^3 \&c.}$$

$$= mx \times \frac{1}{1 + nBx + nCx^2 + nDx^3 \&c.} = mx \times \frac{1}{1 + nBx + nCx^2 + nDx^3 \&c.}$$

$$\&c. \quad P p p \quad \text{In}$$

In which transformed Equation, if you put the complex Coefficients, each after the first, equal to nothing, you'll find the Value of the several indeterminate Coefficients B, C, D , &c. that of the first Term being already found to be A .

The Equations are 1°. $nB - m = 0$. 2°. $nC + \frac{m}{n} \times \frac{n-1}{2} B - \frac{m}{2} \times \frac{m-1}{2} = 0$. 3°. $nD + n \times \frac{n-1}{2} \times 2BC + n \times \frac{n-1}{2} \times \frac{m-2}{2} B^2 - m \times \frac{m-1}{2} \times \frac{m-2}{3} = 0$ &c.

From the first you find $B = +\frac{m}{n}$; by inserting which Value of B into the second, you have $nC + \frac{m^2 n - m^2}{2n} - \frac{m^2 - m}{2} = 0$, that is $C = \left(\frac{m^2 n - m^2}{2n^2} - \frac{m^2 n - m^2}{2n^2} \times \frac{m^2 - mn}{2n^2} \right) \frac{m}{n} \times \frac{m-n}{2n}$; and by inserting the Values of B and C in the third Equation, it becomes $nD + \frac{m^3 n - m^3 - m^2 n^2 + m^2 n}{2n^2} + \frac{m^3 n^2 - 3m^3 n + 2m^3}{6n^2} = \frac{m^3 - 3m^2 n + 2mn^2}{6n^2} \times \frac{m}{n} \times \frac{m-n}{2n}$; that is, by reducing, $D = \left(\frac{m^3 - 3m^2 n + 2mn^2}{6n^3} \right) \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}$; and thus you may proceed to find the Values of the other

indeterminate Coefficients *in inf.* So that it appears that $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n} x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} x^3$ &c.

Wherefore it follows, by what was shewn in Case 1, that $P + PQ = P^{\frac{m}{n}} \times 1 + \frac{m}{n} Q + \frac{m}{n} \times \frac{m-n}{2n} Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} Q^3$ &c.

159. Schol. It is evident that the Series in this second Case must be infinite; since no Repetition of n can be equal to m .

160. Case the third. When the Exponent $\frac{m}{n}$ denotes any negative Number integral or fractional, the Theorem will appear in this Form:

$$P + PQ = P^{\frac{-m}{n}} \times 1 - \frac{m}{n} Q + \frac{m}{n} \times \frac{-m-n}{2n} Q^2 - \frac{m}{n} \times \frac{-m-n}{2n} \times \frac{-m-2n}{3n} Q^3 \text{ \&c.}$$

For

For taking the Binomial $1+x$ as before, I say $(1+x)^{-m} = 1 - \frac{m}{1}x + \frac{m(m+1)}{2}x^2 - \frac{m(m+1)(m+2)}{6}x^3 \&c.$ in *inf.*

To prove which, since $(1+x)^{-m} = \frac{1}{(1+x)^m} =$ (by Cases 1 and 2.)

$\frac{1}{1 + \frac{m}{1}x + \frac{m(m+1)}{2}x^2 + \frac{m(m+1)(m+2)}{6}x^3 \&c.}$, divide the Nume-

rator of this Fraction, *viz.* 1, by the Denominator $1 + \frac{m}{1}x + \frac{m(m+1)}{2}x^2 \&c.$ and the Quotient will come out $1 - \frac{m}{1}x + \frac{m(m+1)}{2}x^2$

$-\frac{m(m+1)(m+2)}{6}x^3 \&c.$ For these two Series multiplied into one another will produce 1, as they ought to do, the other Terms in *inf.* destroying one another. Whence

$P + PQ^{-m} = P^{-m} \times 1 - \frac{m}{1}Q - \frac{m(m+1)}{2}Q^2 - \frac{m(m+1)(m+2)}{6}Q^3 \&c.$ in *inf.* for it is evident that the Series cannot terminate.

From all which it appears, that, whatever be the Value of the numeral Exponent $\frac{m}{n}$, whether integral or fractional, positive or negative,

it will still be true that $P + PQ^{\frac{m}{n}} = P^{\frac{m}{n}} \times 1 + \frac{m}{n}Q + \frac{m(m+1)}{2n^2}Q^2 + \frac{m(m+1)(m+2)}{6n^3}Q^3$

$\&c.$ or $P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m(m+1)}{2n^2}BQ + \frac{m(m+1)(m+2)}{6n^3}CQ \&c.$ Q. E. D.

161. This famous Theorem may be more expeditiously investigated and demonstrated, by the Help of the Doctrine of Fluxions; but it is necessary that the Rule for finding the Fluxion of any Power, perfect or imperfect, of any flowing Quantity, be demonstrated from other Principles, without making use of this Theorem itself for that Purpose: which our Author does *.

Now that Rule may be demonstrated independently of the Binomial Theorem thus: let n denote any positive whole Number, and x a flowing Quantity, then I say the Fluxion of x^n is $nx^{n-1}\dot{x}$.

* Quad.
Curv.
Art. 11.

In order to demonstrate which I proceed thus: let the Fluxion of x^n be supposed to be $nx^{n-1}\dot{x}$, then, if $p = n + 1$, the Fluxion of x^p is $px^{p-1}\dot{x}$. For $x^p = x^n \times x$: wherefore, if z denote that Term in the Geometrical Progression $1, x, x^2, x^3, \&c.$ whose Distance from the first is denoted by n , so that $z = x^n$, then it is $x^p = zx$; therefore their Fluxions are equal: now the Fluxion of zx is $\dot{z}x + z\dot{x}$ (See Art. 76 of the Quadrature of Curves explained) that is, by restoring x^n for z , and $nx^{n-1}\dot{x}$ for $\dot{z}$ (which are equal by the hyp. since $nx^{n-1}\dot{x}$ is supposed to be the Fluxion of x^n) $nx^n\dot{x} + x^n\dot{x}$ or $n + 1 \times x^n\dot{x}$ that is $px^{p-1}\dot{x}$: which therefore is the Fluxion of x^p , if so be $nx^{n-1}\dot{x}$ be the Fluxion of x^n . Now, by the same way of reasoning, if $q = p + 1$, it appears, that, if the Fluxion of $x^p = px^{p-1}\dot{x}$, the Fluxion of x^q shall be $qx^{q-1}\dot{x}$: and so on continually. But if $n = 2$, it is demonstrated that the Fluxion of $x^2 (= x^2)$ is $nx^{n-1}\dot{x} = 2x\dot{x}$, in the forecited Place: therefore it follows, that, if n be 3, 4, 5, &c. that is any positive Integer whatsoever, the fluxion of x^n shall be $nx^{n-1}\dot{x}$. And from what is shewn in the same Place, it appears, that, if $nx^{n-1}\dot{x}$ be the Fluxion of x , when n is a positive Integer, it shall be so likewise when n denotes any other Number, whether it be a negative Integer; or positive or negative Fraction of any Kind: so that the Rule is demonstrated to hold universally.

162. Cor. Hence it appears that the Fluxion of $\sqrt[n]{1+x}$ is $n \times \sqrt[n]{1+x}^{n-1} \dot{x}$; n denoting any numeral Index as before.

163. Upon the preceding Foundation the Binomial Theorem is thus demonstrated: suppose $\sqrt[n]{1+x} = 1 + Bx + Cx^2 + Dx^3 + Ex^4 \&c.$ where n stands for any numeral Index, positive or negative, integral or fractional; and the Letters B, C, D, E, &c. stand for indeterminate Coefficients to be found: it being evident from the Nature of Powers and Roots, that the assumed Series ought to be of the Form here express'd, and have 1 for it's first Term. Now by taking the Fluxions of both Sides of the preceding Equation (by Art. 161, 162.) we have $n \times \sqrt[n]{1+x}^{n-1} \dot{x} = B\dot{x} + 2Cx\dot{x} + 3Dx^2\dot{x} + 4Ex^3\dot{x}, \&c.$ and dividing by $\dot{x}$; $n \times \sqrt[n]{1+x}^{n-1} = B + 2Cx + 3Dx^2 + 4Ex^3 \&c.$ which being true whatever be the Value of the flowing Quantity x , it must be true likewise when x vanishes: wherefore, let x vanish, and you have $n \times 1^{n-1} = B$, that is $n = B$. Again, assuming the last Equation, viz. $n \times \sqrt[n]{1+x}^{n-1} = B + 2Cx + 3Dx^2 + 4Ex^3 \&c.$ and putting it into Fluxion, you have $n \times n-1 \times \sqrt[n]{1+x}^{n-2} \dot{x} = 2C\dot{x} + 2 \times 3Dx\dot{x}$

$3Dxx + 3 \times 4Exx \&c.$ and if you divide both Sides by x , and then suppose x to vanish as before, you find $n \times \frac{n-1}{2} = C$. Again by assuming the Equation $n \times n-1 \times 1+x^{n-2} = 2C + 2 \times 3Dx + 3 \times 4Ex^2 \&c.$ (which arises by dividing the preceding by the Fluxion of x) and putting it into Fluxions, you have $n \times n-1 \times n-2 \times \frac{1}{1+x} x^{n-3} = 2 \times 3Dx + 2 \times 3 \times 4Exx, \&c.$ which being reduced as before, gives $n \times \frac{n-1}{2} \times \frac{n-2}{3} = D$: and by proceeding after this Manner, it will be evident to any one that considers the Operation, that all the Coefficients of the subsequent Terms of the assumed Series $1 + Bx + Cx^2 + Dx^3 + Ex^4 \&c.$ will be found by the continual Multiplication of the Terms of this Series $1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \&c. \text{ in inf.}$ the Coefficient of each Term of the Series $1 + Bx + Cx^2 + Dx^3 + Ex^4 \&c.$ being the Product of as many Terms of the other Series, as it's Place denotes. So that by substituting the Values of $B, C, D, \&c.$ we find $1+x^n = 1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3 \&c.$

164. Whence it appears, from what hath been said formerly, that $P + PQ^n = P^n \times 1 + nQ + n \times \frac{n-1}{2} Q^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} Q^3 \&c.$ or $P^n + nAQ + \frac{n-1}{2} BQ + \frac{n-2}{3} CQ \&c.$ n denoting any numeral Index whatsoever, and $A, B, C, \&c.$ standing for the first, second, third, $\&c.$ Terms in the last Form.

165. What further remains of this Treatise of our celebrated Author with respect to the Application of the Doctrine of infinite Series and Quadratures to other Purposes in Geometry, being either sufficiently plain in itself, to one that understands what hath been said to explain the Foundations upon which those Things are built; or having been mostly explained in the Commentary upon the Quadrature of Curves, I thought it improper to trouble the Reader, now brought this Length, with Repetitions of Things to no Purpose.

GENERAL.

GENERAL SCHOLIUM.

If Mankind be capable of such Degrees of Knowledge, as the Inventions and Discoveries of a Sir Isaac Newton prove they are; how much greater Degrees of Knowledge may we suppose *superior Intelligences* capable of arriving at, whose better Faculties and more extended Views, furnish them with a larger Stock of Ideas, the only *Materials* of Knowledge; and enable them to perceive and discern more *accurately* and quickly their Agreements and various Relations. And how much higher Degrees of Knowledge, shall we ourselves arrive to, when we have got beyond the Confines of Mortality, and shaken off these Tabernacles of Clay, where our Souls being *confined* as in a Prison, are kept from *extending* and *enlarging* their Views; and being *bore down* as by a heavy Weight, are greatly *retarded* in their Progress in the Ways of Knowledge?—Then shall we perceive more and *better* Objects, and judge and reason more accurately, quickly, and distinctly about them. — Happy Time, when the Discovery of Truth shall be attended with no such *Toil* and *Labour*, as it often is here! In this *imperfect* State of Things, we are obliged to pursue long Trains of Reasoning, in order to attain to the Knowledge of those Truths; which, when our Faculties are enlarged and *perfected*, we shall perceive and discern as it were at one View; for as then our Knowledge will be more extensive, so it will be more distinct and more *intuitive*: as a sacred Writer expresses it, *Now we see through a Glass darkly, or in a Riddle; but then Face to Face*¹: Happy Day! when

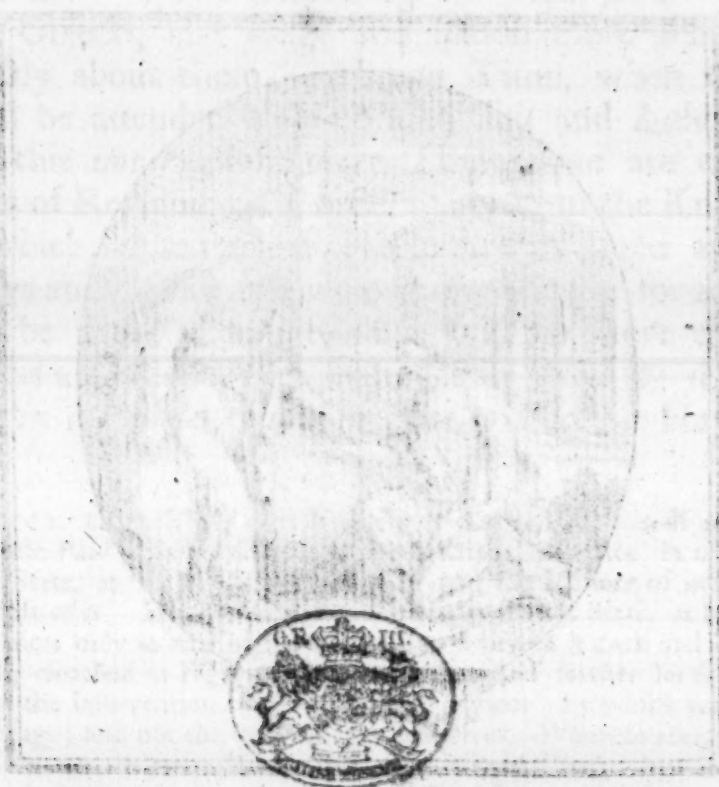
¹ 1 Cor. xiii. 12. Βλέπομεν γὰρ ἄρτι δι' ἰσότητος ἐν ἀινίγματι; τότε δὲ πρόσωπον πρὸς πρόσωπον. Here the Apostle Paul plainly insinuates a remarkable Difference in our Knowledge in this, and the future State, as to the Distinctness of it, and the Manner of attaining it; as well as to the Extensiveness of it. He describes our Knowledge in this State, as being obscure and dark, by saying we know only in an Enigma or Riddle, which is a dark and obscure Representation of some Truth, cloathed in Figure and Metaphor. And further he describes it as acquired δι' ἰσότητος, by the intervention of a Speculum or Mirror; by which we behold the Images or Species's of things; and not the very things themselves. Whereas τότε, in the future State, we shall see and know πρόσωπον πρὸς πρόσωπον; that is immediately, by a direct Intuition of things as they really are in themselves. So that when this Manner of seeing and knowing takes Place, the former shall vanish or be done away: as the same Author asserts in the same Place. And by considering the whole Description St. Paul gives in this Place, of the Knowledge of the future State, compared with that of the present, we shall find the Antithesis to consist in these Things. 1°. It will be vastly greater, more thorough and extensive. 2°. More clear, distinct and satisfactory. 3°. More immediate and intuitive. 4°. Acquired with less Pain and Difficulty. Which Things he illustrates by a beautiful Variety of Expressions.

not only our intellectual, but our *moral* Faculties, shall be extended and enlarged far beyond their present Pitch : and that *blissful* State begin, where our whole reasonable Nature, in all it's Powers and Faculties, shall make further and further Progress and Advancement, in a Likeness to God himself, the *best* and most *perfect* Being.—When we shall be admitted into the *immediate* Presence and *beatifical* Vision of the Divine Majesty ; and for ever employed in searching, contemplating and admiring his Perfections, and his Works : which will afford an inexhaustible Treasure of Knowledge and Delight, encreasing to endless Ages.

The E N D.



not only our intellectual, but our moral Faculties, shall be extended and enlarged far beyond their present limits: and that *Wisdom State* begin, where our whole reasonable Nature, in all its Powers and Faculties, shall make further and further Progress and Advancement, in a likeness to God himself, the best and most perfect Being.—When we shall be admitted into the heavenly Kingdom and blessed Vi-
sion of the living Majesty; and for ever employed in teaching, con-
templating and admiring his Perfections, and his Works: which will afford an inexhaustible Treasure of Knowledge and Delight, ennobling
to exalted Ages:



E. R. R. A. T. A.

In the *Quadrature of Curves* and *Commentary*.

PAG. 7. Line 29. at the Beginning read 21.

p. 10. l. 5. r. $\frac{1}{c}$

p. 16. l. 27. r. $\frac{1}{x}$

20. l. 4. r. $\frac{1}{x^2}$ Ibid. l. 21. Curves r. Cafes

25. to the Figures annex the Nos 1, 2, 3, 4.

Ibid. Tab. Form 1, throughout all the Species's in the Denom^r of the Value of y, r.

$\frac{1}{x^2}$ and in the Area of Species 1, r. $\frac{1}{x^2}$

26. Form 2. throughout all the Species's in the Ordinate of the the Conic Section r.

$$\sqrt{\frac{d}{f} - \frac{e}{f^2} x^2}$$

Ibid. Form 4. Species 2. in the Area, r. $aDGa$

Ibid. Species 3. r. $3aDGa$

27. Form 6. Species 1. Ord. of the second

Conic Section r. $\sqrt{d + \frac{-f-p}{2g} x^2}$

Ibid Form 7. Species 1. in the Denominator of the Area r. $4neg$

29. in the Fig. betwixt K and M place L.

37. l. 16. r. so as to come nearer

39. l. 23. r. wherefore

43. l. 17. r. which the point

44. l. 1. Terms r. Times

45. l. 18. r. $BacC$

48. l. 8. after Area, r. which we have to consider, and this cannot be altered or influenced by any former or future Fluxion of the Absciss

52. l. 1. r. $CEacC$

55. l. 14. r. demonstrated in the

56. l. 6. r. $\frac{n-2}{3} x^{n-3} o^3$

57. l. 31. 11^{th} r. n^{th}

61. l. 14. r. one uniformly flowing Quantity and the Fluxion of another flowing Quantity.

64. Margin 47 r. 46, and 46 r. 51

68. l. 32. r. 15-25

71. l. 3. r. $\frac{x}{x}$ Ibid. l. 5. r. ax

72. l. 13. r. $mx^m y^{m-1} jx^p$

75. l. 11. r. $\frac{4axxy + ax^2y + 6xx^3}{2\sqrt{ay+xx}} = 0$

77. l. 35. dele &c.

78. l. 8. r. $+2cxx=0$ p. 79. l. 35. Times

r. Terms

81. l. 38. or r. and

82. l. penult of the Notes, r. Mr. David Gregory

83. l. 28. BE r. BC

87. in the second Term of both Forms of the Ordinate $2\lambda n$ r. λn

p. 87. in 2^d Term of 2^d Form $2\mu n$ r. μn

88. l. 22. r. $x^{n-1} R^{n-1}$

89. l. 2. r. 26-31

91. Margin r. Art. 114. 117.

95. l. 8. setting r. set.

Ibid. l. 20. r. $\frac{1}{x+1}$

99. l. 11. dele same

101. l. 15. but r. although

Ibid. l. 32. r. may be finite

103. l. 28. r. $e - f^2 x^2$

109. l. 12. r. and so you

Ibid. l. 30. Fraction r. Fluxion

110. l. 28. r. above thrice

112. l. 1. r. $-x^3$ Ibid. l. 9. r. $-\frac{1}{3} - \frac{1}{3}$

117. l. 19. r. $s=1$

119. l. 17. r. be not nothing

121. l. 29. r. $t=s+\lambda$

122. l. 26. r. may be squared

124. l. 29. r. demonstrated

132. l. 29. Forms r. Terms, Terms r. Forms

145. l. 3. r. extending

146. l. 13. r. $\frac{a^2 + x^2}{3}$

147. l. 13. r. $\frac{a^3}{3} = \frac{1}{3} \sqrt{5} - \frac{1}{3}$

148. l. 14. ABC r. ABCD

150. l. 27. r. Now although the Series's above would converge very slowly, or not at all, if it were $x=d$, or $x>d$

151. l. 14. r. $\frac{1}{2} x^2$

153. in Term 2^d of 3^d Ord. $2\mu n$ r. μn

165. l. 30. r. $-\frac{1}{2} + \lambda n$

168. l. 8. R^λ r. $R^{\lambda-1}$ Ibid. l. 15. r. pg

Ibid. l. penult & p. 169. l. 5. r. $-\frac{1}{2} + 2\lambda n \times a$

Ibid. l. 10. f^2 r. f^2

170. l. 14. Before from place a Colon

173. l. 32. r. $\frac{1}{2} + \lambda n - 1$

178. l. 5. r. x^n

180. l. 14. r. $\frac{g + f^2 x^2}{1 + kx^2} \times \frac{1}{1 + kx^2}$

181. l. 7. r. $k + kv^n + mv^{2n} + \&c.$

182. l. 2. r. first, third and fourth

183. l. 29. r. v

184. l. 8. r. $\dot{R}$

186. l. 31. r. gx^{2v}

193. l. 16. of r. for

196. l. ult. after negative, r. (+ only excepted)

199. l. 8. Species r. Form

201. on the Margin opposite to Art. 274. r. Form 1

204. l. penult r. $\frac{4de}{nf} \times \frac{v^3}{2ex} + s$

Qq q

p. 208.

ERRATA.

- p. 208. l. 13. r. $-aDB$
 209. l. 10. r. $+2ngz^nx$
 216. l. 11. r. $\sqrt{\frac{1}{4}pt - \frac{p}{t}x^2}$
 220. l. 4. r. $\frac{d}{ne} \times 2xv - 3s$
 222. l. penult. r. z^{n-0}
 230. l. 24. r. will be an Affymptote
 Ibid. l. 25. are r. and
 231. l. 30. AO r. IO
 233. l. 14, 15, 17. JOHN r. JONH
 Ibid. l. 21. r. $\frac{F. JONH}{HN}$
 Ibid. l. 22. r. $\frac{F. JONH}{HN} = p$
 234. l. 36. $\ddot{x}$ r. $\ddot{z}$
 235. l. 4. x r. z
 236. l. 7. r. the lowest Order
 240. l. 1. r. $\frac{1}{2}xv - s = t$
 242. l. 1. r. $\frac{AB \times BC}{2}$
 243. l. 11, 13. BC r. AC
 244. l. 34, 36. r. AMEGA
 247. l. 12. r. the equilateral Parallelogram
 248. l. 47. r. the Fluxion of the Logarithm
 249. l. 20. DG, Dg, AG, Ag, r. DE, De, AE, Ae
 Ibid. l. 31. DG r. DE, and l. 32 Dg. r. De
 250. l. 4. r. $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = \frac{2}{1}$
 253. l. 12, 14. Aac r. ABE
 Ibid. l. 22. AO^q : r. AO^q
 Ibid. l. 27. r. ΔABE
 254. l. 26. r. and m the measure of
 256. l. 17. r. by a logarithmical
 Ibid. l. 22. r. is to the
 258. l. 25. DB r. BQ
 261. in the Fig. $e r. o$
 Ibid. l. 33. r. $\sqrt{AP^q - aq^q}$
 265. l. 13. r. $\frac{AT \times BD}{2}$
 Ibid. l. 14, 15. r. $\sqrt{e - fz^n}$ Ibid. fx^n r. fx^n
 267. l. 18. r. $DG - DH$
 Ibid. l. 20. r. $\sqrt{fz^n} = AG$
 Ibid. l. 21. r. $DH = \sqrt{-e + fz^n}$
 Ibid. l. 27. $\frac{1}{x^n}$ r. $\frac{1}{\sqrt{x^n}}$
 269. l. 1. in the Denominator r. $\frac{d-2}{d-1}$
 270. l. 11. in the Denominator r. $\frac{d-1}{d-2}$
 274. l. 5. r. $\frac{n-3}{4}$
 Ibid. l. 13. r. Difference of z^n
 Ibid. l. 18. r. In which Case a Difference of any Order is the same with the first Term of that Difference
 277. l. 9. r. when the Fluxions

- p. 280. l. 35. or r. and
 283. l. 16. 3 r. 2 Ibid. l. 18. 2 r. 3
 284. l. 26. r. any impossible
 285. l. 6. r. $2ap - 2px$
 Ibid. l. 7. r. $-2ap + 2px$
 288. l. 16. r. the Chord of the Arch GH measuring the Distance of the Vertex of the Segment from the Circumference of its Base
 289. l. 6. r. 3 = 0 Ibid. l. 27. C r. G
 290. l. 6. r. Ordinate BC
 291. l. 15. x^2 r. a^2
 292. l. 25. Point T. r. Line BT.
 293. l. 4. BC r. BC r. BT
 294. l. 32. SE r. SG
 295. l. 9. r. may^{m-1}
 296. l. 31. y^2 r. y^2
 Ibid. in the Fig. D r. y
 298. l. 20. z r. x Ibid. l. 26. r. $A\beta = \xi$
 301. l. 1. after you r. have
 304. l. penult r. $-\frac{p}{t}x^2$
 305. l. 23. $\frac{1}{2}\sqrt{kl}$ r. $\frac{1}{4}\sqrt{kl}$
 308. l. 29. and r. about
 309. l. 10. r. revolving Parabola.
 Ibid. l. 16. GB r. G β Ibid. l. 30 A β r. G β
 314. l. 17. thence r. then
 315. l. 20. r. than an
 316. l. 24. r. $\frac{aav}{av + uv}$
 317. l. 13. in this and the three next Pages the Numbers of the Articles of Quad. of Curves on the Margin to be encreased by 1.
 Ibid. l. 31. for r. from.

In the Analysis, &c. and Commentary.

322. l. 3. r. $\frac{1}{x^2} (= x^{-2})$
 323. in the Fig. at the End of the Line AB place a
 224. l. 5. r. $= \beta BD\delta$
 327. l. 6. r. $-\frac{1}{2}x^3$
 Ibid. in the Note r. 406 — 408
 329. l. 5. $+2y$ r. $-2y$
 331. l. 1. $4a^2e$ r. $4a^2q$
 339. l. 17. r. $BD = BK$
 Ibid. l. 33. r. Quadratrix
 341. l. 10. r. $3x^2o + 3x^2$
 346. l. 18. r. this Principle
 357. l. 26. r. effected
 377. l. 16. CD r. BD
 378. l. 4. r. AC the half Transverse and AF the half Conjugate
 Ibid. l. 5. AF r. AC, and l. 6. AC r. AF.
 Ibid. l. 16. r. Treatise Art. 406 — 415
 379. l. 3. after the Extraction of the Root, $\frac{1}{2}ax^2$ r. $\frac{1}{2}ax^2$
 380. l. 8. $+P^n$ r. $=P^n$

ERRATA.

- p. 381. l. 11. $r. - \frac{5x^3}{128c^7}$
382. l. 7. $r. \frac{cx^4-x^3}{c^5}; \frac{cx^4-x^3}{c^5}; \frac{cx^4-x^3}{c^5}$
383. l. 2. $r.$ Series. In order to which you may extract the Square Root of the Numerator and Denominator, and then $\mathcal{E}c$.
- Ibid. l. 13. $\frac{bx^2}{a} r. \frac{bx^2}{aa}$
384. l. 8. $-5r r. -5r^3$
387. l. 12. $r.$ a positive Integer
- Ibid. l. ult. dele *in*
388. l. 4. $\frac{dn}{f} r. \frac{dn}{f}$ Ibid. l. 5. $fz^n r. fz^n$
- Ibid. l. 7. $r. \frac{k-n+1}{n}$ Ibid. l. 8. $r. fz^n$
396. at the Bottom dele Values
401. l. 22. $a, \frac{1}{2}x, r. +a, -\frac{1}{2}x$
402. l. 22. after q , $r. \mathcal{E}c$.
404. l. 30. $\frac{3x^2}{r} r. \frac{3x^2}{32}$
406. l. antepen. $r. + \frac{1}{4096} \frac{x^4}{a}$
- p. Ibid. l. penult and p. 407. l. 3. $r. - \frac{1}{4096} \frac{x^4}{a}$
407. l. 12. $r. + 4c^5p$
418. l. ult. whence $r.$ when
420. l. 25. dele cx^2
422. l. 8. Indexes $r.$ Differences of the Indexes Ibid. l. 9. be $r.$ being
423. l. 3. $r. y = + \sqrt[3]{7 \times \sqrt{ax}}$
427. l. 15. $r. + b^{\frac{1}{3}}$
429. l. 10. $r. r' + n'$ and $r'' + n''$
434. l. 6. Sides $r.$ Side
438. l. 32. $r. - x^6$
439. l. 19. dele which do not include x represented here by ay^m ; and another one at least
441. l. ult. $r.$ the Series
445. l. 2. $r.$ to be such, that, after Substitution of that Power of x , &c.
449. l. 5. $r.$ next to one another
466. l. 27. $r. - \frac{80a^3}{x}$

ADVERTISEMENT to the READER.

In revising the Sheets after they were printed off, for discovering the Errors of the Press, I found I had been guilty of some Mistakes, which I thought proper to point out to the Reader, in this Place, with the Corrections; and a few other Additions or Illustrations.

The Reader may be pleased to observe that by the Phrase *Instant of Time* made use of in the following Work, is not meant any Portion of Time, or Part of Duration, how small soever, which must be divisible into Parts, like all other kinds of Quantity; but an Instant of Time must be conceived as having no Parts, and to be with respect to any Portion of Time or Duration, what a Point is with Respect to a Line in Geometry.

Because Lemma 4. Page 41. may be thought not so clearly demonstrated, the following Demonstration may be substituted for the other.

Let V denote any given intermediate Velocity, betwixt M and N any two variable Velocities assumed, through which two Velocities, the variable Velocity, mentioned in this Lemma, passes; so that M be upon the one Side of V, and N upon the other. Imagine M and N to approach continually towards one another, and towards V, which always lies betwixt them: the Time which the variable Velocity takes in passing from M to N, may be so small, by the continual Approach of M and N, as to be less than any given Time: by which Means M and N shall approach so near to one another, as to differ from one another (and consequently from V) by less than any given Difference (as appears from the Nature of the variable Velocity mentioned in this Lemma.) Therefore since M and N approach continually to V, during a finite Time, so as before the End of that Time, to differ from it by less than any given Difference, they must at last be equal to it (by Art. 7.) But V was taken any intermediate Velocity whatsoever. Therefore, $\mathcal{E}c$. as in the Lemma. Q. E. D.

Page 103. Art. 134. The Reader may observe that the first Paragraph of this Article respects the Case in which $\frac{fz^n}{e}$ is less than 1. In which Case there are two Suppositions made; either that s is greater than $r+1$; or less. Now because upon this second Supposition, the Demonstration for proving that the Series there mentioned, must converge, may not appear so plain; it may be more clearly demonstrated thus.

Again if it be $s > r+1$, the Quantity $\frac{s+n}{r+n+1}$ must, at a finite Distance from the Beginning of the Series, become a positive Quantity greater than 1; and after that Time continually diminish, since the Difference betwixt $s+n$ and $r+n+1$, is a given Number: so that the Ratio of

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of $s+n$ to $r+n+1$ approaches nearer to the Ratio of Equality than any given Ratio of greater to less, that is than the Ratio of e to fx^n : therefore $\frac{s+n}{r+n+1} < \frac{e}{fx^n}$: whence $\frac{s+n}{r+n+1} \times \frac{fx^n}{e} < 1$: and therefore the Terms of the Series will continually diminish, and that still in a greater Ratio as the Terms advance, since $\frac{s+n}{r+n+1}$ continually diminishes: and therefore the Series must converge.

Page 104. last Paragraph Art 134. begins thus: "There is a 3^d Case, &c. In which Case the "Convergency depends entirely upon the Coefficients:" whereas it should be said that it depends upon the Coefficients and the Signs of the Terms. Again where it is said, "If $s < r+1$, "both the Series's for the Area will converge," read, One, or both the Series's for the Area will converge, viz. when the Signs of the Terms become alternate; otherwise not:

Page 299. Art. 410. An Oversight having been committed in the Operation in this Article, the Length of the Curve Line is not rightly expressed. Wherefore let that Article run thus.

If we would have the Length of the Curve Line expressed by Logarithms, agreeably to what hath been delivered in §. 10. then (see Fig. Art. 355) which substitute in Place of Fig. 3. Tab 2. Quad. of Curves, supposing the Hyperbola described with the same Latus Transversum and Rectum as before: then it will be $z = \frac{1}{2} \frac{\Delta ABD}{a} - \frac{\frac{1}{2} a A \times a H}{a} \frac{a H}{DC}$: for Example, suppose in the given Parabola, the Length of the Absciss AB (x) = $a = 1$, then in the Hyperbola, by taking $aB = (AB^{\frac{2}{3}} =) 1$, you'll find $AB = \frac{4}{3}$; and from the Property of the Hyperbola, and similar Triangles, $aH = \frac{1}{3}$, $BC = \frac{4}{3}$, $BD = \frac{1}{3}$, $DC = \frac{1}{3}$: therefore the Triangle ABD = $(\frac{AB \times BD}{2} =) \frac{2 \frac{1}{3} \times \frac{1}{3}}{2}$; and the Triangle AaH (which is the Modulus) or $\frac{Aa \times aH}{2} = \frac{2 \frac{1}{3}}{2}$: whence

you'll have $\Delta ABD - \frac{\frac{1}{2} a A \times a H}{a} \frac{a H}{DC} = \frac{2 \frac{1}{3}}{2} - \frac{2 \frac{1}{3}}{2} \times \frac{1}{3}$, that is to say the Measure of the Ratio

of 3 to 1 to the Modulus $\frac{2 \frac{1}{3}}{2}$ subtracted from $\frac{2 \frac{1}{3}}{2}$. But whereas the Modulus of the Tabular Logarithms is 0.434294 &c. (see Art. 349.) therefore the same Quantity expressed by Means of the

Tab. Log. will be (see Art. 350) $\frac{2 \frac{1}{3}}{2} - \frac{27}{512 \times 0.434294 \&c.} \times \text{Log. } \frac{1}{3} = \frac{2 \frac{1}{3}}{2} - \frac{27}{512 \times 0.434294 \&c.} \times 0.477122 \&c. =$ (by reducing all to Decimals) 1.0816293 nearly. Wherefore if you multiply this last Number by $\frac{3}{2}$, you shall have 1.622444 p. min. for the Length of the Curve Line AC required: which, by Means of a Table of Logarithms of more Places of Figures, may be carried to a proportionably greater Number of Figures.

Page 439. Art. 120. near the Beginning: it is said that in the general Equation there mentioned, there will be one Term at least which doth not include x ; and another one at least that includes not y . Which is an Oversight: for it is true that there will be one Term at least that doth not include x : since if all the Terms of the proposed Equation do involve x , you may always free one Term of x by dividing by the lowest Dimension of x : and for the like Reason there will be one Term at least that doth not include y . But it will not always happen that these Terms are distinct: for the same Term that doth not include x , may be the only one likewise that doth not include y . However this Oversight makes no Alteration in the Reasoning there used. But it may be corrected by making it run thus: in which there will be one Term at least which doth not include y , which is here represented by bx^v , omitting what is in the middle.

Page 445. Art. 123. not far from the Beginning of the Article, there is a Mistake in supposing the Letter D to represent the Exponent of the Power of x contained in the first Term of the Value of p ; and then, as a Consequence of that and what was supposed before that, making $A+D$, $B+D$, $E+D$, &c. to represent the Dimensions of x in the penult Term. Because the Dimensions of x in the penult Term will not exceed these in the ultimate by so great a Difference as is the Dimension of x in the first Term of the Value of p , as appears by Rem. 3. Art. 120. preceding. Yet this Mistake doth not affect the Reasoning: and it may be rectified thus.

Instead of saying, "Then suppose the Exponent of the Power of x , which is contained in "the first Term of the Value of p , found as above, to be D: so that after Substitution of such "Power of x , &c." read thus, Then suppose the Exponent of the Power of x , which is contained in the first Term of the Value of p , found as above, to be such, that, after Substitution of that Power of x &c. as it was formerly. And then the Reasoning proceeds and the Symbols answer.



